Topos Theory, Spring 2022
Hand-In Exercises

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1 Exercises

Exercise 1 (Deadline: March 3) Let $\mathcal{E}$ be a topos, X an object of $\mathcal{E}$ and $A \stackrel{m}{\to} X$ a subobject of X such that the classifying map $\chi_A : X \to \Omega$ is monic.

a) (3 pts) Show that the unique map $! : A \to 1$ is monic.

b) (3 pts) Suppose for a pair of maps $f, g : Y \to X$ there is a subobject $B \to Y$ such that both squares

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle m} \\ Y & \underset{f}{\longrightarrow} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle m} \\ Y & \underset{g}{\longrightarrow} & X \end{array}$$

are pullbacks. Show that $f = g$.

c) (2+2 pts) We call a category *well-powered* if for every object, its collection of subobjects is a set. As you know, a category is *locally small* if for every pair Y, X of objects, the collection of arrows $Y \to X$ is a set.

Prove that a topos is well-powered if and only if it is locally small.

[Hint: use Exercise 1 of the lecture notes]

Exercise 2 (Deadline: March 17) Let $\mathcal{E}$ be a category with finite limits. For $X \in \mathcal{E}$ and a subobject U of X, we define a *map from* U *to* Y (where $Y \in \mathcal{E}$) to be an equivalence class of diagrams $X \xleftarrow{m} Z \xrightarrow{f} Y$ where m is a representative of U; two such diagrams (m, f) and (m', f') sre equivalent if there is an isomorphism $\sigma : Z \to Z'$ satisfying $m'\sigma = m$ and $f'\sigma = f$.

Now, we define a *partial map* $f : X \rightharpoonup Y$ as a map from U to Y where U is a subobject of X.

a) (3 pts) Show that there is a category $\mathcal{E}_p$ with the same objects as $\mathcal{E}$, but with partial maps as arrows.

b) (3 pts) Show that there is a functor $I : \mathcal{E} \to \mathcal{E}_p$ which is the identity on objects.

c) (4 pts) Show that in $\mathcal{E}$, partial maps are representable if and only if the functor I of part b) has a right adjoint.

Exercise 3 (Deadline: April 4) Call an object A of a locally small category $\mathcal{C}$ *connected* if the representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathrm{Set}$ preserves finite coproducts. From now on, we work in a topos $\mathcal{E}$ and we assume a geometric morphism $f = (f^* \dashv f_*) : \mathcal{E} \to \mathrm{Set}$.

a) (3 pts) An object A is connected if and only if A is non-initial and A is not a coproduct of two non-initial subobjects.

b) (2 pts) Suppose the inverse image functor $f^* : \mathrm{Set} \to \mathcal{E}$ has a left adjoint $f_!$. Prove that an object A of $\mathcal{E}$ is connected precisely if $f_!(A) \simeq 1$.

c) (2 pts) Let $\begin{array}{ccc} X & \xrightarrow{g} & f^*(A) \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle f^*(m)} \\ Y & \xrightarrow[h]{} & f^*(B) \end{array}$ be a pullback diagram in $\mathcal{E}$. Prove that the transposed diagram:

$$\begin{array}{ccc} f_!(X) & \xrightarrow{\tilde{g}} & A \\ {\scriptstyle f_!(n)}\downarrow & & \downarrow{\scriptstyle m} \\ f_!(Y) & \xrightarrow[\tilde{h}]{} & B \end{array}$$

is a pullback diagram in Set. [Hint: in Set, every object is a coproduct of copies of 1]

d) (3 pts) We still assume the existence of the left adjoint $f_!$. Prove that in $\mathcal{E}$, every object is a coproduct of connected objects. [Hint: for an object A of $\mathcal{E}$ and element $s \in f_!(A)$, regarded as arrow $1 \to f_!(A)$ in Set, consider the pullback diagram

$$\begin{array}{ccc} U_s & \xrightarrow{p} & f^*(1) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f^*(s)} \\ A & \xrightarrow[\eta_A]{} & f^* f_!(A) \end{array}$$

where η is the unit of the adjunction $f_! \dashv f^*$]

