

# Topos Theory, Spring 2024

## Hand-In Exercises

Jaap van Oosten

February–May 2024

### 1 Exercises

**Exercise 1 (Deadline: February 29)** We consider a small category  $\mathcal{C}$  and a monoid  $M$ . An  $M$ -presheaf on  $\mathcal{C}$  is a presheaf  $F$  on  $\mathcal{C}$  endowed with, for every object  $C \in \mathcal{C}$ , a right  $M$ -action  $F(C) \times M \rightarrow F(C)$  (written:  $(x, m) \mapsto xm$ ) which, besides the usual axioms for an  $M$ -action, also satisfies:  $f^*(xm) = f^*(x)m$  for  $f : D \rightarrow C$ ,  $x \in F(C)$  and  $m \in M$ . A morphism of  $M$ -presheaves  $F \rightarrow G$  is a natural transformation  $\mu : F \Rightarrow G$  such that  $\mu_C(xm) = \mu_C(x)m$  for all  $C \in \mathcal{C}$ ,  $x \in F(C)$ . Clearly, we have a category  $M\text{-}\widehat{\mathcal{C}}$  of  $M$ -presheaves and morphisms.

- a) Let  $\Delta : \widehat{\mathcal{C}} \rightarrow M\text{-}\widehat{\mathcal{C}}$  be the functor which endows each presheaf  $F$  with the trivial (identity)  $M$ -action. Show that  $\Delta$  has a right adjoint, and describe it explicitly.
- b) Show that  $M\text{-}\widehat{\mathcal{C}}$  is a topos.

**Exercise 2 (Deadline: March 14)** Recall the definition (before Proposition 3.14) of the map  $\exists_f : \Omega^X \rightarrow \Omega^Y$  for any monomorphism  $f : X \rightarrow Y$ .

- a) (4 pts) Show that  $\exists_f$  induces a function  $\sum_f : \text{Sub}(X) \rightarrow \text{Sub}(Y)$ , and describe this function explicitly.
- b) (6 pts) Show that for any subobject  $A$  of  $X$ , the inequality  $A \leq f^*(\sum_f(A))$  holds.

**Exercise 3 (Deadline: March 28)** Let  $T : \mathcal{E} \rightarrow \mathcal{F}$  be a logical functor between toposes.

- a) (4 pts) Let  $X$  be an object of  $\mathcal{E}$ . Show that the functor  $T/X : \mathcal{E}/X \rightarrow \mathcal{F}/TX$  which sends  $(Y \xrightarrow{f} X)$  to  $(TY \xrightarrow{Tf} TX)$  (with the straightforward action on morphisms) is logical.
- b) (3 pts) Suppose the functor  $T$  has a left adjoint  $F$ . Show that  $T/X$  has a left adjoint.

- c) (3 pts) Under the assumption in b), show that  $T/X$  has a right adjoint. Can you describe it explicitly?

**Exercise 4 (Deadline: April 11)** We consider a universal closure operation  $c$  on a topos  $\mathcal{E}$ .

- a) (2 pts) Let

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ m \downarrow & & \downarrow n \\ A & \xrightarrow{f} & B \end{array}$$

be a commutative square with  $m$  a dense mono and  $n$  a closed mono. Prove that there is a unique “filler”  $g : A \rightarrow B'$  (i.e., a map such that  $gm = f'$  and  $ng = f$ ).

- b) (2 pts) For a subobject  $A'$  of  $A$ , show that  $c_A(A')$  is the unique subobject  $A''$  of  $A$  with the property that  $A' \rightarrow A''$  is dense and  $A'' \rightarrow A$  is closed.
- c) (3 pts) Show that the composition of two dense monos is dense; and the same for closed monos.
- d) (3 pts) Show that for  $A', A'' \in \text{Sub}(A)$  we have:  $c_A(A' \cap A'') = c_A(A') \cap c_A(A'')$ . [Hint: one inclusion is clear since  $c$  is order-preserving. For the other, show that it suffices to prove that  $(A' \cap A'') \rightarrow (c_A(A') \cap c_A(A''))$  is dense and  $(c_A(A') \cap c_A(A'')) \rightarrow A$  is closed.]

