Topos Theory, Spring 2021
Hand-In Exercises

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1 Exercises

Exercise 1 (Deadline: March 7) Let $\mathcal{C}$ be a small category. Suppose $\mathcal{R}$ is an operation that assigns, to each object C of $\mathcal{C}$, a family $\mathcal{R}(C)$ of sieves on C.

Given a presheaf X on $\mathcal{C}$ and a subobject A of X with classifying map $\chi_A : X \to \Omega$, we define a subobject $\overline{A}$ of X by putting

$$\overline{A}(C) \;=\; \{x \in X(C) \,|\, \chi_A(x) \in \mathcal{R}(C)\}$$

Prove that the operation $A \mapsto \overline{A}$ is a universal closure operation on $\widehat{\mathcal{C}}$ if and only if $\mathcal{R}$ is a Grothendieck topology on $\mathcal{C}$.

Exercise 2 (Deadline: March 21) Let $\mathcal{E}$ be a topos, X an object of $\mathcal{E}$ and $(A \stackrel{a}{\to} X)$ an object of the slice category $\mathcal{E}/X$.

a) Show that there is a bijection between subobjects of $(A \stackrel{a}{\to} X)$ in $\mathcal{E}/X$, and subobjects of A in $\mathcal{E}$.

b) Show that the forgetful functor $\mathcal{E}/X \to \mathcal{E}$, which sends $(A \stackrel{a}{\to} X)$ to A, preserves and reflects monomorphisms.

c) Show that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\langle t, \mathrm{id}_X \rangle} & \Omega \times X \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & X & \end{array}$$

(where p_1 is the projection on the second coordinate, and t is the composition $X \stackrel{!}{\to} 1 \stackrel{t}{\to} \Omega$) is a subobject classifier in $\mathcal{E}/X$.

Exercise 3 (Deadline: April 4) Give the details of the proof of Proposition 1.44, which says that there is a 1-1 correspondence between universal closure operations and Lawvere-Tierney topologies in a topos.

Exercise 4 (Deadline: April 18) Suppose $\mathcal{E}$ is a topos, and $\overline{(\cdot)}$ is a universal closure operation on $\mathcal{E}$.

For a morphism $f : X \to Y$ in $\mathcal{E}$ we let $\forall_f : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ be the restriction of the functor $\prod_f : \mathcal{E}/X \to \mathcal{E}/Y$ to the subcategories Mon/X and Mon/Y of monos into X, Y respectively. We have $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ by pullback, and f^* is left adjoint to $\forall_f$.

a) Prove: if $A \in \mathrm{Sub}(X)$ is closed in X, then $\forall_f(A)$ is closed in Y.

b) Prove that for every pair A, B of subobjects of X there exists a subobject $A \Rightarrow B$ of X which satisfies, for each $C \in \mathrm{Sub}(X)$:

$$C \leq (A \Rightarrow B) \text{ if and only if } C \cap A \leq B$$

Hint: if A is given by the mono $a : A \to X$, consider the subobject $\forall_a(a^*(B))$.

c) Show that for the subobject $A \Rightarrow B$ of X of part b), we always have: if B is closed, then $A \Rightarrow B$ is closed.

Exercise 5 (Deadline: May 9) We consider the presheaf topos $\widehat{\mathcal{C}}$ for a small category $\mathcal{C}$; let C be a fixed object of $\mathcal{C}$. Let $\mathrm{ev}_C : \widehat{\mathcal{C}} \to \mathrm{Set}$ denote the functor which sends a presheaf X to $X(C)$.

a) Show that ev_C preserves all small limits and colimits.

b) Let $A : \mathcal{C} \to \mathrm{Set}$ be the representable functor on C, i.e. the functor which sends an object D to the set $\mathcal{C}(C, D)$. Prove that A is flat.

c) Give a concrete description of the functor $G : \mathrm{Set} \to \widehat{\mathcal{C}}$ which is right adjoint to ev_C.

d) Show that the geometric morphism $\mathrm{Set} \to \widehat{\mathcal{C}}$ determined by the adjunction $\mathrm{ev}_C \dashv G$ is essential.

