Exam Category Theory and Topos Theory
May 30, 2016; 10:00–13:00
With solutions and comments on the grading

THIS EXAM CONSISTS OF FIVE PROBLEMS. All problems have equal weight (10 points); where a problem consists of more than one part, it is indicated what each part is worth

Exercise 1.Let $\mathcal{C}$ be a locally small category. For an object X of $\mathcal{C}$ we define the *representable* functor $R_X : \mathcal{C} \to \mathrm{Set}$ by

$$R_X(A) \;=\; \mathcal{C}(X, A)$$

(and on arrows by composition)

a) (3 pts) Prove that the functor R_X preserves monomorphisms.

b) (4 pts) Assume that the category $\mathcal{C}$ has all small coproducts. Show that R_X has a left adjoint.

c) (3 pts) Suppose $F : \mathcal{C} \to \mathrm{Set}$ is a functor and $\mu : R_X \Rightarrow F$ a natural transformation. Show that μ is completely determined by the element $\mu_X(\mathrm{id}_X)$ of $F(X)$.

Exercise 2. For each of the functors given below, determine whether it preserves all limits, and whether it preserves all colimits. Give a short argument in each case.

a) (2 pts) The forgetful functor Ring $\to$ Mon, which sends each ring to its underlying multiplicative monoid.

b) (3 pts) The domain functor: $\mathcal{C}/A \to \mathcal{C}$, where $\mathcal{C}$ is a category with finite products, and A is a non-terminal object of $\mathcal{C}$ (recall that an object of $\mathcal{C}/A$ is an arrow with codomain A).

c) (3 pts) The forgetful functor from preorders to sets.

d) (2 pts) The "poset reflection functor" functor from preorders to posets: it sends a preorder P to the poset of isomorphism classes in P.

Exercise 3. In this exercise we work in a regular category $\mathcal{C}$. Suppose $f : X \to Y$ is an arrow in $\mathcal{C}$; we denote by $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ the function on subobjects defined by pullback along f.

a) (5 pts) Prove, for $M \in \mathrm{Sub}(X)$ and $N \in \mathrm{Sub}(Y)$ the following identity:

$$\exists_f(M \wedge f^*N) \;=\; \exists_f(M) \wedge N$$

b) (5 pts) Now suppose that f is a regular epimorphism. Prove, for subobjects M, N of Y, that $f^*M \leq f^*N$ implies $M \leq N$.

Exercise 4. Let G be a group with more than 1 element, considered as a category. We consider the category $\mathrm{Set}^{G^{\mathrm{op}}}$, which we may identify with the category of G-sets.

a) (3 pts) Show that in $\mathrm{Set}^{G^{\mathrm{op}}}$, the terminal object is not projective. Recall that an object P is projective if, whenever we have an arrow $f : P \to Y$ and an epimorphism $g : X \to Y$, there is an arrow $h : P \to X$ such that $f = gh$.

b) (4 pts) Show that in $\mathrm{Set}^{G^{\mathrm{op}}}$, the object $\{0, 1\}$ with trivial G-action is a subobject classifier.

c) (3 pts) Show that $\mathrm{Set}^{G^{\mathrm{op}}}$ is Boolean. That is, every subobject lattice is a Boolean algebra.

Exercise 5. Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$.

a) (2 pts) Let U be a subobject of 1. Show that U determines a *sieve on* $\mathcal{C}$, that is: a set of objects $\mathcal{D}$ with the property that for any morphism $X \to Y$, if $Y \in \mathcal{D}$ then $X \in \mathcal{D}$.

b) (3 pts) We define, using the sieve $\mathcal{D}$ on $\mathcal{C}$ from part a), a morphism $c(U) : \Omega \to \Omega$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ by putting, for a sieve R on an object C:

$$c(U)_C(R) \;=\; R \cup \{f : C' \to C \,|\, C' \in \mathcal{D}\}$$

Prove that $c(U)$ is a Lawvere-Tierney topology on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

c) (2 pts) Let F be a subpresheaf of a presheaf G on $\mathcal{C}$. Prove that F is dense for $c(U)$ if and only if for all $C \in \mathcal{C}_0$ and $x \in G(C)$ we have: $x \in F(C)$ or $C \in \mathcal{D}$.

d) (3 pts) Prove that the category of sheaves for $c(U)$ is equivalent to the category of presheaves on some subcategory of $\mathcal{C}$.