**Exercise 5 (Deadline: April 25)** This exercise is about the Heyting algebra structure of subobject lattices.  $\text{Sub}(X)$  denotes the lattice of subobjects of  $X$ .

- a) Let  $i : X \rightarrow Y$  be a monomorphism. Prove: if  $\text{Sub}(Y)$  is a Boolean algebra, then so is  $\text{Sub}(X)$ .
- b) Let  $p : X \rightarrow Y$  be an epimorphism. Prove: if  $\text{Sub}(X)$  is a Boolean algebra, then so is  $\text{Sub}(Y)$ .

**Exercise 6 (Deadline: May 17)** Let  $L$  be a one-sorted first-order language and suppose  $R$  is an  $L$ -structure in a topos  $\mathcal{E}$ .

- a) Construct, for every object  $X$  of  $\mathcal{E}$ , an  $L$ -structure  $R_X$  on the set  $\mathcal{E}(X, R)$ .
- b) Let  $\phi$  be an  $L$ -formula with  $n$  free variables  $x_1, \dots, x_n$ , and suppose the only logical symbols occurring in  $\phi$  are  $\wedge$  and  $\rightarrow$ . Prove: for any  $n$ -tuple  $\alpha_1, \dots, \alpha_n : X \rightarrow R$  the following are equivalent:
- $X \Vdash \phi[\alpha_1, \dots, \alpha_n]$
  - For all arrows  $f : Y \rightarrow X$ ,  $R_Y \models \phi(\alpha_1 f, \dots, \alpha_n f)$

## 2 Solutions

**Exercise 1** a) Clearly, if  $\mu : \Delta(F) \rightarrow G$  is any morphism in  $M\text{-}\widehat{\mathcal{C}}$  then by the definition of such morphisms, for all  $C \in \mathcal{C}$ ,  $x \in F(C)$  and  $m \in M$  we have  $(\mu_C(x))m = \mu_C(x)$ , so  $\mu$  lands in the part of  $G$  which is invariant under the  $M$ -action. We have a functor from  $M\text{-}\widehat{\mathcal{C}}$  to  $\widehat{\mathcal{C}}$  which sends each  $M$ -presheaf to its invariant part. This is right adjoint to  $\Delta$ , the verification of which is left to you.

b) This is most easily done by observing that  $M\text{-}\widehat{\mathcal{C}}$  is equivalent to a presheaf category: it is equivalent to the category of presheaves on the product category  $\mathcal{C} \times M$ .

**Exercise 2** a) Elements of  $\text{Sub}(X)$  are in 1-1 correspondence with maps  $1 \rightarrow \Omega^X$ : take the exponential transpose of the classifying map.

Define  $\sum_f$  as follows: for  $A \in \text{Sub}(X)$ , corresponding to the map  $a : 1 \rightarrow \Omega^X$ , define  $\sum_f(A) \in \text{Sub}(Y)$  as the subobject corresponding to the composition  $\exists_f \circ a : 1 \rightarrow \Omega^Y$ .

One can prove that this is simply the subobject  $A \rightarrow X \xrightarrow{f} Y$ , although it is hard to argue that this operation is *induced by*  $\sum_f$ .

b) Let  $\widetilde{\exists}_f$  be the exponential transpose of  $\exists_f$ . Then  $\sum_f(A)$  is classified by the composition

$$Y \xrightarrow{\langle a, \text{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists}_f} \Omega.$$

Then  $f^*(\sum_f(A))$  is classified by

$$X \xrightarrow{f} Y \xrightarrow{\langle a, \text{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists}_f} \Omega.$$

and the inequality  $A \leq f^*(\sum_f(A))$  holds if and only if the composition

$$A \rightarrow X \xrightarrow{f} Y \xrightarrow{\langle a, \text{id}_Y \rangle} \Omega^X \times Y \xrightarrow{\widetilde{\exists}_f} \Omega$$

factors through the subobject classifier  $1 \xrightarrow{t} \Omega$ .