Exercise 6 (Deadline: May 30) Let $\mathcal{C}$ be a poset and X a presheaf on $\mathcal{C}$ with the property that for every inequality $k' \leq k$ in $\mathcal{C}$, the map $X(k) \to X(k')$ is an inclusion of sets. We consider interpretations of a 1-sorted language where X is the interpretation of the unique sort. We study formulas of the form

$$(D) \qquad\qquad \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$$

where A and B are arbitrary formulas, but the formula B does not contain the variable x free.

Prove that all formulas of the form (D) are always true in X, precisely when X is a constant presheaf.

2 Solutions

Exercise 1 Some notations, which occur also in the lecture notes: for a presheaf X on a category $\mathcal{C}$, an arrow $f : C' \to C$ and an element $x \in X(C)$ we write xf or $X(f)(x)$ for the action of X on f and x. If R is a sieve on C we write f^*R for the set $\{g \in \mathcal{C}_1 \,|\, \mathrm{cod}(g) = C', fg \in R\}$. So if we regard R as element of $\Omega(C)$ then $f^*R = Rf = \Omega(f)(R)$. We write $\max(C)$ for the maximal sieve on C.

We start with some simple remarks:
1. For a sieve R on C, we have $R = \max(C)$ if and only if $\mathrm{id}_C \in R$.
2. For a sieve R on C and $f : C' \to C$, we have $f^*R = \max(C')$ if and only if $f \in R$.
3. Let T be the subobject of Ω which is represented by the mono $1 \xrightarrow{t} \Omega$. Then T is classified by the identity on Ω: $\chi_T = \mathrm{id}_\Omega$.
First, let us assume that the operation $A \mapsto \overline{A}$ is a universal closure operation. For the subobject T of Ω defined in remark 3., we have $\overline{T}(C) = \{R \in \Omega(C) \,|\, (\chi_T)_C(R) \in \mathcal{R}(C)\} = \mathcal{R}(C)$ (by remark 2.). Since $\overline{T}$ is a presheaf, we must have that if $R \in \mathcal{R}(C)$ and $f : C' \to C$ then $f^*R \in \mathcal{R}(C')$, which is the second requirement ("stability") for $\mathcal{R}$ to be a Grothendieck topology.

Since $T \subseteq \overline{T}$ we must have $\max(C) \in \mathcal{R}(C)$, which is the second requirement.

Now let us consider a sieve S on C as subobject of the representable presheaf y_C. It is classified by the map $\chi_S : y_C \to \Omega$ which sends $f : C' \to C$ to f^*S. Now for the closure $\overline{S} \subseteq y_C$ we have

$$\begin{array}{rcl} \overline{S}(C') & = & \{f : C' \to C \,|\, (\chi_S)_{C'}(f) \in \mathcal{R}(C')\} \\ & = & \{f : C' \to C \,|\, f^*S \in \mathcal{R}(C')\} \end{array}$$

From this, using remarks 1. and 3., we deduce that $S \in \mathcal{R}(C)$ if and only if $\overline{S} = y_C$. Now we see that if R and S are sieves on C and $R \in \mathcal{R}(C)$, then also $S \in \mathcal{R}(C)$; for, we have $y_C = \overline{R} \subseteq \overline{S}$. It remains to prove local character for $\mathcal{R}$. So suppose R is a sieve on C, $S \in \mathcal{R}(C)$ and for all $g : C' \to C$ in S we have $g^*R \in \mathcal{R}(C')$. We need to see that $R \in \mathcal{R}(C)$.

The classifying map $\chi_{\mathcal{R}}$ for $\mathcal{R}$ as subobject of Ω sends $R \in \Omega(C)$ to the sieve $\{g : C' \to C \,|\, g^*R \in \mathcal{R}(C')\}$. So,

$$\begin{array}{rcl} \overline{\mathcal{R}}(C) & = & \{R \in \Omega(C) \,|\, (\chi_{\mathcal{R}})_C(R) \in \mathcal{R}(C)\} \\ & = & \{R \in \Omega(C) \,|\, \{g : C' \to C \,|\, g^*R \in \mathcal{R}(C')\} \in \mathcal{R}(C)\} \end{array}$$

Hence, our assumptions on R and S imply that $S \subseteq (\chi_{\mathcal{R}})_C(R)$. Since $S \in \mathcal{R}(C)$, we have $(\chi_{\mathcal{R}})_C(R) \in \mathcal{R}(C)$, which means that $R \in \overline{\mathcal{R}}(C)$. Now $\overline{\mathcal{R}} = \overline{\overline{T}} = \overline{T} = \mathcal{R}$, so $R \in \overline{R}(C)$, as desired. We conclude that $\mathcal{R}$ is a Grothendieck topology.

