Exam Topos Theory, June 10, 2021

with model solutions

This exam consists of 4 exercises. Every exercise is worth 10 points; if an exercise consists of more than one part, it is indicated what each part is worth. The grade W for the written exam is your total number of points divided by 4. You final grade is the maximum of W and $\frac{7W+3H}{10}$ where H is your result from the homework exercises.

Advice: first do those exercises for which you see a solution right away. Then start thinking about the harder ones. Good luck!

Exercise 1 We consider a topological space X and a point $x \in X$; we shall also write x for the induced geometric morphism $\mathrm{Set} \to \mathrm{Sh}(X)$.

Recall the construction, for a sheaf F on X, of the *stalk* F_x of F *at* x: F_x consists of equivalence classes of elements $s \in F(U)$ for open neighborhoods U of x; given two such, $s \in F(U)$ and $t \in F(V)$, s and t are equivalent if there is an open neighborhood W of x such that $W \subset U \cap V$ and $s{\restriction}W = t{\restriction}W$ in $F(W)$.

a) (4pts) Show that, up to isomorphism, the inverse image functor x^* of the geometric morphism x sends a sheaf F to F_x.

b) (3pts) Give a concrete description of the direct image functor $x_* : \mathrm{Set} \to \mathrm{Sh}(X)$

c) (3pts) The *specialization order* on the space X is defined as follows: $x \leq y$ if every open neighborhood of x also contains y. Show, that every inequality $x \leq y$ induces a natural transformation $x^* \Rightarrow y^*$.

Solution: a) The category $\mathrm{Sh}(X)$ is equivalent to the category of étale maps into X (see section 0.3 of the lecture notes). Under this equivalence, a sheaf F on X corresponds to the space of pairs (x, s_x) with $x \in X$ and s_x a germ of F at x. The étale map to X is the first projection.

For a continuous function $f : Y \to X$ of spaces, we have a geometric morphism $f : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, the inverse image of which, $f^* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ sends an étale space $E \to X$ to the étale space over Y given by the pullback

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow[f]{} & X \end{array}$$

Now if Y is a single point space $\{*\}$ and f sends $*$ to $x \in X$, then the recipy above gives that $f^*(F)$ is the set of germs of F at x, that is: the stalk of F at x.

b) For the geometric morphism $f : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ induced by the continuous function f discussed in a), the direct image is given by

$$f_*(F)(U) = F(f^{-1}(U))$$

for a sheaf F on Y and open set $U \subset X$.

Now if Y is a one-point space $\{*\}$ then $\mathrm{Sh}(Y)$ is equivalent to Set by an equivalence which sends a set A to the sheaf $\widehat{A}$ on $\{*\}$ given by

$$\begin{array}{rcl} \widehat{A}(\emptyset) & = & 1 \\ \widehat{A}(\{*\}) & = & A \end{array}$$

Hence, if we now denote the continuous map $\{*\} \to X$ which sends $*$ to x also by x, then we have

$$x_*(A)(U) = \widehat{A}(x^{-1}(U)) = \begin{cases} A & \text{if } x \in U \\ 1 & \text{else} \end{cases}$$

c) If $x \leq y$ for the specialization order, then for any sheaf F on X we have a natural map $F_x \to F_y$: every representative $s \in F(U)$ of a germ of F at x is also a representative of a germ at y, and if two such representatives, $s \in F(U)$, $t \in F(V)$ represent the same germ at x, then for some open neighbourhood W of x with $W \subset U \cap V$ we have $s{\restriction}W = t{\restriction}W$; since W is also a neighbourhood of y, s and t represent the same germ at y.

This natural map determines a natural transformation $x^* \Rightarrow y^*$.

Exercise 2 In this exercise we work in the category $\widehat{\mathcal{C}}$ of presheaves on a small category $\mathcal{C}$.

Let P be an arbitrary presheaf on $\mathcal{C}$. We construct the presheaf $\widetilde{P}$ as follows: for an object C of $\mathcal{C}$, $\widetilde{P}(C)$ consists of all pairs.(R, f) where R is a sieve on C (considered as a subobject of the representable presheaf y_C) and $f : R \to P$ is an arrow in $\widehat{\mathcal{C}}$. If $\alpha : C' \to C$ is a morphism in $\mathcal{C}$ then $\widetilde{P}(\alpha)$ sends $(R, f) \in \widetilde{P}(C)$ to $(\alpha^*(R), f{\circ}v)$ where

$$\begin{array}{ccc} \alpha^*(R) & \xrightarrow{v} & R \\ \downarrow & & \downarrow \\ y_{C'} & \xrightarrow[y_\alpha]{} & y_C \end{array}$$

is a pullback.

a) (4pts) Show that $\widetilde{P}$ is indeed a well-defined presheaf on $\mathcal{C}$.

b) (6pts) Show that there is a map of presheaves $\eta : P \to \widetilde{P}$ which is a partial map classifier on P.

