Exam Gödel's Incompleteness Theorems

June 4, 2015, 10.00–13.00
With solutions

Exercise 1.

a) Let F be a primitive recursive function. Prove that the function

$$x \mapsto \underbrace{F(F(\cdots F}_{F(x)\text{ times}}(x))\cdots)$$

is primitive recursive too.

b) Let $\log(y, x)$ denote the largest number z such that $y^z < x$. Prove that $\log(y, x)$ is a primitive recursive function of two variables.

Solution: **a)**. Define the function $G(x, y)$ by $G(x, 0) = x$ and $G(x, y + 1) = F(G(x, y))$. Then G is defined by primitive recursion from the identity function, projections and F, so G is primitive recursive. The given function is the function $x \mapsto G(x, F(x))$, defined from G by composition, so this function is primitive recursive.

b). The given function is not defined for $y < 2$, so let's agree on a default value: put $\log(y, x) = 0$ for $y < 2$. Now for $y \geq 2$, $y^z > z$ for all z, so we see that $\log(y, x) < x$ in that case. So we can replace the search for a largest number by a bounded minimisation: for $y \geq 2$, put

$$\log(y, x) \; = \; (\mu z \leq x . y^z \geq x) - 1$$

Now $\log(y, x)$ is defined by case distinction on $y \geq 2$, bounded minimisation and the exponential function (which is primitive recursive), so it is primitive recursive.

Exercise 2. Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is *provably recursive* if there is a Σ_1-formula $F(x, y)$ satisfying the following conditions:

$$\begin{array}{ll} \mathrm{PA} \vdash F(\overline{n}, \overline{f(n)}) & \text{for every natural number } n \\ \mathrm{PA} \vdash \forall x \exists ! y F(x, y) & \end{array}$$

Prove that the set of provably recursive functions is closed under composition.

Solution: the exercise only speaks of functions of one variable, so we can limit ourselves to that case. Let f and g be provably recursive functions, and F, G two Σ_1-formulas such that for all natural numbers n, $\text{PA} \vdash F(\overline{n}, \overline{f(n)}) \wedge G(\overline{n}, \overline{g(n)})$, and moreover $\text{PA} \vdash \forall x \exists ! y F(x, y)$ and $\text{PA} \vdash \forall u \exists ! v G(u, v)$. Let $H(x, v)$ be the formula

$$\exists w (F(x, w) \wedge G(w, v))$$

Then $H(x, v)$ is a Σ_1-formula (beware! If we had put $H(x, v) \equiv \exists ! w (F(x, w) \wedge G(w, v))$, then we had *not* obtained a Σ_1-formula!), since Σ_1-formulas are closed under conjunctions and existential quantifications. Moreover since we have $\text{PA} \vdash F(\overline{n}, \overline{f(n)}) \wedge G(\overline{f(n)}, \overline{g(f(n))})$, we see that $\text{PA} \vdash H(\overline{n}, \overline{g(f(n))})$.

For the other property, reason inside PA (or, equivalently, in an arbitrary model of PA). Given x, there is y with $F(x, y)$; for such a y there is v with $G(y, v)$, so there is v with $H(x, v)$. So we see $\text{PA} \vdash \forall x \exists v H(x, v)$. For uniqueness, suppose $H(x, v) \wedge H(x, v')$. Then there are y and y' with $F(x, y), F(x, y'), G(y, v)$ and $G(y', v')$. But by the uniqueness satisfied by F and G, we see $y = y'$ and hence $v = v'$. So we have in fact $\text{PA} \vdash \forall x \exists ! v H(x, v)$, as desired.

Exercise 3. Recall that the notation $\Box \phi$ stands for $\exists x \overline{\text{Prf}}(x, \ulcorner \phi \urcorner)$ and that for $\Box$ the following three "derivability conditions" hold:

D1 $\text{PA} \vdash \phi$ implies $\text{PA} \vdash \Box \phi$

D2 $\text{PA} \vdash \Box(\phi \to \psi) \to (\Box \phi \to \Box \psi)$

D3 $\text{PA} \vdash \Box \phi \to \Box \Box \phi$

You may use without proof, that conditions D1 and D2 imply
$\text{PA} \vdash \Box(\phi \wedge \psi) \leftrightarrow \Box \phi \wedge \Box \psi$.

a) Let PA' be the theory $\text{PA} + \Box \chi$ for some sentence χ. Show that property D1 also holds for PA': if $\text{PA}' \vdash \phi$, then $\text{PA}' \vdash \Box \phi$.

b) Prove *Formalised Löb's Theorem*, which is the statement

$$\text{PA} \vdash \Box(\Box \phi \to \phi) \to \Box \phi$$

for arbitrary ϕ.

[Hint: given ϕ, show that there is a sentence ψ such that $\text{PA} \vdash \psi \leftrightarrow (\Box \psi \to \phi)$. Let PA' be $\text{PA} + \Box(\Box \phi \to \phi)$. Prove that $\text{PA}' \vdash \Box \psi$ and conclude that $\text{PA}' \vdash \Box \phi$.]

