Topos Theory, Spring 2024 Hand-In Exercises

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1 Exercises

Exercise 1 (Deadline: February 29) We consider a small category \mathcal{C} and a monoid M. An M-presheaf on \mathcal{C} is a presheaf F on \mathcal{C} endowed with, for every object $C \in \mathcal{C}$, a right M-action $F(C) \times M \to F(C)$ (written: $(x,m) \mapsto xm$) which, besides the usual axioms for an M-action, also satisfies: $f^*(xm) = f^*(x)m$ for $f: D \to C$, $x \in F(C)$ and $m \in M$. A morphism of M-presheaves $F \to G$ is a natural transformation $\mu: F \Rightarrow G$ such that $\mu_C(xm) = \mu_C(x)m$ for all $C \in \mathcal{C}$, $x \in F(C)$. Clearly, we have a category M- $\widehat{\mathcal{C}}$ of M-presheaves and morphisms.

- a) Let $\Delta: \widehat{\mathcal{C}} \to M$ - $\widehat{\mathcal{C}}$ be the functor which endows each presheaf F with the trivial (identity) M-action. Show that Δ has a right adjoint, and describe it explicitly.
- b) Show that $M-\widehat{\mathcal{C}}$ is a topos.

Exercise 2 (Deadline: March 14) Recall the definition (before Proposition 3.14) of the map $\exists_f: \Omega^X \to \Omega^Y$ for any monomorphism $f: X \to Y$.

- a) (4 pts) Show that $\exists f$ induces a function $\sum_f : \operatorname{Sub}(X) \to \operatorname{Sub}(Y)$, and describe this function explicitly.
- b) (6 pts) Show that for any subobject A of X, the inequality $A \leq f^*(\sum_f(A))$ holds

Exercise 3 (Deadline: March 28) Let $T: \mathcal{E} \to \mathcal{F}$ be a logical functor between toposes.

- a) (4 pts) Let X be an object of \mathcal{E} . Show that the functor $T/X: \mathcal{E}/X \to \mathcal{F}/TX$ which sends $(Y \xrightarrow{f} X)$ to $(TY \xrightarrow{Tf} TX)$ (with the straightforward action on morphisms) is logical.
- b) (3 pts) Suppose the functor T has a left adjoint F. Show that T/X has a left adjoint.

c) (3 pts) Under the assumption in b), show that T/X has a right adjoint. Can you describe it explicitly?

Exercise 4 (Deadline: April 11) We consider a universal closure operation c on a topos \mathcal{E} .

a) (2 pts) Let

$$A' \xrightarrow{f'} B'$$

$$\downarrow^n$$

$$A \xrightarrow{f} B$$

be a commutative square with m a dense mono and n a closed mono. Prove that there is a unique "filler" $g:A\to B'$ (i.e., a map such that gm=f' and ng=f).

- b) (2 pts) For a subobject A' of A, show that $c_A(A')$ is the unique subobject A'' of A with the property that $A' \to A''$ is dense and $A'' \to A$ is closed.
- c) (3 pts) Show that the composition of two dense monos is dense; and the same for closed monos.
- d) (3 pts) Show that for $A', A'' \in \operatorname{Sub}(A)$ we have: $c_A(A' \cap A'') = c_A(A') \cap c_A(A'')$. [Hint: one inclusion is clear since c is order-preserving. For the other, show that it suffices to prove that $(A' \cap A'') \to (c_A(A') \cap c_A(A''))$ is dense and $(c_A(A') \cap c_A(A'')) \to A$ is closed.]

Exercise 5 (Deadline: April 25) This exercise is about the Heyting algebra structure of subobject lattices. Sub(X) denotes the lattice of subobjects of X.

- a) Let $i: X \to Y$ be a monomorphism. Prove: if Sub(Y) is a Boolean algebra, then so is Sub(X).
- b) Let $p: X \to Y$ be an epimorphism. Prove: if Sub(X) is a Boolean algebra, then so is Sub(Y).

Exercise 6 (Deadline: May 17) Let L be a one-sorted first-order language and suppose R is an L-structure in a topos \mathcal{E} .