This, however, follows from the commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & X & \xrightarrow{f} & Y \\ & & \downarrow \langle a, \text{id} \rangle & & \downarrow \langle a, \text{id}_Y \rangle \\ & & \Omega^X \times X & \xrightarrow{\text{id} \times f} & \Omega^X \times Y \\ & & \downarrow \text{ev}_X & \nearrow \widetilde{\exists}_f & \\ 1 & \xrightarrow{t} & \Omega & & \end{array}$$

The lower right-hand triangle commutes by Proposition 3.14, and the left hand square commutes because the composite  $\text{ev}_X \circ \langle a, \text{id} \rangle$  equals the transpose of  $a$ , that is: the map which classifies  $A$  as subobject of  $X$ .

**Exercise 3** a) Assume  $T : \mathcal{E} \rightarrow \mathcal{F}$  is logical; this means that  $T$  preserves finite limits, subobject classifiers and exponentials. So  $T(1 \xrightarrow{t} \Omega)$  is a subobject classifier in  $\mathcal{F}$  and moreover, if  $\chi_A : X \rightarrow \Omega$  classifies the subobject  $A$  of  $X$  in  $\mathcal{E}$ , then  $T(\chi_A)$  classifies  $T(A) \in \text{Sub}(TX)$  in  $\mathcal{F}$ . It follows that the map  $\Delta : X \times X \rightarrow \Omega$ , which classifies the diagonal  $\delta : X \rightarrow X \times X$ , is preserved by  $T$ . Also,  $T$  commutes with taking exponents and also with exponential transposes. So, for example, the singleton map  $\{\cdot\} : X \rightarrow \Omega^X$  is preserved by  $T$ . We see that partial map classifiers are preserved by  $T$ . It is now a matter of inspection to see that the whole topos structure of  $\mathcal{E}/X$  is preserved by  $T/X$ . We conclude that  $T/X$  is logical.

b) Let  $F \dashv T$ . Define  $F^X : \mathcal{F}/TX \rightarrow \mathcal{E}/X$  as follows: for an object  $(Y \xrightarrow{g} TX)$  of  $\mathcal{F}/TX$  let  $F^X(g)$  be the map  $FY \xrightarrow{\tilde{g}} X$ , the transpose of  $g$  along the adjunction  $F \dashv T$ . On morphisms

$$\begin{array}{ccc} Y & \xrightarrow{h} & Y' \\ & \searrow g & \swarrow g' \\ & TX & \end{array}$$

the image  $F^X(h)$  is the map  $F(h) : \tilde{g} \rightarrow \tilde{g}'$  obtained by transposing. The adjunction is straightforward.

c) The existence of a right adjoint is an immediate application of Corollary 3.20:  $T/X$  is logical and has a left adjoint, so it has a right adjoint by 3.20.

In order to exhibit the right adjoint, we use the assumption in b) once more, and conclude that  $T$  has a right adjoint. Let  $G : \mathcal{F} \rightarrow \mathcal{E}$  be right adjoint to  $T$ . Define  $G_X : \mathcal{F}/TX \rightarrow \mathcal{E}/X$  as follows:  $G_X(Y \xrightarrow{g} TX)$  is the map  $Y' \xrightarrow{f} X$ , from the pullback diagram

$$\begin{array}{ccc} Y' & \longrightarrow & GY \\ f \downarrow & & \downarrow G(g) \\ X & \xrightarrow{\eta} & GTX \end{array}$$

where  $\eta : X \rightarrow GTX$  is the unit of the adjunction  $T \dashv G$ . Again, the adjunction  $T/X \dashv G_X$  is left to you.