Solution: a) We shall take a sieve R on an object C of $\mathcal{C}$ as a sub-presheaf of the representable presheaf y_C (so $R(D)$ is a subset of $\mathcal{C}(D, C)$). So if $\alpha : C' \to C$ is an arrow in $\mathcal{C}$ and R is a sieve on C, then $\alpha^*(R)$ is the subpresheaf of $y_{C'}$ defined by

$$\alpha^*(R)(D) = \{g : D \to C' \mid \alpha g \in R(D)\}$$

This implies that $\mathrm{id}^*(R) = R$ and for $C'' \stackrel{\beta}{\to} C' \stackrel{\alpha}{\to} C$ that $\beta^*(\alpha^*(R)) = (\alpha\beta)^*(R)$, and using this it is not hard to show that $\widetilde{P}$ is a well-defined presheaf.

b) Define $\eta : P \to \widetilde{P}$ by: $\eta_C(x) = (\max(C), \hat{x})$, where $\max(C)$ denotes the maximal sieve on C (that is, the object y_C) and $\hat{x}$ is the map $y_C \to P$ which sends the identity map on C to $x \in P(C)$ (i.e., it is the map which corresponds to $x \in P(C)$ under the Yoneda Lemma).

Now suppose $U \to Y$ is a subobject in $\widehat{\mathcal{C}}$ (we assume that U is a subpresheaf of Y); and suppose we have an arrow $\phi : U \to P$. We define a map $\tilde{\phi} : Y \to \widetilde{P}$ as follows: for $y \in Y(C)$, $\tilde{\phi}_C(y) = (R_y, f_y)$ where: $R(y)$ is the sieve on C consisting of those arrows $\alpha : C' \to C$ which satisfy $Y(\alpha)(y) \in U(C')$, and for $(C' \stackrel{\alpha}{\to} C) \in R(y)$ we put $(f_y)_C(\alpha) = Y(\alpha)(y)$. This gives a well-defined morphism of presheaves $f_y : R_y \to P$.

It is left to you to verify that this completes the definition of a map $\tilde{\phi} : Y \to \widetilde{P}$, and that this map has the property that we have a pullback diagram

$$\begin{array}{ccc} U & \longrightarrow & Y \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\phi}} \\ P & \xrightarrow[\eta]{} & \widetilde{P} \end{array}$$

Now this is the universal property characterizing the partial map classifier of P.

Exercise 3 a) (4pts) Let $\mathcal{C}$ be.a small category with finite limits. Prove that the functor $y : \mathcal{C} \to \widehat{\mathcal{C}}$ has the following property: $\widehat{\mathcal{C}}$ is cocomplete and for every finite-limit preserving functor $F : \mathcal{C} \to \mathcal{E}$ where $\mathcal{E}$ is a cocomplete topos, there is an essentially unique functor $\overline{F} : \widehat{\mathcal{C}} \to \mathcal{E}$ which preserves finite limits and all colimits, and satisfies $\overline{F} \circ y \simeq F$.

b) (6pts) Now assume that $\mathcal{C}$ is a small regular category. Prove that there is a topos $\widetilde{\mathcal{C}}$ and a functor $y_R : \mathcal{C} \to \widetilde{\mathcal{C}}$ which is regular and has the following property: for every regular functor $F : \mathcal{C} \to \mathcal{E}$ from $\mathcal{C}$ to a cocomplete topos $\mathcal{E}$, there is an essentially unique functor $\widetilde{F} : \widetilde{\mathcal{C}} \to \mathcal{E}$ which preserves finite limits and all colimits, and satisfies $\widetilde{F} \circ y_R \simeq F$. [Hint: a functor is regular if it preserves finite limits and regular epimorphisms. Pick a suitable Grothendieck topology on $\mathcal{C}$ to make y_R preserve regular epimorphisms]

Solution: a) The category $\widehat{\mathcal{C}}$ is cocomplete since colimits are calculated pointwise (and Set is cocomplete). We know from the theory of geometric morphisms that for a cocomplete topos $\mathcal{E}$ there is a natural 1-1 correspondence between:

i) functors $\widehat{\mathcal{C}} \to \mathcal{E}$ which preserve colimits and finite limits, and

ii) flat functors $\mathcal{C} \to \mathcal{E}$.