Conversely, assume that $\mathcal{R}$ is a Grothendieck topology, so it satisfies conditions i), ii) ("stability") and iii) ("local character") of Definition 0.16. We define the operation $A \mapsto \overline{A}$ as in the exercise, and prove that this is a universal closure operation. We check the conditions of Definition 0.18.

First we see that if R, S are sieves on C, with $R \subseteq S$ and $R \in \mathcal{R}(C)$, then $S \in \mathcal{R}(C)$. This follows from i) and iii) of 0.16, since for each $f : C' \to C$ in R, $f^*S = \max(C')$.

Since $\max(C) \in \mathcal{R}(C)$, we see at once that $A \subseteq \overline{A}$, which is requirement i) of 0.18.

If $A \subseteq B$ for subobjects A, B of X, then $(\chi_A)_C(x) \subseteq (\chi_B)_C(x)$ always, and hence $\overline{A}(C) \subseteq \overline{B}(C)$, which is iii) of 0.18.

Condition iv) of 0.18 follows from stability of $\mathcal{R}$.

Finally, in order to prove condition ii), we calculate:

$$\begin{array}{rcl} \overline{\overline{A}}(C) & = & \{x \in X(C) \,|\, (\chi_{\overline{A}})_C(x) \in \mathcal{R}(C)\} \\ & = & \{x \in X(C) \,|\, \{g : C' \to C \,|\, (\chi_A)_{C'}(xg) \in \mathcal{R}(C')\} \in \mathcal{R}(C)\} \end{array}$$

Suppose $x \in \overline{\overline{A}}(C)$. Then $(\chi_{\overline{A}})_C(x) \in \mathcal{R}(C)$, and for any element g of this sieve we have: $g^*((\chi_A)_C(x)) = (\chi_A)_{C'}(xg)$, which is an element of $\mathcal{R}(C')$. By local character of $\mathcal{R}$ we conclude $(\chi_A)_C(x) \in \mathcal{R}(C)$, which means $x \in \overline{A}(C)$. So $\overline{\overline{A}} = \overline{A}$ and we are done.

Exercise 2. It seems best to start with part b). Let $\sum_X : \mathcal{E}/X \to \mathcal{E}$ denote the forgetful functor. Let $f : (B \stackrel{b}{\to} X) \to (A \stackrel{a}{\to} X$ be an arrow in $\mathcal{E}/X$. Suppose $\sum_X(f)$, which is $f : B \to A$, is mono in $\mathcal{E}$ and suppose g, h are arrows $(C \stackrel{c}{\to} X) \to (B \stackrel{b}{\to} X)$ sauch that $fg = fh$. Then since f is mono in $\mathcal{E}$ we have $g = h$ in $\mathcal{E}$, but then also $g = h$ in $\mathcal{E}/X$. So $\sum_X$ reflects monos. Now if f is mono in $\mathcal{E}/X$ and $fg = fh$ in $\mathcal{E}$, for a parallel pair $g, h : C \to B$, then we have $bg = afg = afh = bh : C \to X$, so for $c = bg = bh$ we have that g, h are arrows $(C \stackrel{c}{\to} X) \to (B \stackrel{b}{\to} X)$, so since f is mono in $\mathcal{E}/X$, $g = h$. So f is mono in $\mathcal{E}$ and $\sum_X$ preserves monos, as claimed.

For a), we define a bijection between monos to $A \stackrel{a}{\to} X$ in $\mathcal{E}/X$ and monos to A in $\mathcal{E}$. For a mono $f : (B \stackrel{b}{\to} X) \to (A \stackrel{a}{\to} X)$ let $\phi(f) = f : B \to A$. For a mono $g : B \to A$ in $\mathcal{E}$, let $\chi(g) = g : (B \stackrel{ag}{\to} X) \to (A \stackrel{a}{\to} X)$. It is immediate that $\phi\chi(g) = g$ and $\chi\phi(f) = f$. Moreover, $\phi(f)$ in mono since $\sum_X$ preserves monos, and $\chi(g)$ is mono since $\sum_X$ reflects monos.

