Hand-in Exercise 2 - O-minimal Structures

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Problem 1.
Let F denote an ordered field and let R be a nontrivial ordered F-linear space as defined in (7.2). Construe R as a model-theoretic structure for the language $L_F = \{<, 0, -, +\} \cup \{\lambda\cdot : \lambda \in F\}$ of ordered abelian groups augmented by a unary function symbol $\lambda\cdot$ for each $\lambda \in F$, to be interpreted as multiplication by the scalar λ. Prove:

1. The subsets of R^m definable in the L_F-structure $\mathcal{R}$ using constants are exactly the semilinear sets in R^m.

2. The maps $R \to R$ that are additive and definable using constants are exactly the scalar multiplications by elements of F. A map f is additive iff

$$\forall r_1, r_2 \in R : f(r_1 + r_2) = f(r_1) + f(r_2).$$

Solution

1. (2 points)
This little exercise is a good example of the concepts defined in paragraph 5, model-theoretic structures. We have to prove that the subsets of R^m definable in the structure $\text{Def}(\mathcal{R}_R)$ are exactly the semilinear sets in R^m. Since every affine function is definable using constants from R, we conclude that all basic semilinear sets in R^m are definable using constants. This means that the basic semilinear sets are definable using constants in every structure on R that contains the relations and the functions of the L_F-structure. Since every structure has to be a boolean algebra on every level of the structure, we conclude that every structure, containing the basic semilinear sets, defines the semilinear sets. Hence the semilinear sets are defined in $\text{Def}(\mathcal{R}_R)$. Furthermore, Corollary (7.6) shows that $(\mathcal{S}_m)_{m\in\mathbb{N}}$, with $\mathcal{S}_m$ the boolean algebra of semilinear subsets of R^m, is actually a structure. We conclude that $\text{Def}(\mathcal{R}_R)$ is said structure and that the subsets of R^m definable in $\text{Def}(\mathcal{R}_R)$ are exactly the semilinear sets in R^m. □

2. (8 points)
Notice that the scalar multiplications are indeed definable and additive. (Additivity is a property of scalar multiplication in a vector space).
Let $f : R \to R$ be an additive map, definable in the L_F-structure $\mathcal{R}$ using constants. Following definition (7.2), we see that R is an ordered additive group and using proposition (4.2), we conclude that R is abelian, divisible and torsion-free. Writing the identity element of R as 0, we see that $f(0) = f(0+0) = f(0) + f(0)$. Since R is torsion-free, $f(0)$ has

to be the identity element, so $f(0) = 0$. Furthermore, writing the additive inverse of an element $r \in R$ as $-r$, we see that $0 = f(0) = f(r + (-r)) = f(r) + f(-r)$, which means that $-f(r) = f(-r)$.
In point 1 of the exercise, we saw that $\mathrm{Def}(\mathcal{R}_R)$ is the structure defined in corollary (7.6), so we can apply the same corollary to see that there is a partition of R into basic semilinear sets A_i, $(1 \leq i \leq k)$, such that $f|A_i$ is the restriction to A_i of an affine function on R, for each $i \in \{1, \ldots, k\}$. Using this we can write $f(x) = \lambda_i x + a_i$ for all $x \in A_i$, with $\lambda_i \in F$, $a_i \in R$, $i \in \{1, \ldots, k\}$. Since R is infinite (for example because it is torsion-free) and our partition finite of definable subsets, there is at least one A_i, such that A_i contains an interval. Take WLOG A_1 as such an element in our partition and let $y, z \in R$ s.t. $(y, z) \subset A_1$. Let $x \in (y, z)$ and $r \in R$, s.t. $x + r \in (y, z)$. We then have for all $r' \in (0, r)$, (so $x + r' \in (y, z) \subset A_1$), the following:

$$f(r') = f(x + r' - x) = f(x + r') + f(-x) = f(x + r') - f(x) = \lambda_1(x + r') + a_1 - (\lambda_1 x + a_1)$$

$$= \lambda_1 x + \lambda_1 r' + a_1 - \lambda_1 x - a_1 = \lambda_1 r'.$$

Here we used the usual properties of scalar multiplication in a vector space. Write this λ_1 as λ. We'll now first prove that for every A_i containing an interval, $\lambda_i = \lambda$. Next we'll prove that for all $x \in R$, $x \in A_j$, that $f(x) = \lambda_j x + a_j = \lambda x$, concluding our prove that every additive and definable map: $R \to R$ is a scalar multiplication by elements of F.
Let A_i be an element in our partition containing an interval. Then there are $x \in A_i$, $r' \in (0, r)$ s.t. $x + r' \in A_i$. Now we have that $\lambda_i x + \lambda_i r' + a_i = \lambda_i(x + r') + a_i = f(x + r') = f(x) + f(r') = \lambda_i x + a_i + \lambda r'$. This means that $\lambda r' = \lambda_i r'$, which implies that $\lambda = \lambda_j$. Suppose not and assume WLOG that $\lambda > \lambda_i$, because we have a linear order on F. Then $(\lambda - \lambda_i)r' = 0$, but $\lambda - \lambda_i > 0$ and $r' > 0$. This is in direct contradiction with definition 7.2 of an ordered F-linear space. We conclude that for every A_i containing an interval $\lambda_i = \lambda$.
Next let A_j be any element of our partition and let $x \in A_j$. Notice that we have a finite number of sets in our partition, each being a finite union of intervals and points. Since R is torsion-free and since F has an infinite number of elements, we conclude that there exist two different $n_1, n_2 \in F$ s.t. $n_1 x = (1 + \cdots + 1)x = x + \cdots + x \in A_k$ and $n_2 x \in A_k$, where A_k is an element in our partition containing an interval. Hence we have that $n_1(\lambda_j x + a_j) = n_1 f(x) = f(x) + \cdots + f(x) = f(x + \cdots + x) = f(n_1 x) = \lambda n_1 x + a_k$ and $n_2(\lambda_j x + a_j) = n_2 f(x) = f(x) + \cdots + f(x) = f(x + \cdots + x) = f(n_2 x) = \lambda n_2 x + a_k$. If we subtract these expressions from each other, we find that $(n_1 - n_2)(\lambda_j x + a_j) = n_1(\lambda_j x + a_j) - n_2(\lambda_j x + a_j) = \lambda n_1 x + a_k - (\lambda n_2 x + a_k) = (n_1 - n_2)\lambda x$. Again we have used the usual properties of scalar multiplication in a vector space. Furthermore we used that F has commutative multiplication, since it is by definition an ordered field. Now multiplying with the multiplicative inverse of $(n_1 - n_2)$, which exists since $n_1 \neq n_2$, we find that $f(x) = \lambda_j x + a_j = \lambda x$. This holds for every A_j in our partition and every $x \in A_j$, so it holds for every $x \in R$.
We conclude that the maps $R \to R$ that are additive and definable using constants are exactly the scalar multiplications by elements of F. $\square$