Exercise 4 (Deadline: April 18) We are working in a topos $\mathcal{E}$ with a Lawvere-Tierney topology (and associated universal closure operation).

a) Suppose

$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ \downarrow & & \downarrow g \\ N & \xrightarrow[n]{} & Y \end{array}$$

is a pullback square with m, n mono and g epi. Show: M is closed in X if and only if N is closed in Y.

b) Suppose that R is an equivalence relation on X and $R \rightrightarrows X \longrightarrow M$ is a coequalizer diagram. Show that M is separated if and only if the mono $R \to X \times X$ is closed.

Exercise 5 (Deadline: May 19) a) (3 pts) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between cartesian closed categories; suppose F has a left adjoint L. Show that F is a cartesian closed functor (i.e., preserves finite products and exponentials) if and only if the natural morphism

$$\langle L\pi_0, \varepsilon_A L\pi_1 \rangle : L(B \times FA) \to LB \times A$$

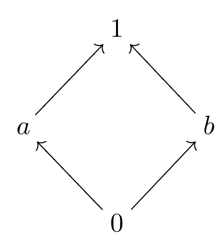
is an isomorphism for all $A \in \mathcal{C}$, $B \in \mathcal{D}$ (here, ε is the counit of $L \dashv F$, and π_0, π_1 are projections).

b) (2 pts) Let F and L be as in a). Show that if F is cartesian closed and L preserves 1, then F is full and faithful.

c) (3 pts) Let again F and L be as in a). Show: if F is full and faithful and L preserves binary products, then F is cartesian closed.

d) (2 pts) Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism between toposes. Show that f is an inclusion if and only if f_* is cartesian closed.

Exercise 6 (Deadline: June 2) Let $\mathcal{C}$ be the following preorder:



a) (5 pts) Show that the presheaf category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a classifying topos for "pairs of subobjects of 1".

b) (5 pts) Give a Grothendieck topology J on $\mathcal{C}$ such that $\mathrm{Sh}(\mathcal{C}, J)$ is a classifying topos for "complemented subobjects of 1" (recall that a subobject A of an object X is complemented if there is a subobject B of X such that $A \cup B = X$ and $A \cap B = 0$).

2 Solutions

Exercise 1. Part a): since χ_A is monic, the composition $\chi_A \circ m = t \circ !$ is monic; so ! is monic.
Part b): the subobject $B \to Y$ is classified by both $\chi_A \circ f$ and $\chi_A \circ g$; by uniqueness of classifying maps, $\chi_A \circ f = \chi_A \circ g$; since χ_A is monic, $f = g$.
Part c): by Exercise 1 of the lecture notes, the map $\mathcal{E}(X, Y) \to \mathrm{Sub}(Y \times X)$ which sends an arrow $f : X \to Y$ in $\mathcal{E}$ to the graph of f as subobject of $Y \times X$, is injective. So if $\mathcal{E}$ is well-powered, then $\mathrm{Sub}(Y \times X)$ is a set, so $\mathcal{E}(X, Y)$ is a set; so $\mathcal{E}$ is locally small.

For the converse, if $\mathcal{E}$ is locally small then $\mathrm{Sub}(X)$, which is in bijective correspondence with $\mathcal{E}(X, \Omega)$, must be a set; so $\mathcal{E}$ is well-powered.

Exercise 2. Part a): we have to show that there are identities and a well-defined notion of composition on partial maps, which make $\mathcal{E}_p$ a category.