Then, we should show that for monos $f : B \to A$ and $f' : B' \to A$ in $\mathcal{E}$ we have: f factors through f' (in which case the subobject represented by f is $\leq$ the subobject represented by f') if and only if $\chi(f)$ factors through $\chi(f')$ in $\mathcal{E}/X$. This is evident.

c) Given a mono $g : (B \stackrel{b}{\to} X) \to (A \stackrel{a}{\to} X)$ is mono in $\mathcal{E}/X$, by b) we know that g is mono in $\mathcal{E}$; let $\chi_g : A \to \Omega$ its classifying map. We have a pullback diagram

$$\begin{array}{ccc} B & \xrightarrow{g} & A \\ \downarrow & & \downarrow{\scriptstyle \chi_g} \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

in $\mathcal{E}$. It is left to you to show that the diagram

$$\begin{array}{ccc} B & \xrightarrow{g} & A\langle a, \chi_g \rangle \\ {\scriptstyle b}\downarrow & & \downarrow \\ X & \xrightarrow[\langle t, \mathrm{id} \rangle]{} & \Omega \times X \end{array}$$

is also a pullback diagram in $\mathcal{E}$, and this is a diagram in $\mathcal{E}/X$. So the map $\langle a, \chi_g \rangle$ classifies the mono g in $\mathcal{E}/X$. Uniqueness is left to you.

Exercise 3. Let UCl be the set of universal closure operations on $\mathcal{E}$, and let LT be the set of Lawvere-Tierney topologies on $\mathcal{E}$. We define operations $\Phi : UCl \to LT$ and $\Psi : LT \to UCl$ as follows:

for a universal closure operation $c_{(\cdot)}$, $\Phi(c_{(\cdot)}) = j$, where j classifies $c_\Omega(1 \stackrel{t}{\to} \Omega)$.

for a Lawvere-Tierney topology j we define $\Psi(j) = c_{(\cdot)}$, where, if $M \in \mathrm{Sub}(X)$ is classified by $\chi : X \to \Omega$, $c_X(M)$ is classified by $j\chi$.

I show first that the pair Φ, Ψ gives a 1-1 correspondence. So, let $c_{(\cdot)}$ be a universal closure operation. For $M \in \mathrm{Sub}(X)$ classified by $\chi : X \to \Omega$, we have that $M = \chi^*(1 \stackrel{t}{\to} \Omega)$ and therefore, by stability of $c_{(\cdot)}$, $c_X(M) = \chi^*(c_\Omega(1 \stackrel{t}{\to} \Omega))$. If we denote the latter by $J \stackrel{a}{\to} \Omega$, then inspecting the diagram

$$\begin{array}{ccc} c_X(M) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \chi} \\ J & \longrightarrow & \Omega \\ \downarrow & & \downarrow{\scriptstyle j} \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

we see that $c_X(M)$ is classified by $j\chi$. This shows that $\Psi\Phi(c_{(\cdot)} = c_{(\cdot)}$.

In the other direction, if j is a Lawvere-Tierney topology and $c_{(\cdot)} = \Psi(j)$ then $c_X(M)$ is classified by $j\chi$ (if χ classifies M), and therefore $c_\Omega(1 \stackrel{t}{\to} \Omega)$ is classified by j. Hence $\Phi\Psi(j) = j$.

Of course, we must show that Φ and Ψ are well-defined: $\Phi(c_{(\cdot)})$ is a Lawvere-Tierney topology if $c_{(\cdot)}$ is a universal closure operation, and $\Psi(j)$ is a universal closure operation if j is a Lawvere-Tierney topology.

So, assume that $c_{(\cdot)}$ is a universal closure operation, and let j classify $c_\Omega(1 \stackrel{t}{\to} \Omega)$, which we write as $J \stackrel{a}{\to} \Omega$. We check i)–iii) of Definition 1.42.

i) By i) of definition 1.43, $(1 \stackrel{t}{\to} \Omega) \leq (J \stackrel{a}{\to} \Omega$, so we have a map $* : 1 \to J$ such that $a* = t$. Since 1 is terminal, $!* = \mathrm{id}_1$. Then $jt = t!* = t$, as desired.

ii) Since j classifies $c_\Omega(1 \stackrel{t}{\to} \Omega)$, jj classifies $c_\Omega(c_\Omega(1 \stackrel{t}{\to} \Omega)$. By iii) of Definition 1.43, $jj = j$.

iii) Let M, N be subobjects of X, classified by ϕ, χ respectively. Then $j \circ \wedge \circ \langle \phi, \chi \rangle$ classifies $c_X(M \cap N)$; and $\wedge \circ (j \times j) \langle \phi, \psi \rangle$ classifies $c_X(M) \cap c_X(N)$. By Exercise 28, these two are equal. So $j \circ \wedge = \wedge \circ (j \times j)$, which is requirement iii) of Definition 1.42.

