Everything is Relative – Some Remembrances of Pieter Hofstra's Personality and Work

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Partial Combinatory Algebras (PCAs)

A *Partial Combinatory Algebra* is a set A together with a partial map $A \times A \rightharpoonup A$ (the *application map*). Also here we write, for elements $a, b \in A$, $ab\downarrow$ to indicate that the pair (a, b) is in the domain of the application map.

Moreover, a PCA A should have elements k and s satisfying:

$$\begin{gathered} \mathsf{k}x\downarrow \\ (\mathsf{k}a)b = a \\ (\mathsf{s}a)b\downarrow \end{gathered}$$

and: if $(ac)(bc)\downarrow$ then $((\mathsf{s}a)b)c\downarrow$ and

$$((\mathsf{s}a)b)c = (ac)(bc)$$

.

PCAs are building blocks of toposes. For each PCA A we have a category $\mathrm{Asm}(A)$ of *assemblies* on A:
An assembly over A is a pair (X, E) where X is a set and $E(x)$ is a *nonempty* subset of A, for each $x \in X$.
A *morphism of assemblies* $(X, E) \to (Y, F)$ is a function $f : X \to Y$ of sets, for which there is an element $a \in A$ such that for all $x \in X$ and all $b \in E(x)$, $ab \in F(f(x))$. One says that a *tracks* the function f.

The category Asm(A) is locally cartesian closed, regular, has a weak subobject classifier (is a quasi-topos). Moreover, Asm(A) comes with an adjunction

$$(\Gamma : \mathsf{Ass}(A) \to \mathrm{Set}) \dashv (\nabla : \mathrm{Set} \to \mathsf{Ass}(A))$$

$\Gamma(X, E) = X$; $\nabla(X) = (X, \lambda x.A)$.
The category Asm(A) also has a *natural numbers object*.

Theorem (Pitts 1980; Carboni, Freyd, Scedrov 1988): the exact completion of $\mathsf{Asm}(A)$, $\mathsf{Asm}(A)_{\mathrm{ex/reg}}$, is a topos, the *realizability topos* over A.

We now wish to understand: how functorial is the construction $A \mapsto \mathrm{RT}(A)$?

It turns out that there is a very nice categorical structure on the class of PCAs, which was first explored by John Longley in his thesis (1995).

Let A, B be PCAs. An *applicative morphism* $A \to B$ is a total relation γ (we think of γ as a function from A to the set of nonempty subsets of B, so (A, γ) is an assembly over B) for which there is an element $\mathrm{r} \in B$ which satisfies:

For each pair a, a' of elements of A and $b \in \gamma(a), b' \in \gamma(a')$, if $aa' \downarrow$ in A then $\mathrm{r}bb' \downarrow$ in B, and $\mathrm{r}bb' \in \gamma(aa')$.

The element r *realizes* the morphism γ. Composition of morphisms is composition of total relations.

We think of γ as a *simulation* in B of computations in A; the element r is a machine that translates code for an A-program into code for a B-program.

Theorem (Longley 1995): every applicative morphism $A \stackrel{\gamma}{\to} B$ gives rise to a regular functor $\mathsf{Asm}(\gamma) : \mathsf{Asm}(A) \to \mathsf{Asm}(B)$ which makes the diagram

$$\begin{array}{ccc} \mathsf{Asm}(A) & \xrightarrow{\mathsf{Asm}(\gamma)} & \mathsf{Asm}(B) \\ & \underset{\Gamma}{\searrow} & \downarrow \Gamma \\ & & \mathrm{Set} \end{array}$$

commute. Conversely, every regular functor making this diagram commute, is of the form $\mathsf{Ass}(\gamma)$ for an essentially unique applicative morphism $\gamma : A \to B$.
In fact the functor $\mathrm{Asm} : \mathsf{PCA} \to \mathsf{REG}/\mathrm{Set}$, which sends an assembly A to the functor $\Gamma_A : \mathrm{Asm}(A) \to \mathrm{Set}$, is locally an equivalence.

I. Computationally dense morphisms

What do geometric morphisms between realizability toposes look like?

Fundamental observation by Peter Johnstone (2013): Every geometric morphism $\mathrm{RT}(A) \to \mathrm{RT}(B)$ restricts to an adjunction between the categories of assemblies.