Solutions and comments on the grading

The exam seems to have been a bit tough, or in any case a lot of work for most students; no one handed in a *perfect* solution. In order to obtain a decent result I have decided to count, for each student, only his/her four best exercises.

Exercise 1

a) For an arrow $f : A \to B$, we have $R_X(f) : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ sending $g : X \to A$ to the composition fg in $\mathcal{C}$. If f is mono and $g, h : X \to A$ are elements of $\mathcal{C}(X, A)$, then clearly $fg = fh$ implies $g = h$, so $R_X(f)$ is injective, which means it is a monomorphism in Set.

b) An arrow $Y \to R_X(A)$ in Set is just a Y-indexed family of arrows $\{f_y : X \to A \,|\, y \in Y\}$. Since $\mathcal{C}$ has all small coproducts, this corresponds uniquely to an arrow from the Y-indexed coproduct of copies of X, $\sum_{y\in Y} X$, to A. For every function $\phi : Y \to Z$ we have an arrow

$$\sum_{y\in Y} X \to \sum_{z\in Z} X$$

which sends the y-th cofactor to the $\phi(y)$-th one. This determines a functor Set $\to \mathcal{C}$ which is left adjoint to R_X.

c) This is just the Yoneda lemma, and saying so would have given you 3 points. Concretely, if μ is as given and $f \in R_X(A)$, then $f = R_X(f)(\mathrm{id}_X)$, so by naturality

$$\mu_A(f) = \mu_A(R_X(f)(\mathrm{id}_X)) = F(f)(\mu_X(\mathrm{id}_X))$$

which proves the claim.

Exercise 2

a) This functor has a left adjoint, which sends each monoid M to the ring $\mathbb{Z}[M]$ of finite expressions $n_1m_1 + \cdots n_km_k$ (for $k \geq 0$) with $n_i \in \mathbb{Z}$ and $m_i \in M$. Therefore, it preserves all limits.

It does *not* preserve all colomits; for example, the initial object of Ring, the ring $\mathbb{Z}$, is not sent to the initial monoid (which is a one-element monoid).

b) This functor has a right adjoint, which sends an object X of $\mathcal{C}$ to the projection $X \times A \to A$. It therefore preserves all colimits.

It does *not* preserve all limits; for example, the terminal object of $\mathcal{C}/A$ is the identity on A, which is sent to A, which as stated is non-terminal in $\mathcal{C}$ (if A were terminal, then the given functor would be an equivalence).

c) This functor has *both* adjoints: the left adjoint sends a set X to the discrete preorder on X ($x \leq y$ iff $x = y$) and the right adjoint sends X to the *indiscrete* preorder on X ($x \leq y$ always). So it preserves all limits and all colimits.

d) The poset reflection functor is, as is easily seen, left adjoint to the inclusion of Posets into Preorders; it preserves all colimits. It does *not* preserve all limits; for example, look at equalizers. Consider the two maps $f, g : 1 \to A$ where A is the indiscrete preorder on a two-element set. The equalizer is the empty set (you may apply c) here). However, upon poset reflection the two maps become equal, and the equalizer is 1 itself.

Some students (even among the very best) remarked that this functor is "an equivalence of categories". Although this is quite erroneous, it is, in the heat of the fight, a plausible error to make; and I have therefore decided to award you 1.5 pts if you wrote this down. After all, every preorder is, as a category, equivalent to its poset reflection; however, equivalence is not isomorphism, and in any case pseudo-inverses need choice and cannot be natural.

Exercise 3

a) Let us abuse notation and write the same symbol for a subobject and the domain of a representing monomorphism. So we have subobjects $M \stackrel{m}{\to} X$ and $N \stackrel{n}{\to} Y$; we have the regular epi-mono factorization $M \to \exists_f M \to Y$, and pullback diagrams

$$
\begin{array}{ccc}
M \wedge f^*N & \longrightarrow & M \\
\downarrow & & \downarrow{\scriptstyle m} \\
f^*N & \longrightarrow & X
\end{array}
\qquad
\begin{array}{ccc}
f^*N & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle f} \\
N & \underset{n}{\longrightarrow} & Y
\end{array}
\qquad
\begin{array}{ccc}
(\exists_f M) \wedge N & \longrightarrow & \exists_f M \\
\downarrow & & \downarrow \\
N & \underset{n}{\longrightarrow} & Y
\end{array}
$$