Finally, assume that j is a Lawvere-Tierney topology, and $c_{(\cdot)} = \Psi(j)$. So, $c_X(M)$ is classified by $j\chi$, if M is classified by χ. Again, we write $J \xrightarrow{a} \Omega$ for $c_\Omega(1 \xrightarrow{t} \Omega)$. We check i)–iv) of Definition 1.43.

i) We have a pullback

$$\begin{array}{ccc} J & \xrightarrow{a} & \Omega \\ \downarrow & & \downarrow{\scriptstyle j} \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

Since $jt = t$ (requirement i) of Definition 1.42) we see that t factors through j, i.e. $(1 \xrightarrow{t} \Omega) \leq c_\Omega(1 \xrightarrow{t} \Omega)$ in $\mathrm{Sub}(\Omega)$. It follows that the inequality $M \leq c_X(M)$ always holds, since pullback functors $f^* : \mathrm{Sub}(\Omega) \to \mathrm{Sub}(X)$ are order-preserving.

ii) Using iii) of Definition 1.42 we deduce that $c_X(M \cap N) = c_X(M) \cap c_X(N)$: let ϕ and χ classify M and N, respectively. Then $\wedge \circ \langle \phi, \psi \rangle$ classifies $M \cap N$ so $j \circ \wedge \circ \langle \phi, \chi \rangle$ classifies $c_X(M \cap N)$, whereas $\wedge \circ (j \times j) \circ \langle \phi, \psi \rangle$ classifies $c_X(M) \cap c_X(N)$. So equality must hold.

Now $M \leq N$ means $M = M \cap N$. This implies $c_X(M) = c_X(M \cap N) = c_X(M) \cap c_X(N)$. So $c_X(M) \leq c_X(N)$, as desired.

iii) This follows straightforwardly from ii) of Definition 1.42.

iv) This is also straightforward, since if $f : Y \to X$ is any arrow and $M \in \mathrm{Sub}(X)$ is classified by ϕ, then $f^*(M)$ is classified by ϕf. Hence $c_Y(f^*M)$ is classified by $j\phi f$; but this arrow also classifies $f^*(c_X(M))$.

Exercise 4.a) Suppose A is closed as subobject of X. Then $c_X(A) \leq A$. By the adjunction $f^* \dashv \forall_f$, $f^*\forall_f(A) \leq A$, so by monotonicity of the closure operation, $c_X(f^*\forall_f(A)) \leq A$. Stability of the closure operation gives $f^*(c_Y(\forall_f(A))) \leq A$. Finally, applying the adjunction once again, we get $c_Y(\forall_f(A)) \leq \forall_f(A)$ and we conclude that $\forall_f(A)$ is closed, as desired.

b) Defining $A \Rightarrow B$ as $\forall_a(a^*B)$ as in the hint (where a is a monomorphism into X which represents A), we have, for an arbitrary subobject C of A:

$$\begin{array}{cc} C \leq (A \Rightarrow B) & \Leftrightarrow \\ a^*(C) \leq a^*(B) & \Leftrightarrow \\ C \cap A \leq B \cap A & \Leftrightarrow \\ C \cap A \leq B & \end{array}$$

c) This follows at once from the construction of $A \Rightarrow B$ in the hint of part b), and part a) of the exercise: if B is closed, so is $a^*(B)$ by stability of the closure operation, and hence so is $\forall_a(a^*(B))$ by part a).