The left adjoint of such a restriction is always a regular functor commuting with the Γ's, and therefore corresponds to an applicative morphism $B \stackrel{\gamma}{\to} A$. The question then is:

For which applicative morphisms $\gamma : B \to A$ does the regular functor $\mathrm{Ass}(\gamma) : \mathrm{Ass}(B) \to \mathrm{Ass}(A)$ have a right adjoint?

A generalization: ordered PCAs.

An ordered PCA (OPCA) is a poset $(A, \leq)$ with a partial application function $a, b \mapsto ab$ for which the following hold:

- the domain of the application function is downwards closed and application is order-preserving on its domain;
- there exist k and s in A such that $\mathsf{k}ab \leq a$ and $\mathsf{s}abc \preceq ac(bc)$ (i.e., if $ac(bc)\downarrow$ then $\mathsf{s}abc\downarrow$ and $\mathsf{s}(abc) \leq ac(bc)$).

Main example: given an ordinary PCA A, its powerset $\mathcal{P}(A)$ becomes an OPCA if we put: $\alpha\beta\downarrow$ iff for all $a \in \alpha, b \in \beta$, $ab\downarrow$, in which case we let $\alpha\beta$ be the subset $\{ab \mid a \in \alpha, b \in \beta\}$.

An applicative morphism between OPCAs A and B is a function $f : A \to B$ for which there is some $r \in B$ satisfying: whenever $aa'{\downarrow}$ in A, then $rf(a)f(a'){\downarrow}$ in B, and $rf(a)f(a') \leq f(aa')$.
An assembly over an OPCA A is a pair (X, E) with X a set and $E(x)$ a *nonempty* downwards closed subset of A, for each $x \in X$.
Similarly to Longley's treatment we have a local equivalence $\mathrm{Asm} : \mathsf{OPCA} \to \mathsf{REG}/\mathrm{Set}$. Also, Pitts' theorem generalizes: $\mathrm{Asm}(A)$ is a regular category, and $\mathrm{Asm}(A)_{\mathrm{ex/reg}}$ is a realizability topos $\mathsf{RT}(A)$.
We are interested in applicative morphisms $f : B \to A$ for which $\mathrm{Asm}(f) : \mathrm{Asm}(B) \to \mathrm{Asm}(A)$ has a right adjoint. Because then, applying the exact completion we obtain a geometric morphism of realizability toposes.

Call an applicative morphism $f : B \to A$ between OPCAs *computationally dense* if for all $a \in A$ there exists $b \in B$ such that whenever $af(c)\downarrow$ for $c \in B$, we have $bc\downarrow$ in B, and $f(bc) \leq af(c)$.
It says: every endomap on B which is realized (modulo f) in A, is already (up to order) realized in B.
Theorem (Hofstra 2003): $\mathrm{Asm}(f) : \mathrm{Asm}(B) \to \mathrm{Asm}(A)$ has a right adjoint if and only if f is computationally dense.

A further generalization: relative OPCAs.
If A is an OPCA, a *filter* on A is a subset F which:

- is upwards closed
- is closed under the application map
- contains elements k and s which satisfy the axioms for A being an OPCA.

We call the pair (A, F) a *relative PCA*.
For example: let $B = \mathbb{N}^{\mathbb{N}}$, $A = \mathcal{P}(B)$. We could take F the set of those elements of A which contain at least one computable function (Kleene-Vesley 1965).
An assembly over a relative PCA (A, F) is just an assembly over A, but a morphism of such assemblies has to be tracked by an element of F. Again, $\mathrm{Asm}(A, F)_{\mathrm{ex/reg}}$ is a topos, $\mathrm{RT}(A, F)$.
It turns out that for important closure properties of realizability toposes one has to move to these relative realizability toposes (Zoethout 2022)

II. BCOs and triposes

In a very nice paper (Hofstra 2006), Pieter analyzed the notion of a relative OPCA from a more primitive notion. The central definition is that of a *basic combinatorial object* (BCO).