The bijection (i)$\Rightarrow$(ii) sends a functor $F : \widetilde{\mathcal{C}} \to \mathcal{E}$ to $F \circ y : \mathcal{C} \to \mathcal{E}$.

We also have that if $\mathcal{C}$ has finite limits, then a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.

So if $F : \mathcal{C} \to \mathcal{E}$ preserves finite limits, it is flat, and hence the colimit preserving functor $\overline{F} : \widehat{\mathcal{C}} \to \mathcal{E}$ which satisfies $\overline{F} \circ y = F$, preserves finite limits.

Conversely, if a functor $\widehat{\mathcal{C}} \to \mathcal{E}$ preserves colimits and finite limits then its composition with $y : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves finite limits since y preserves all limits which exist in $\mathcal{C}$.
b): this was possibly the hardest exercise. Every regular functor $F : \mathcal{C} \to \mathcal{E}$ preserves finite limits, so by a), F has an extension $\overline{F} : \widehat{C} \to \mathcal{E}$ which preserves finite limits and all colimits. However, the Yoneda embedding $y : \mathcal{C} \to \widehat{\mathcal{C}}$ is not regular. We see, however, that if $e : A \to B$ in $\mathcal{C}$ is a regular epi and $\varepsilon : y(A) \to X$ is the coequalizer in $\widehat{\mathcal{C}}$ of the kernel pair of $y(e)$, then $\overline{F}(y(a))$ and $\overline{F}(\varepsilon)$ coincide, since F is regular and $\overline{F}$ preserves coequalizers of kernel pairs.

We need a Grothendieck topology on $\mathcal{C}$ which is as small as possible and is yet such that the composition

$$\mathcal{C} \stackrel{y}{\to} \widehat{\mathcal{C}} \stackrel{L}{\to} \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$$

(where L is sheafification w.r.t. Cov) preserves regular epimorphisms. The least topology doing this, has for an arbitrary object $C \in \mathcal{C}$ the collection of those sieves on C which contain a regular epimorphism.

Note, that a functor $F : \mathcal{C} \to \mathcal{E}$ is regular, precisely wjen it is flat and continuous (for Cov); which holds if and only if $\overline{F}$ factors through $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

Exercise 4 In this exercise we consider the category $\mathbf{2}$ which has two objects 0 and 1 and one non-identity arrow $a : 0 \to 1$.

a) (2pts) Show that $\mathbf{2}$ is isomorphic to $\mathbf{2}^{\mathrm{op}}$.

b) (4pts) Show that for any category $\mathcal{E}$ with finite limits there is a bijection between the following two collections:

   i) The collection of subobjects of 1 in $\mathcal{E}$

   ii) The collection of isomorphism classes of functors $\mathbf{2} \to \mathcal{E}$ which preserve finite limits

c) (4pts) Show that the category $\mathrm{Set}^{\mathbf{2}}$ is a "classifying topos for subobjects of 1"; that is: for any cocomplete topos $\mathcal{E}$ there is a natural bijection between geometric morphisms $\mathcal{E} \to \mathrm{Set}^{\mathbf{2}}$ and subobjects of the terminal object in $\mathcal{E}$.

Solution: a) Left to you.
b): Left to you.
c): First, suppose $F : \mathbf{2} \to \mathcal{E}$ preserves finite limits. In $\mathbf{2}$, the object 1 is terminal and the arrow a is monic, so we must have that $F(a) : F(0) \to 1 \simeq F(1)$ is a subobject of 1. Clearly, a natural isomorphism $F \simeq G$ of two such functors gives the same subobject of 1.
d): We have: (writing Top for the category of toposes and geometric morphisms)

$$\begin{array}{rcl} \mathrm{Top}(\mathcal{E}, \mathrm{Set}^{\mathbf{2}}) & \simeq & \text{by a)} \\ \mathrm{Top}(\mathcal{E}, \mathrm{Set}^{\mathbf{2}^{\mathrm{op}}}) & \simeq & \text{by theory} \\ \mathrm{Flat}(\mathbf{2}, \mathcal{E}) & \simeq & \text{by c)} \\ \mathrm{Sub}_{\mathcal{E}}(1) & & \end{array}$$