Exercise 5.a) This follows at once from the fact that limits and colimits in $\widehat{\mathcal{C}}$ are calculated 'point-wise'.

b) We have $\mathrm{ev}_C(y_D) = y_D(C) = \mathcal{C}(C, D) = A(D)$, so the following diagram commutes:

$$\begin{array}{ccc} & & \widehat{\mathcal{C}} \\ & \overset{y}{\nearrow} & \downarrow{\scriptstyle \mathrm{ev}_C} \\ \mathcal{C} & \xrightarrow[A]{} & \mathrm{Set} \end{array}$$

Since, moreover, the functor ev_C preserves colimits by a), we have that ev_C is equal to the functor $(-) \otimes_{\mathcal{C}} A$. And this functor preserves finite limits by a), so A is flat, as desired.

c) Suppose $G : \mathrm{Set} \to \widehat{\mathcal{C}}$ is right adjoint to ev_C. Then by the Yoneda Lemma we calculate:

$$\begin{array}{rcl} G(X)(D) & \simeq & \widehat{\mathcal{C}}(y_D, G(X)) \\ & \simeq & \mathrm{Set}(\mathrm{ev}_C(y_D), X) \\ & \simeq & \mathrm{Set}(\mathcal{C}(C, D), X) \end{array}$$

Now it is easy to see that indeed the assignment $X \mapsto (D \mapsto X^{\mathcal{C}(C,D)})$ defines a functor $\mathrm{Set} \to \widehat{\mathcal{C}}$, which is right adjoint to ev_C.

d) We need to show that the functor ev_C also has a left adjoint. I claim that the functor $F : \mathrm{Set} \to \mathcal{C}$ defined by

$$F(X) = \sum_{x \in X} y_C$$

(the functor which sends X to the coproduct of X many copies of y_C) does the job. Indeed, we calculate:

$$\begin{array}{rcl} \widehat{\mathcal{C}}(\sum_{x \in X} y_C, P) & \simeq & \prod_{x \in X} \widehat{\mathcal{C}}(y_C, P) \\ & \simeq & \prod_{x \in X} P(C) \\ & \simeq & \mathrm{Set}(X, \mathrm{ev}_C(P)) \end{array}$$

Exercise 6 The sentence "all formulas of the form (D) are always true in X" means that for every subpresheaf A of X (interpreting the relation symbol A) and every subpresheaf B of 1 (interpreting the 0-ary relation symbol, or propositional constant B), we have that $k \Vdash (D)$.

First, suppose the presheaf X is constant, so every map $X(k) \to X(k')$ is an identity (in Set). Suppose $k \in \mathcal{C}$. We need to prove that $k \Vdash (D)$, so assume $k' \leq k$ is such that $k' \Vdash \forall x(A(x) \vee B)$. We need to prove that $k' \Vdash \forall x A(x) \vee B$. The assumption on k' tells us that for all $k'' \leq k'$ and all $a \in X(k) = X(k'')$, we have $k'' \Vdash A(a) \vee B$. In particular, $k' \Vdash A(a) \vee B$. If $k' \Vdash B$ then we are done. If $k' \nVdash B$, then we must have $k' \Vdash A(a)$ for all $a \in X(k)$, but then by stability (downwards persistence) of $\Vdash$ and the assumption that X is constant, we have $k'' \Vdash A(a)$ for all $a \in X(k'')$, for every $k'' \leq k'$. But this means that $k' \Vdash \forall x A(x)$ and hence $k' \Vdash \forall x A(x) \vee B$, as required.

For the converse, we argue by contraposition so assume X is not constant. Fix k_0, k_1 such that $k_1 < k_0$ and $X(k_0)$ is a proper subset of $X(k_1)$. Define the subpresheaf A of X by: $A(k) = X(k_0)$ if $k \leq k_0$ (and $A(k) = \emptyset$ elsewhere).

Define the subpresheaf B of 1 (so, B is a downwards closed subset of $\mathcal{C}$) by: $k \in B$ if and only if $k \leq k_0$ and $X(k) \neq X(k_0)$.

Now for any $k \leq k_0$ we have: if it is not the case that for every $a \in X(k)$ we have $k \Vdash A(a)$, then $X(k)$ cannot be equal to $X(k_0)$. But then $k \Vdash B$. We conclude that $k_0 \Vdash \forall x(A(x) \vee B)$. However, $k_0 \nVdash \forall x A(x)$ since $k_1 \nVdash \forall x A(x)$, and $k_0 \nVdash B$ is evident. We conclude that $k_0 \nVdash (D)$.