**Exercise 4** a) Commutativity of the square gives that  $m \leq f^*(n)$  in  $\text{Sub}(A)$ , so by the order-preservingness of the closure operation, we have  $c_A(m) \leq c_A(f^*(n))$ . Now  $c_A(m) = \text{id}_A$  since  $m$  is dense, and  $c_A(f^*(n)) = f^*(c_X(n)) = f^*(n)$  by stability of closure and the assumption that  $n$  is closed. The resulting

inequality  $\text{id}_A \leq f^*(n)$  in  $\text{Sub}(A)$  gives us a commutative diagram

$$\begin{array}{ccccc}
 & & A' & & \\
 & & \downarrow k & \searrow f' & \\
 A & \xrightarrow{a} & B'' & \xrightarrow{b} & B' \\
 & \searrow \text{id} & \downarrow b' & & \downarrow n \\
 & & A & \xrightarrow{f} & B
 \end{array}$$

where the square is a pullback, and  $m : A' \rightarrow A$  is the composite  $b'k$ . Note that  $b'ab' = b'$  hence  $ab' = \text{id}$  since  $b'$  is mono. Therefore the map  $g = ba : A \rightarrow B'$  satisfies the stated equalities.

b) We have inclusions  $A' \rightarrow c_A(A') \rightarrow A$ . Clearly, the second one is closed. To see that the first one is dense we must prove the equality

$$c_{c_A(A')}(A') = c_A(A').$$

Consider the pullback

$$\begin{array}{ccc}
 A' & \longrightarrow & A' \\
 \downarrow & & \downarrow \\
 c_A(A') & \xrightarrow{i} & A
 \end{array}$$

The desired equality is now clear from:

$$c_{c_A(A')}(A') = c_{c_A(A')}(i^*A') = i^*(c_A(A')) = c_A(A')$$

Now, we need to see that  $c_A(A')$  is unique with the stated property. So assume  $A''$  is such that  $A' \rightarrow A''$  is dense and  $A'' \rightarrow A$  is closed. We have commutative diagrams

$$\begin{array}{ccc}
 A' & \longrightarrow & A'' \\
 \downarrow & & \downarrow \\
 c_A(A') & \longrightarrow & A
 \end{array}
 \quad
 \begin{array}{ccc}
 A' & \longrightarrow & c_A(A') \\
 \downarrow & & \downarrow \\
 A'' & \longrightarrow & A
 \end{array}$$

which in turn, by applying part a), yield  $c_A(A') \leq A''$  and  $A'' \leq c_A(A')$ . We conclude that  $A'' = c_A(A')$ .

c) Let  $N \rightarrow M \rightarrow X$  be subobjects. First assume both inclusions are dense; we show that  $N \rightarrow X$  is dense. Let  $i : M \rightarrow X$  the inclusion. We have:

$$c_X(N) \cap M = i^*(c_X(N)) = c_M(i^*N) = c_M(M \cap N) = c_M(N)$$

Now since  $N \rightarrow M$  is dense we have

$$M = c_M(N) = c_X(N) \cap M,$$

so  $M \subseteq c_X(N)$ . Since  $c$  is order preserving and idempotent we have  $X = c_X(M) \subseteq c_X(c_X(N)) = c_X(N)$ , giving that  $N \rightarrow X$  is dense as required.

Now assume both inclusions are closed. We have (as used before)  $c_X(N) \cap M = c_M(N) = N$ , so we have a pullback:

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ c_X(N) & \longrightarrow & X \end{array}$$

Since  $M \rightarrow X$  is closed,  $N \rightarrow c_X(N)$  is closed. But  $N \rightarrow c_X(N)$  is also dense. We conclude  $N = c_X(N)$ .

d) First a little remark: if  $A \xrightarrow{i} B \xrightarrow{j} X$  is a composition of monos and  $c_X(A) = B$ , then  $i$  is dense. Indeed,

$$c_B(A) = c_X(A) \cap B = j^*(c_X(A)) = B.$$

For the proof that  $c_A(A' \cap A'') = c_A(A') \cap c_A(A'')$ , we prove:

d1) the map  $A' \cap A'' \rightarrow c_A(A') \cap c_A(A'')$  is dense;

d2) the map  $c_A(A') \cap c_A(A'') \rightarrow A$  is closed.