Given representatives ($X \xleftarrow{m} Z \xrightarrow{f} Y$) and ($Y \xleftarrow{n} W \xrightarrow{g} Z$) of partial maps $f : X \rightharpoonup Y$, $g : Y \rightharpoonup Z$ respectively, let

$$\begin{array}{ccc} V & \xrightarrow{v} & Z \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[n]{} & Y \end{array}$$

be a pullback. Let the composition $gf : X \rightharpoonup Z$ be represented by ($X \xleftarrow{mv} V \xrightarrow{g\phi} Z$).

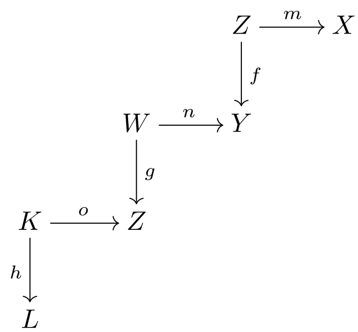
It is easy to see that this is well-defined: if ($X \xleftarrow{m'} Z' \xrightarrow{f'} Y$) and ($Y \xleftarrow{n'} W' \xrightarrow{g'} Z$) are other representatives of the same partial maps, then there are appropriate isomorphisms $\sigma : Z \to Z'$ and $\tau : W \to W'$ which ensure that the pullback diagrams defining the composition will be isomorphic.

For the identity $\mathrm{id} : X \rightharpoonup X$ we take the diagram ($X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$). If ($X \xleftarrow{v} W \xrightarrow{g} Z$) represents a partial map $g : X \rightharpoonup Z$ then by the above definition, $g \circ \mathrm{id}$ is represented by ($X \xleftarrow{v} V \xrightarrow{g\phi} Z$) where

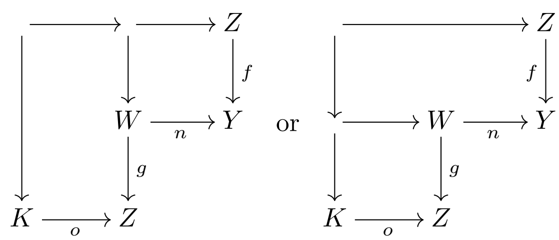
$$\begin{array}{ccc} V & \xrightarrow{v} & X \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ W & \xrightarrow[n]{} & X \end{array}$$

is a pullback. We see that ϕ is an isomorphism and that modulo this isomorphism, $n = v$; so $g \circ \mathrm{id} = g$ as partial maps $X \rightharpoonup Z$. The other identity law is, of course, similar.

It remains to prove that composition is associative; I do this sketchily. We have a diagram



and clearly, in order to define the compositions $f(gh)$ and $(fg)h$, one needs to "fill out" the upper left hand part of this by taking appropriate pullbacks:

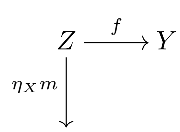


Clearly, for both pullbacks there is an isomorphism between the vertices which commutes with the vertical and horizontal "legs" of the diagram.

Part b): define $I(X) = X$; for $f : X \to Y$ in $\mathcal{E}$ let $I(f) : X \rightharpoonup Y$ be represented by the diagram ($X \xleftarrow{\text{id}} X \xrightarrow{f} Y$). Now obviously, I preserves identities; that I preserves composition is left to you.

Part c): if $\tilde{X}$ represents partial maps into X (for $X \in \mathcal{E}$) then there is a natural 1-1 correspondence between partial maps $Y \rightharpoonup X$ and morphisms $Y \to \tilde{X}$; that is, between $\mathcal{E}_p(I(Y), X)$ and $\mathcal{E}(Y, \tilde{X})$. So the adjunction is clear once we see that $\widetilde{(\cdot)}$ is a functor.

Given an arrow $X \rightharpoonup Y$ in $\mathcal{E}_p$, represented by ($X \xleftarrow{m} Z \xrightarrow{f} Y$), let $\tilde{f}$ be the morphism $\tilde{X} \to \tilde{Y}$ which represents the partial map



Here, $\eta_X : X \to \tilde{X}$ is the universal arrow which belongs to the partial map classifier structure.