- a) Construct, for every object X of \mathcal{E} , an L-structure R_X on the set $\mathcal{E}(X,R)$.
- b) Let ϕ be an L-formula with n free variables x_1, \ldots, x_n , and suppose the only logical symbols occurring in ϕ are \wedge and \to . Prove: for any n-tuple $\alpha_1, \ldots, \alpha_n : X \to R$ the following are equivalent:
 - i) $X \Vdash \phi[\alpha_1, \ldots, \alpha_n]$
 - ii) For all arrows $f: Y \to X$, $R_Y \models \phi(\alpha_1 f, \dots, \alpha_n f)$

2 Solutions

Exercise 1 a) Clearly, if $\mu: \Delta(F) \to G$ is any morphism in $M - \widehat{\mathcal{C}}$ then by the definition of such morphisms, for all $C \in \mathcal{C}$, $x \in F(C)$ and $m \in M$ we have $(\mu_C(x))m = \mu_C(x)$, so μ lands in the part of G which is invariant under the M-action. We have a functor from $M - \widehat{\mathcal{C}}$ to $\widehat{\mathcal{C}}$ which sends each M-presheaf to its invariant part. This is right adjoint to Δ , the verification of which is left to vou.

b) This is most easily done by observing that $M-\widehat{\mathcal{C}}$ is equivalent to a presheaf category: it is equivalent to the category of presheaves on the product category $\mathcal{C} \times M$.

Exercise 2 a) Elements of Sub(X) are in 1-1 correspondence with maps $1 \to \Omega^X$: take the exponential transpose of the classifying map.

Define \sum_f as follows: for $A \in \operatorname{Sub}(X)$, corresponding to the map $a: 1 \to \Omega^X$, define $\sum_f (A) \in \operatorname{Sub}(Y)$ as the subobject corresponding to the composition $\exists_f \circ a: 1 \to \Omega^Y$.

One can prove that this is simply the subobject $A \to X \xrightarrow{f} Y$, although it is hard to argue that this operation is *induced by* \sum_{f} .

b) Let $\widetilde{\exists_f}$ be the exponential transpose of \exists_f . Then $\sum_f(A)$ is classified by the composition

$$Y \stackrel{\langle a, \operatorname{id}_Y \rangle}{\longrightarrow} \Omega^X \times Y \stackrel{\widetilde{\exists_f}}{\longrightarrow} \Omega.$$

Then $f^*(\sum_f(A))$ is classified by

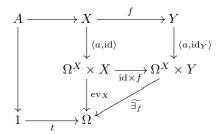
$$X \xrightarrow{f} Y \xrightarrow{\langle a, \mathrm{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists_f}} \Omega.$$

and the inequality $A \leq f^*(\sum_f(A))$ holds if and only if the composition

$$A \to X \xrightarrow{f} Y \xrightarrow{\langle a, \operatorname{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists_f}} \Omega$$

factors through the subobject classifier $1 \stackrel{t}{\to} \Omega$.

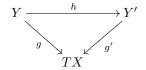
This, however, follows from the commutative diagram:



The lower right-hand triangle commutes by Proposition 3.14, and the left hand square commutes because the composite $\operatorname{ev}_X \circ \langle a, \operatorname{id} \rangle$ equals the transpose of a, that is: the map which classifies A as subobject of X.

Exercise 3 a) Assume $T: \mathcal{E} \to \mathcal{F}$ is logical; this means that T preserves finite limits, subobject classifiers and exponentials. So $T(1 \stackrel{t}{\to} \Omega)$ is a subobject classifier in \mathcal{F} and moreover, if $\chi_A: X \to \Omega$ classifies the subobject A of X in \mathcal{E} , then $T(\chi_A)$ classifies $T(A) \in \operatorname{Sub}(TX)$ in \mathcal{F} . It follows that the map $\Delta: X \times X \to \Omega$, which classifies the diagonal $\delta: X \to X \times X$, is preserved by T. Also, T commutes with taking exponents and also with exponential transposes. So, for example, the singleton map $\{\cdot\}: X \to \Omega^X$ is preserved by T. We see that partial map classifiers are preserved by T. It is now a matter of inspection to see that the whole topos structure of \mathcal{E}/X is preserved by T/X. We conclude that T/X is logical.