Now the compositions $M \wedge f^*N \to f^*N \to N \to Y$ and $M \wedge f^*N \to M \to \exists_f M \to Y$ are clearly equal, so by the third pullback diagram we have a unique arrow $g : M \wedge f^*N \to (\exists_f M) \wedge N$, making the diagrams

$$\begin{array}{ccc} M \wedge f^*N & \xrightarrow{g} & (\exists_f M) \wedge N \\ \downarrow & & \downarrow \\ M & \longrightarrow & \exists_f M \end{array} \qquad \begin{array}{ccc} M \wedge f^*N & \xrightarrow{g} & (\exists_f M) \wedge N \\ \downarrow & & \downarrow \\ f^*N & \longrightarrow & N \end{array}$$

commute.

Consider now the diagrams

$$\begin{array}{ccc} M \wedge f^*N & \longrightarrow & M \\ {\scriptstyle g}\downarrow & & \downarrow \\ (\exists_f M) \wedge N & \longrightarrow & \exists_f M \\ \downarrow & & \downarrow \\ N & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} M \wedge f^*N & \longrightarrow & M \\ \downarrow & & \downarrow \\ f^*N & \longrightarrow & X \\ \downarrow & & \downarrow \\ N & \longrightarrow & Y \end{array}$$

The outer squares of both are equal, and the right-hand diagram is a composite of two pullbacks; therefore the left hand outer square is a pullback. Since also the left hand lower square is a pullback, the left-hand upper square must be a pullback. That means that g, being pullback of the regular epi $M \to \exists_f M$, is a regular epi; but now we see that

$$M \wedge f^*N \xrightarrow{g} (\exists_f M) \wedge N \longrightarrow Y$$

is a regular epi-mono factorization. This establishes that

$$\exists_f(M \wedge f^*N) \;=\; (\exists_f M) \wedge N$$

as desired.

One inequality was easy to prove: we have $M \wedge f^*N \leq M$ and $M \wedge f^*N \leq f^*N$, so since $\exists_f$ is order-preserving we have $\exists_f(M \wedge f^*N) \leq (\exists_f M) \wedge \exists_f f^*N$; now $\exists_f f^*N \leq N$ by the adjunction $\exists_f \dashv f^*$, hence $\exists_f(M \wedge f^*N) \;\leq\; (\exists_f M) \wedge N$. If you had only this, you got 3 points.

You got 3.5 points if you proved the equality assuming that $\mathrm{Sub}(X)$ and $\mathrm{Sub}(Y)$ were Heyting algebras and f^* preserved the Heyting structure; although of course, we cannot in general assume this in a regular category.

b) Suppose $f : X \to Y$ is regular epi. Now for any subobject N of Y, we have a pullback

$$\begin{array}{ccc} f^*N & \longrightarrow & N \\ \downarrow & & \downarrow \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

so the map $f^*N \to N$ is regular epi, and hence the regular epi part of the composite map $f^*N \to Y$. Therefore $N = \exists_f f^*N$ for all subobjects N of Y; from this (and the fact that $\exists_f$ is order-preserving) the required implication follows at once.

Exercise 4

a) The presheaf category $\mathrm{Set}^{G^{\mathrm{op}}}$ is isomorphic to the category G-Set of sets X together with a right G-action $X \times G \to X$, and G-equivariant functions (functions commuting with the G-action). Modulo this isomorphism, the fact that in any presheaf category, limits and colimits are calculated point-wise, translates a.o. into the statements that the terminal object is a one-element set (with unique G-action); that epis are surjective functions and monos are injective functions. Furthermore, the one representable presheaf corresponds to the object G, with G-action given by multiplication in G.