Solution: **a)**. Suppose $\mathrm{PA}' \vdash \phi$. Then $\mathrm{PA} \vdash \Box\chi \to \phi$, so by D1 for PA, $\mathrm{PA} \vdash \Box(\Box\chi \to \phi)$. Applying D2 we get $\mathrm{PA} \vdash \Box\Box\chi \to \Box\phi$. Using D3 on χ ($\mathrm{PA} \vdash \Box\chi \to \Box\Box\chi$) we obtain $\mathrm{PA} \vdash \Box\chi \to \Box\phi$, which is equivalent to $\mathrm{PA}' \vdash \Box\phi$.

b). Apply the Diagonalisation Lemma to the formula $(\exists x \overline{\mathrm{Prf}}(x, v)) \to \phi$: we obtain a sentence ψ satisfying $\mathrm{PA} \vdash \psi \leftrightarrow (\Box\psi \to \phi)$. Let PA' be the theory $\mathrm{PA} + \Box(\Box\phi \to \phi)$. We note that the properties D1,D2 and D3 hold true for PA' as well (using part a) of the exercise). We now get:

By D2 and choice of ψ,	$\mathrm{PA}' \vdash \Box\psi \leftrightarrow \Box(\Box\psi \to \phi)$	(1)
By D3 on ψ,	$\mathrm{PA}' \vdash \Box\psi \to \Box\Box\psi$	(2)
By D2 and (1),	$\mathrm{PA}' \vdash \Box\psi \to (\Box\Box\psi \to \Box\phi)$	(3)
By (2) and (3),	$\mathrm{PA}' \vdash \Box\psi \to \Box\phi$	(4)
By (4) and D1,	$\mathrm{PA}' \vdash \Box(\Box\psi \to \Box\phi)$	(5)
By (5) and definition of PA',	$\mathrm{PA}' \vdash \Box(\Box\psi \to \phi)$	(6)
By (1),(6) and choice of ψ,	$\mathrm{PA}' \vdash \Box\psi$	(7)
By (4),	$\mathrm{PA}' \vdash \Box\phi$	

And the last line just means $\mathrm{PA} \vdash \Box(\Box\phi \to \phi) \to \Box\phi$.

Exercise 4. Let $f : \mathbb{N} \to \mathbb{N}$ be a provably recursive function (see Exercise 2). We work in a conservative extension of PA which has a function symbol for f, and axiom $\forall x F(x, f(x))$, where $F(x, y)$ is the Σ_1-formula representing f. Note, that any model of PA has a unique interpretation of the function symbol f making the axiom true.

Now assume that f is strictly increasing. Let $\mathcal{M}$ be a nonstandard model of PA; by $\mathcal{N}$ we denote, as usual, the standard model. Furthermore, assume that $\mathcal{N}$ is a Π_1-elementary submodel of $\mathcal{M}$.

Prove that the following two statements are equivalent:

i) In $\mathcal{M}$ there exists a copy of $\mathbb{Z}$ which contains no elements of the form $f(x)$

ii) $\mathcal{N} \models \forall y \exists x (f(x + 1) > f(x) + y)$

Solution. i)⇒ii): Suppose ii) fails, so $\mathcal{N} \not\models \forall y \exists x (f(x+1 > f(x)+y)$. Then there is a standard number k such that

$$\mathcal{N} \models \forall x (f(x + 1) \leq f(x) + k)$$

Now $\forall x(f(x+1) \leq f(x)+k)$ is a Π_1-sentence, so since the inclusion $\mathcal{N} \subset \mathcal{M}$ is supposed to be Π_1-elementary, we get

$$\mathcal{M} \models \forall x(f(x+1) \leq f(x)+k)$$

In order to see that i) fails, let $x \in \mathcal{M}$ be nonstandard. Because f is strictly increasing, $x \leq f(x)$ so there is a least y such that $x \leq f(y)$. Then y cannot be 0, for $f(0)$ is a standard number. Now we have $f(y) - f(y-1) \leq k$ and $f(y-1) < x \leq f(y)$, so the element $f(y)$ lies in the same copy of $\mathbb{Z}$ as x. The element $x \in \mathcal{M}$ was an arbitrary nonstandard number, so we see that i) fails.

ii)$\Rightarrow$i): Suppose $\mathcal{N} \models \forall y \exists x(f(x+1) > f(x)+y)$. Then for all standard numbers m we have

$$\mathcal{N} \models \exists x(f(x+1) > f(x)+m)$$

and since this is a Σ_1-sentence (we don't need the assumption $\mathcal{N} \prec_{\Pi_1} \mathcal{M}$ here!) it holds in $\mathcal{M}$:

$$\mathcal{M} \models \exists x(f(x+1) > f(x)+m)$$

This holds for all standard m, so by Overspill there is a nonstandard element c satisfying $\mathcal{M} \models \exists x(f(x+1) > f(x)+c)$. Pick $a \in \mathcal{M}$ such that $f(a+1) > f(a)+c$. We see then, that $f(a)$ and $f(a+1)$ lie in different copies of $\mathbb{Z}$. Since the ordering of these copies is dense, there is a copy of $\mathbb{Z}$ in between. Now that copy cannot contain an element of the form $f(x)$, because f is strictly increasing.