Definition: a BCO is a poset $(\Sigma, \leq)$ together with a set F_Σ of partial endofunctions on Σ, which satisfies the following axioms:

1. Every function in F_Σ has downwards closed domain and is order-preserving on its domain;
2. there is a total function $i \in F_\Sigma$ such that $i(a) \leq a$ for all $a \in \Sigma$ (i is a "weak identity");
3. For each pair $f, g \in F_\Sigma$ there is $h \in F_\Sigma$ satisfying: $\mathrm{domain}(gf) \subseteq \mathrm{domain}(h)$ and $h(a) \leq g(f(a))$ for $a \in \mathrm{domain}(gf)$ (we have some sort of "weak composition").

Note that every poset is a BCO, as is every monoid, every partial combinatory algebra. More importantly, every relative OPCA is a BCO in a natural way.

A morphism between BCOs $(\Sigma, \leq, F_\Sigma) \to (\Theta, \leq, F_\Theta)$ is a function $\phi : \Sigma \to \Theta$ satisfying:

1. there exists $u \in F_\Theta$ such that for each inequality $a \leq a'$ in Σ we have $u(\phi(a)) \leq \phi(a')$ in Θ ("ϕ is order-preserving modulo u");
2. for all $f \in F_\Sigma$ there is $g \in F_\Theta$ with $g(\phi(a)) \leq \phi(f(a))$ ("g simulates the functional behaviour of f relative to ϕ").

The category BCO is order-enriched and has a monad on it, the Downset monad $\mathcal{D}$, which is Kock-Zöberlein (algebras are left adjoint to units).

Moreover, for every BCO Σ we have a Set-indexed preorder $[-,\Sigma]$: $[X,\Sigma]$ is the set of functions from X to Σ, and for $\phi,\psi \in [X,\Sigma]$ we have $\phi \leq \psi$ if and only if there is some $f \in F_\Sigma$ such that $f(\phi(x)) \leq \psi(x)$ for all $x \in X$.

We shall be interested in the question: when is $[-,\Sigma]$ a tripos?

Theorem (Hofstra 2006): Let Σ be a BCO, and $[-, \Sigma]$ its associated Set-indexed preorder. Then the following two statements are equivalent:

1. Σ is an OPCA with filter Φ, so the preorder on $[X, \Sigma]$ is given by: $\alpha \leq \beta$ iff there is $a \in \Phi$ such that for all $x \in X$ and $b \in \alpha(x)$, $ab \in \beta(x)$.
2. $[-, \mathcal{D}\Sigma]$ is a tripos.

A refinement: a *pre-implicative* OPCA is a filtered OPCA A together with suitable maps $\bigwedge : \mathcal{P}(A) \to A$ and $\Rightarrow: A \times A \to A$.
Theorem (vO–Zou 2016): $[-, \Sigma]$ is a tripos if and only if Σ is a pre-implicative OPCA.

III. Dialectica Monads.

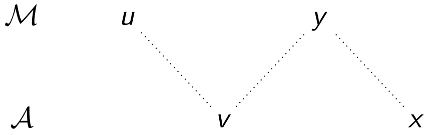
In 1958, Gödel published a paper about which he had been mulling since the early 1940's: *On a hitherto unused extension of the finitary point of view*, in which he sought to reduce the consistency of Peano Arithmetic to that of the theory of quantifier-free equations involving primitive recursive functionals of finite type. In 2002, Martin Hyland (following De Paiva's Ph.D. thesis) gave a categorical construction of this interpretation. Suppose we have a posetal fibration $p : \mathbb{P} \to \mathbb{T}$, where $\mathbb{T}$ is a category with finite products. We construct a new category $\mathrm{Dial}(p)$:

- objects are triples (U, X, α) with $U, X \in \mathbb{T}$ and $\alpha \in p^{U \times X}$;
- maps $(U, X, \alpha) \to (V, Y, \beta)$ are pairs $f : U \to V, F : U \times Y \to X$ of morphisms in $\mathbb{T}$ such that for the morphisms

$$\begin{aligned} \langle \pi_0, F \rangle : &\quad U \times Y \to U \times X \\ \langle f\pi_0, \pi_1 \rangle : &\quad U \times Y \to V \times Y \end{aligned}$$

  we have $\langle \pi_0, F \rangle^*(\alpha) \le \langle f\pi_0, \pi_1 \rangle^*(\beta)$ in $p^{U \times Y}$