If 1 were projective, we would have a section for the epi $G \to 1$. However, such a section must send the unique element of 1 to an element of G which is invariant under the action; i.e., an element $g \in G$ for which $gh = g$ for all $h \in G$. This is clearly impossible if G has more than one element.

b) A subobject of a G-set X is just a subset $A \subset X$ which is closed under the action. Since G is a group, this means that then also $X - A$ is closed under the G-action. We have therefore a classifying map $\chi_A : X \to \{0, 1\}$ such that $\chi_A(x) = 0$ iff $x \in A$; i.e. a map such that

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \chi_A} \\ 1 & \underset{t}{\longrightarrow} & \{0,1\} \end{array}$$

is a pullback (if $t(\star) = 0$), and χ_A is clearly unique with this property. Note, that by our remarks above, χ_A is a morphism of G-sets.

c) We know that in any topos, $\mathrm{Sub}(X)$ is a Heyting algebra, so it only remains to see that complements exist in $\mathrm{Sub}(X)$. Basically, the argument is the same as for b); observe that for a subobject A of X, $X - A$ is a complement (again using that limits and colimits are calculated point-wise).

Exercise 5.

a) In $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, 1 is the presheaf with $1(C) = \{*\}$ for all C. If U is a subobject of 1, we may regard U as a subpresheaf of 1, and U determines the set of objects

$$\mathcal{D} \;=\; \{C \in \mathcal{C}_0 \,|\, * \in U(C)\}$$

Clearly, if $C \in \mathcal{D}$ and $f : C' \to C$ then $C' \in \mathcal{D}$.

b) Let $\Phi_C \;=\; \{f : C' \to C \,|\, C' \in \mathcal{D}\}$, so $c(U)_C(R) = R \cup \Phi_C$. The properties $R \subseteq c(U)_C(R)$, $c(U)_C(R \cap S) = c(U)_C(R) \cap c(U)_C(S)$ and $c(U)_C(c(U)_C(R)) = c(U)_C(R)$ all follow trivially from this.

c) A subpresheaf F of G is dense if and only if for each object C of $\mathcal{C}$ and every $x \in G(C)$, we have that $c(U)_C((\chi_F)_C(x))$ is the maximal sieve on C, where $\chi_F(x)$ is the sieve on C consisting of precisely those $f : C' \to C$ for which $G(f)(x) \in F(C')$. So, F is dense in G if and only if for each C and x, the arrow id_C is an element of $(\chi_F)_C(x) \cap \Phi_C$; but $\mathrm{id}_C \in (\chi_F)_C(x)$ means that $x \in F(C)$ and $\mathrm{id}_C \in \Phi_C$ means that $C \in \mathcal{D}$, so we are done.

d) We use the property which characterizes sheaves w.r.t. dense inclusions: a presheaf X is a sheaf if and only if for every dense inclusion $F \subseteq G$, every map $F \to X$ has a unique extension to a map $G \to X$.

We claim that X is a sheaf for $c(U)$ if and only if for every object $C \in \mathcal{D}$, $X(C)$ is a singleton set.

Clearly, the condition above is sufficient: suppose that for every object $C \in \mathcal{D}$, $X(C)$ is a singleton set. Let $F \subseteq G$ be dense, and $\mu : F \to X$ a map. Since for $C \in \mathcal{D}$ there is nothing to choose (by the condition) and for $C \notin \mathcal{D}$, $F(C) = G(C)$, we have a clearly unique extension of μ. So X is a sheaf.

Conversely, suppose X is a sheaf. Let us write $\mathcal{D}$ also for the full subcategory of $\mathcal{C}$ on the set of objects $\mathcal{D}$, and write $X{\restriction}\mathcal{D}$ for the restriction of the functor X to $\mathcal{D}^{\mathrm{op}}$. Let Y be any presheaf on $\mathcal{D}$ and $\hat{Y}$ be the presheaf on $\mathcal{C}$ defined by:

$$\hat{Y}(C) = \begin{cases} Y(C) & \text{if } C \in \mathcal{D} \\ \emptyset & \text{otherwise} \end{cases}$$

Let $\mathbf{0}$ be the empty presheaf on $\mathcal{C}$. Now $\mathbf{0} \subseteq \hat{Y}$ is dense, so the unique map $\mathbf{0} \to X$ has a unique extension $\hat{Y} \to X$; in other words, there is, for any presheaf Y on $\mathcal{D}$ exactly one arrow from Y to $X{\restriction}\mathcal{D}$. This means $X{\restriction}\mathcal{D}$ is terminal in $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$, so the given condition holds.

We now see that the category of sheaves for $c(U)$ is equivalent to the category of presheaves on $\mathcal{E}$, where $\mathcal{E}$ is the full subcategory of $\mathcal{C}$ on the objects *not* in $\mathcal{D}$.