Exercise 3. Part a): If A is connected then $\mathcal{E}(A, 0)$ must be initial in Set (since $\mathcal{E}(A, -)$ preserves the empty coproduct), so A is non-initial in $\mathcal{E}$. If $A = B \sqcup C$

with B and C non-initial then $\mathcal{E}(A,A) \simeq \mathcal{E}(A,B) \sqcup \mathcal{E}(A,C)$, so the identity on A factors through a proper subobject of A, which is impossible.

Conversely, suppose A is non-initial and not a coproduct of two non-initial subobjects. Since 0 is strict in any topos, $\mathcal{E}(A,0) = \emptyset$. Consider a map $f : A \to B \sqcup C$. If f does not factor through either B or C then $f^{-1}B$ and $f^{-1}C$ are non-initial and $A = f^{-1}B \sqcup f^{-1}C$; contradicting the assumption on A. We conclude that $\mathcal{E}(A,-)$ preserves finite coproducts.

Part b): first, let us remark that $\mathcal{E}(A,0) \simeq \mathcal{E}(A, f^*(\emptyset) \simeq \mathrm{Set}(f_!A, \emptyset)$, so A is non-initial precisely when $f_!A$ is nonempty.

Suppose $f_!A = 1$. Then A is non-initial by the remark; moreover, if $A = B \sqcup C$ with B and C non-initial, then $1 \simeq f_!B \sqcup f_!C$ so 1 is a coproduct of two nonempty sets; this contradiction shows that A is connected.

Conversely, suppose A is connected. Then $f_!A$ is nonempty by the remark. Moreover, we have a chain of equalities (using, in turn, the adjunction $f_! \dashv f^*$, the fact that f^* preserves 1 and coproducts, and the assumption that A is connected):

$$\begin{array}{ccccccc} 2^{|f_!A|} & = & |\mathrm{Set}(f_!A, 1+1)| & = & |\mathcal{E}(A, f^*(1+1))| & = \\ |\mathcal{E}(A, 1+1)| & = & |\mathcal{E}(A,1)| + |\mathcal{E}(A,1) & = & 2 & \end{array}$$

so $|f_!A| = 1$ and hence $f_!A \simeq 1$.

Part c): consider the commutative diagram

$$(*) \qquad \begin{array}{ccc} X & \xrightarrow{g} & f^*A \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle f^*m} \\ Y & \xrightarrow[h]{} & f^*B \end{array} \qquad \text{for a map of sets } m : A \to B$$

By the hint, B is a coproduct $\coprod_{b\in B} 1$ so $f^*B = \coprod_{b\in B} 1$. Similarly, $f^*A = \coprod_{a\in A} 1$ and f^*m sends the a-th summand of f^*A into the $f(a)$-th summand of f^*B.

Since coproducts are preserved by pullback functors, we have that Y is isomorphic to a coproduct $\coprod_{b\in B} Y_b$ and likewise, X is a coproduct $\coprod_{b\in B} X_b$. For each $b \in B$ we have a pullback square

$$\begin{array}{ccc} X_b & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle n} \\ Y_b & \longrightarrow & Y \end{array}$$

Now the diagram $(*)$ is a pullback precisely when for each $b \in B$, the object X_b is a coproduct of $|n^{-1}(b)|$ many isomorphic copies of Y_b. But if this is the case, then this is preserved by the functor $f_!$. Hence the transposed diagram is a pullback in Set.