A simple interpretation. Let us think of a fibration of sets. Consider the following two-move game, between Merlin (evil) and Arthur (the good guy). Merlin starts by picking $u \in U$, Arthur responds by picking $v \in V$. Merlin now picks $y \in Y$, and Arthur picks $x \in X$. End of the game:



Now we have special subsets $\alpha \subseteq U \times X$, $\beta \subseteq V \times Y$, and the stipulation is that Arthur wins if $(v, y) \in \beta$ whenever $(u, x) \in \alpha$. Note that Arthur's choice of x may depend on both Merlin's moves u and y. Hence a *strategy* for Arthur consists of a pair of functions $(f : U \to V, F : U \times Y \to X)$, and if this satisfies $(f(u), y) \in \beta$ whenever $(u, F(u, y) \in \alpha$ for all u, y, then (f, F) is a *winning strategy*.

Now suppose that α and β are the complements of graphs of functions A, B respectively. Then taking the contrapositive of the winning condition, we see that (f, F) is a winning strategy if $F(u, y) = A(u)$ whenever $y = B(f(u))$, that is: *the pair* (f, F) *determines a one-query oracle computation of* A *with oracle* B.

This game can be analyzed further: for a function A we have the one-move game G_A: $\mathcal{M}$erlin picks some ξ, $\mathcal{A}$rthur responds with σ, and wins if $A(\xi) = \sigma$. The "oracle game" above is now a cut-off version of the "implication game" $G_B \Rightarrow G_A$ (in the sense of Hyland-Ong).

Pieter set out to analyse the Hyland-De Paiva Dialectica construction as a composition of canonical constructions on fibrations.
Let $p : E \to B$ be a fibration; we assume B has finite products.
Say *p has simple coproducts* if for every projection $I \times J \xrightarrow{\pi} I$ in B, the functor $\pi^* : p^I \to p^{I \times J}$ has a left adjoint, and these left adjoints satisfy the Beck-Chevalley condition.
Similarly, one defines *simple products*. Let p^{op} be the opposite fibration (i.e. the fibration over B such that $(p^{\mathrm{op}})^I = (p^I)^{\mathrm{op}}$): then p has simple products if and only if p^{op} has simple coproducts.

To any fibration $p : E \to B$ one can add simple coproducts in a universal way: let

$$\begin{array}{ccc} \mathrm{Fam}(E) & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ B^{\to} & \xrightarrow[\mathrm{dom}]{} & B \end{array}$$

be a pullback. Let $\mathrm{Fam}(p)$ be the composition

$$\mathrm{Fam}(E) \longrightarrow B^{\to} \xrightarrow{\mathrm{cod}} B$$

$\mathrm{Fam}(p)$ is the free fibration on p with coproducts; it has a subfibration $\mathrm{Sum}(p)$ which is universal (w.r.t. p) with simple coproducts.
Similarly, we have $\mathrm{Prod}(p) = \mathrm{Sum}(p^{\mathrm{op}})^{\mathrm{op}}$.
We have: the operations Sum and Prod have the structure of pseudo-monads on Fib(p), the category of fibrations on p.
Moreover, there is an appropriate distributive law between them, guaranteeing that also the composition $\mathrm{Sum}{\circ}\mathrm{Prod}$ has a pseudo-monad structure.

Lemma: there is a natural isomorphism of fibrations

$$\mathrm{Dial}(p) \simeq \mathrm{Sum}(\mathrm{Prod}(p))$$

Theorem: Assume B is cartesian closed. Then the pseudo-algebras for the pseudomodad Dial on $\mathrm{Fib}(B)$ are the fibrations with simple products and coproducts satisfying the distributivity

$$\forall u \exists x \alpha(i, u, x) \simeq \exists f \forall u \alpha(i, fu, u)$$

Pieter Hofstra was my first PhD student, but also a friend. I recall with gratitude his hospitality in 2006 when I stayed with him and Miyoung in Calgary, and had an unforgettable ride in the Rocky Mountains.

I was also deeply moved when he organized (together with Benno van den Berg, who was a PhD student of Moerdijk roughly the same time as Pieter was working with me) a special PSSL celebrating my and Thomas Streicher's 60th birthdays, in 2018.

His death is still unthinkable.

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