Let us first see that this is enough. We first show that statements d1) and d2) also hold for  $c_A(A' \cap A'')$  in the place of  $c_A(A') \cap c_A(A'')$ , so that by the uniqueness of part b) we will be done after proving d1) and d2).

We have that  $c_A(A' \cap A'') \rightarrow A$  is clearly closed, and  $c_A(A' \cap A'') \rightarrow c_A(A') \cap c_A(A'')$  is dense by the little remark, since

$$\begin{aligned} c_{c_A(A'')}(c_A(A') \cap c_A(A'')) &= c_{c_A(A'')}(j'^*(c_A(A'))) = \\ (j')^*c_A(c_A(A')) &= (j')^*c_A(A') = c_A(A') \cap c_A(A'') \end{aligned}$$

(where  $j'$  is the mono  $c_A(A'') \rightarrow A$ ).

Now for the proof of d1) and d2).

d1): this arrow is a composite of  $A' \cap A'' \rightarrow c_A(A') \cap A'' \rightarrow c_A(A') \cap c_A(A'')$ . Let  $j$  be the mono  $A'' \rightarrow A$ . Then

$$c_{A''}(A' \cap A'') = c_{A''}(j^*(A')) = j^*(c_A(A')) = c_A(A') \cap A''$$

so the first arrow in the composite is dense; the second one is dense because it is a pullback (intersection with  $c_A(A')$ ) of the dense map  $A'' \rightarrow c_A(A'')$ . By part c) we conclude that d1) has been proved.

For d2), we split this as  $c_A(A') \cap c_A(A'') \rightarrow c_A(A'') \rightarrow A$ . For the first of these arrows, we have (again, let  $j'$  be the arrow  $c_A(A'') \rightarrow A$ ):

$$c_A(A') \cap c_A(A'') = (j')^*(c_A(A')) = c_{c_A(A'')}(j'^*(A'))$$

We see that  $c_A(A') \cap c_A(A'')$  is closed in  $c_A(A'')$ . The second arrow of the composite is trivially closed, so (invoking once again part c)) we are done.

**Exercise 5.** Of course, we are talking about maps in a topos.

a) If  $i : X \rightarrow Y$  is a monomorphism then  $X$  is a subobject of  $Y$  and has therefore a complement  $X^c \in \text{Sub}(Y)$  since  $\text{Sub}(Y)$  is Boolean by assumption. If  $A$  is a subobject of  $X$  then  $A$  is also a subobject of  $Y$  and has complement  $A^c$  in  $\text{Sub}(Y)$ ; let  $A' = A^c \cap X$ . Then  $A'$  is a subobject of  $X$ ,  $A \cap A' = 0$  and

$$A \cup A' = A \cup (A^c \cap X) = (A \cup A^c) \cap (A \cup X) = Y \cap X = X$$

so  $A'$  is a complement of  $A$  in  $\text{Sub}(X)$ , which is therefore Boolean.

b) We work with the geometric morphism  $\mathcal{E}/X \rightarrow \mathcal{E}/Y$ . Its inverse image functor  $f^*$  is logical. It restricts to a map between the lattices of subobjects of 1 in the respective toposes. Note that  $\text{Sub}(X)$  in  $\mathcal{E}$  is isomorphic to  $\text{Sub}(1)$  in  $\mathcal{E}/X$ , and ditto for  $Y$ ; modulo these isomorphisms, the restriction of  $f^*$  to the lattices of subobjects of 1 is the pullback functor  $f^* : \text{Sub}(Y) \rightarrow \text{Sub}(X)$ . This functor preserves the Heyting structure. Note also that a Heyting algebra  $H$  is Boolean if and only if  $x = \neg\neg x$  holds for all  $x \in H$ . Now if  $A \in \text{Sub}(Y)$  then  $f^*(\neg\neg A) = \neg\neg f^*(A) = f^*(A)$  (the last equality since  $\text{Sub}(X)$  is Boolean; hence  $\neg\neg A = A$  because  $f$  is a surjection. So  $\text{Sub}(Y)$  is Boolean.