b) Let $F \dashv T$. Define $F^X : \mathcal{F}/TX \to \mathcal{E}/X$ as follows: for an object $(Y \xrightarrow{g} TX)$ of \mathcal{F}/TX let $F^X(g)$ be the map $FY \xrightarrow{\tilde{g}} X$, the transpose of g along the adjunction $F \dashv T$. On morphisms



the image $F^X(h)$ is the map $F(h): \tilde{g} \to \tilde{g'}$ obtained by transposing. The adjunction is straightforward.

c) The existence of a right adjoint is an immediate application of Corollary 3.20: T/X is logical and has a left adjoint, so it has a right adjoint by 3.20.

In order to exhibit the right adjoint, we use the assumption in b) once more, and conclude that T has a right adjoint. Let $G: \mathcal{F} \to \mathcal{E}$ be right adjoint to T. Define $G_X: \mathcal{F}/TX \to \mathcal{E}/X$ as follows: $G_X(Y \stackrel{g}{\to} TX)$ is the map $Y' \stackrel{f}{\to} X$, from the pullback diagram

$$Y' \longrightarrow GY$$

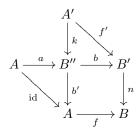
$$f \downarrow \qquad \qquad \downarrow G(g)$$

$$X \longrightarrow GTX$$

where $\eta: X \to GTX$ is the unit of the adjunction $T \dashv G$. Again, the adjunction $T/X \dashv G_X$ is left to you.

Exercise 4 a) Commutativity of the square gives that $m \leq f^*(n)$ in Sub(A), so by the order-preservingness of the closure operation, we have $c_A(m) \leq c_A(f^*(n))$. Now $c_A(m) = \mathrm{id}_A$ since m is dense, and $c_A(f^*(n)) = f^*(c_X(n)) = f^*(n)$ by stability of closure and the assumption that n is closed. The resulting

inequality $id_A \leq f^*(n)$ in Sub(A) gives us a commutative diagram



where the square is a pullback, and $m:A'\to A$ is the composite b'k. Note that b'ab'=b' hence $ab'=\mathrm{id}$ since b' is mono. Therefore the map $g=ba:A\to B'$ satisfies the stated equalities.

b) We have inclusions $A' \to c_A(A') \to A$. Clearly, the second one is closed. To see that the first one is dense we must prove the equality

$$c_{c_A(A')}(A') = c_A(A').$$

Consider the pullback

The desired equality is now clear from:

$$c_{c_A(A')}(A') = c_{c_A(A')}(i^*A') = i^*(c_A(A')) = c_A(A')$$

Now, we need to see that $c_A(A')$ is unique with the stated property. So assume A'' is such that $A' \to A''$ is dense and $A'' \to A$ is closed. We have commutative diagrams

$$A' \longrightarrow A'' \qquad A' \longrightarrow c_A(A')$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$c_A(A') \longrightarrow A \qquad A'' \longrightarrow A$$

which in turn, by applying part a), yield $c_A(A') \leq A''$ and $A'' \leq c_A(A')$. We conclude that $A'' = c_A(A')$.

c) Let $N \to M \to X$ be subobjects. First assume both inclusions are dense; we show that $N \to X$ is dense. Let $i: M \to X$ the inclusion. We have:

$$c_X(N) \cap M = i^*(c_X(N)) = c_M(i^*N) = c_M(M \cap N) = c_M(N)$$

Now since $N \to M$ is dense we have

$$M = c_M(N) = c_X(N) \cap M$$

so $M \subseteq c_X(N)$. Since c is order preserving and idempotent we have $X = c_X(M) \subseteq c_X(c_X(N)) = c_X(N)$, giving that $N \to X$ is dense as required.