Part d): we follow the hint. Let $A \in \mathcal{E}$, $s \in f_! A$, and

$$\begin{array}{ccc} U_s & \xrightarrow{p} & f^*(1) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f^*(s)} \\ A & \xrightarrow[\eta_A]{} & f^* f_!(A) \end{array} \quad \text{be a pullback in } \mathcal{E}$$

By part c), the transposed diagram

$$\begin{array}{ccc} f_!(U_s) & \xrightarrow{\tilde{p}} & 1 \\ {\scriptstyle f_!(q)}\downarrow & & \downarrow{\scriptstyle s} \\ f_!(A) & \xrightarrow[\text{id}]{} & f_! A \end{array} \quad \text{is a pullback diagram in Set.}$$

We see that $\tilde{p}$ must be an isomorphism, so $f_!(U_s) \simeq 1$. Since A is the coproduct of the objects U_s, we see that A is a coproduct of connected objects, as desired.

Exercise 4. Part a): let $\overline{M} \stackrel{\bar{m}}{\to} X$, $\overline{N} \stackrel{\bar{n}}{\to} Y$ be the closures of m in $\mathrm{Sub}(X)$, n in $\mathrm{Sub}(Y)$ respectively. Then by stability of the closure operation we have a pullback diagram

$$\begin{array}{ccc} \overline{M} & \longrightarrow & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{N} & \longrightarrow & Y \end{array}$$

and hence the diagram

$$\begin{array}{ccc} M & \longrightarrow & \overline{M} \\ \downarrow & & \downarrow{\scriptstyle h} \\ N & \longrightarrow & \overline{N} \end{array}$$

is also a pullback.

Moreover, h is an epimorphism. In any regular category, the pullback functor along an epimorphism is faithful, and hence reflects monos and epis. Therefore in a topos it reflects isomorphisms (since a topos is balanced). So we have equivalences:

$$\begin{array}{rc} M \text{ is closed} & \Leftrightarrow \\ M \to \overline{M} \text{ is an isomorphism} & \Leftrightarrow \\ N \to \overline{N} \text{ is an isomorphism} & \Leftrightarrow \\ N \text{ is closed} & \end{array}$$

Part b): we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & X \times X \\ \downarrow & & \downarrow \\ M & \xrightarrow[\delta_M]{} & M \times M \end{array}$$

We have: M is separated if and only if δ_M is closed. Since the map $X \times X \to M \times M$ is epi, by part a) this is equivalent to: R is closed as a subobject of $X \times X$, as required.

Exercise 5. Part a): suppose the natural map $\langle L\pi_0, \varepsilon_A L\pi_1 \rangle : L(B \times FA) \to LB \times A$ is an isomorphism. Since F has a left adjoint, F preserves finite products. To see that F preserves exponentials, we have the following natural bijections for an arbitrary object X of $\mathcal{D}$:

$$\begin{array}{rcl} \mathcal{D}(X, F(B^A)) & \simeq & \mathcal{D}(LX, B^A) \\ & \simeq & \mathcal{D}(LX \times A, B) \\ & \simeq & \mathcal{D}(L(X \times FA), B) \\ & \simeq & \mathcal{D}(X \times FA, FB) \\ & \simeq & \mathcal{D}(X, FB^{FA}) \end{array}$$

(where the third bijection is by application of the assumption), so that $F(B^A)$ is naturally isomorphic to FB^{FA} by the Yoneda Lemma.

Conversely: if F is cartesian closed, we calculate for an arbitrary object X of $\mathcal{C}$:

$$\begin{array}{rcl} \mathcal{C}(L(B \times FA), X) & \simeq & \mathcal{D}(B \times FA, FX) \\ & \simeq & \mathcal{D}(B, FX^{FA}) \\ & \simeq & \mathcal{D}(B, F(X^A)) \\ & \simeq & \mathcal{C}(LB, X^A) \\ & \simeq & \mathcal{C}(LB \times A, X) \end{array}$$

so we have an isomorphism $L(B \times FA) \simeq LB \times A$, again by the Yoneda Lemma (here the third bijection is by cartesian closedness of F). That the *given* morphism is an isomorphism is explicitly shown (by exhibiting an inverse) in the **Elephant**, Lemma A1.5.8.

Part b): Assume F is cartesian closed and L preserves 1. We calculate:

$$\begin{array}{rcl} \mathcal{C}(A, B) & \simeq & \mathcal{C}(1, B^A) \\ & \simeq & \mathcal{C}(L1, B^A) \\ & \simeq & \mathcal{D}(1, F(B^A)) \\ & \simeq & \mathcal{D}(1, FB^{FA}) \\ & \simeq & \mathcal{D}(FA, FB) \end{array}$$

so F is full and faithful.

Part c): Assume F is full and faithful and L preserves binary products.

First, we show that for objects A and B of $\mathcal{C}$, B^{LFA} is isomorphic to B^A: for U arbitrary, we calculate

$$\begin{array}{rcl} \mathcal{C}(U, B^{LFA}) & \simeq & \mathcal{C}(LFA, B^U) \\ & \simeq & \mathcal{D}(FA, F(B^U)) \\ & \simeq & \mathcal{C}(A, B^U) \\ & \simeq & \mathcal{C}(U, B^A) \end{array}$$

Next, we see that we have natural bijective correspondences

$$\begin{array}{rcl} \mathcal{D}(X, F(B^A)) & \simeq & \mathcal{C}(LX, B^A) & \simeq & \mathcal{C}(LX, B^{LFA}) \\ & \simeq & \mathcal{C}(LX \times LFA, B) & \simeq & \mathcal{C}(L(X \times FA), B) \\ & \simeq & \mathcal{D}(X \times FA, FB) & \simeq & \mathcal{D}(X, FB^{FA}) \end{array}$$

so F is cartesian closed.

Part d): If f is an inclusion then f_* is full and faithful. Since f^* preserves finite limits, we can apply part c) and conclude that f_* is cartesian closed. Conversely, if f_* is cartesian closed then since f^* preserves 1 always, by part b) we see that f_* is full and faithful, so f is an inclusion.

Exercise 6. Part a): we must show that for an arbitrary cocomplete topos $\mathcal{E}$, we have a natural bijection between geometric morphisms from $\mathcal{E}$ to $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and pairs of subobjects of 1 in $\mathcal{E}$. Now we know that geometric morphisms $\mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ correspond to flat functors $\mathcal{C} \to \mathcal{E}$. Since $\mathcal{C}$ is finitely complete, flat functors coincide with finite-limit preserving functors $\mathcal{C} \to \mathcal{E}$. In $\mathcal{C}$ we have the following finite limit structure: 1 is the terminal object, $0 = a \wedge b$, and all arrows are monic. Hence a flat functor $\mathcal{C} \to \mathcal{E}$ sends a and b to objects A and B for which the unique morphism to 1 is monic, and is completely determined by this.

Part b): since $\mathcal{C}$ is a poset, we may identify a sieve on some object X of $\mathcal{C}$ with a downwards closed subset of $\{Y \in \mathcal{C} \,|\, Y \leq X\}$. Consider the following Grothendieck topology on $\mathcal{C}$: for a sieve R on 1, $R \in J(1)$ if and only if $\{a, b\} \subset R$; for a sieve R on a, $R \in J(a)$ if and only if $a \in R$ and for a sieve R on b, $R \in J(b)$ if and only if $b \in R$; finally, every sieve on 0 (including the empty sieve) is in $J(0)$.

Now we know that a geometric morphism $\mathcal{E} \to \mathrm{Sh}(\mathcal{C}, J)$ correspond with flat (i.e., finite-limit preserving as we saw in part a)) and continuous functors $\mathcal{C} \to \mathcal{E}$. The continuity now means (for such a functor F) that $F(1) = F(a) \cup F(b)$ and that $F(0) = 0$. So we get that $F(a)$ and $F(b)$ are subobjects of 1, that $F(a) \cap F(b) = 0$ and $F(a) \cup F(b) = 1$. This means that F is (up to isomorphism) completely determined by $F(a)$, which is a complemented subobject of 1.