Now assume both inclusions are closed. We have (as used before) $c_X(N) \cap M = c_M(N) = N$, so we have a pullback:

$$\begin{array}{ccc}
N & \longrightarrow M \\
\downarrow & & \downarrow \\
c_X(N) & \longrightarrow X
\end{array}$$

Since $M \to X$ is closed, $N \to c_X(N)$ is closed. But $N \to c_X(N)$ is also dense. We conclude $N = c_X(N)$.

d) First a little remark: if $A \xrightarrow{i} B \xrightarrow{j} X$ is a composition of monos and $c_X(A) = B$, then i is dense. Indeed,

$$c_B(A) = c_X(A) \cap B = j^*(c_X(A)) = B.$$

For the proof that $c_A(A' \cap A'') = c_A(A') \cap c_A(A'')$, we prove:

- d1) the map $A' \cap A'' \to c_A(A') \cap c_A(A'')$ is dense;
- d2) the map $c_A(A') \cap c_A(A'') \to A$ is closed.

Let us first see that this is enough. We first show that statements d1) and d2) also hold for $c_A(A' \cap A'')$ in the place of $c_A(A') \cap c_A(A'')$, so that by the uniqueness of part b) we will be done after proving d1) and d2).

We have that $c_A(A' \cap A'') \to A$ is clearly closed, and $c_A(A' \cap A'') \to c_A(A') \cap c_A(A'')$ is dense by the little remark, since

(where j' is the mono $c_A(A'') \to A$).

Now for the proof of d1) and d2).

d1): this arrow is a composite of $A' \cap A'' \to c_A(A') \cap A'' \to c_A(A') \cap c_A(A'')$. Let j be the mono $A'' \to A$. Then

$$c_{A''}(A' \cap A'') = c_{A''}(j^*(A')) = j^*(c_A(A')) = c_A(A') \cap A''$$

so the first arrow in the composite is dense; the second one is dense because it is a pullback (intersection with $c_A(A')$) of the dense map $A'' \to c_A(A'')$. By part c) we conclude that d1) has been proved.

For d2), we split this as $c_A(A') \cap c_A(A'') \to c_A(A'') \to A$. For the first of these arrows, we have (again, let j' be the arrow $c_A(A'') \to A$):

$$c_A(A') \cap c_A(A'') = (j')^*(c_A(A')) = c_{c_A(A'')}((j')^*(A'))$$

We see that $c_A(A') \cap c_A(A'')$ is closed in $c_A(A'')$. The second arrow of the composite is trivially closed, so (invoking once again part c)) we are done.

Exercise 5. Of course, we are talking about maps in a topos.

a) If $i: X \to Y$ is a monomorphism then X is a subobject of Y and has therefore a complement $X^c \in \operatorname{Sub}(Y)$ since $\operatorname{Sub}(Y)$ is Boolean by assumption. If A is a subobject of X then A is also a subobject of Y and has complement A^c in $\operatorname{Sub}(Y)$; let $A' = A^c \cap X$. Then A' is a subobject of X, $A \cap A' = 0$ and

$$A \cup A' = A \cup (A^c \cap X) = (A \cup A^c) \cap (A \cup X) = Y \cap X = X$$

so A' is a complement of A in Sub(X), which is therefore Boolean.

b) We work with the geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$. Its inverse image functor f^* is logical. It restricts to a map between the lattices of subobjects of 1 in the respective toposes. Note that $\mathrm{Sub}(X)$ in \mathcal{E} is isomorphic to $\mathrm{Sub}(1)$ in \mathcal{E}/X , and ditto for Y; modulo these isomorphisms, the restriction of f^* to the lattices of subobjects of 1 is the pullback functor $f^*: \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$. This functor preserves the Heyting structure. Note also that a Heyting algebra H is Boolean if and only if $x = \neg \neg x$ holds for all $x \in H$. Now if $A \in \mathrm{Sub}(Y)$ then $f^*(\neg \neg A) = \neg \neg f^*(A) = f^*(A)$ (the last equality since $\mathrm{Sub}(X)$ is Boolean; hence $\neg \neg A = A$ because f is a surjection. So $\mathrm{Sub}(Y)$ is Boolean.