Exam Gödel's Incompleteness Theorems

May 26, 2010, 14.00–17.00

THIS EXAM CONSISTS OF 4 PROBLEMS; SEE ALSO BACK SIDE
Advice: first do those problems you can do right away; then, start thinking about the others. Good luck!

1. Let ϕ, ψ be sentences of PA. Define $\mathcal{C}$ by the following abstract syntax:

$$\mathcal{C} := \phi \mid \psi \mid \mathcal{C} \wedge \mathcal{C} \mid \neg\mathcal{C}$$

More precisely, $\mathcal{C}$ is the smallest class of sentences such that

$$\begin{array}{rcl} & & \phi, \psi \in \mathcal{C} \\ \chi, \theta \in \mathcal{C} & \Rightarrow & (\chi \wedge \theta) \in \mathcal{C} \\ \chi \in \mathcal{C} & \Rightarrow & (\neg\chi) \in \mathcal{C}. \end{array}$$

(a) Show precisely that $\mathcal{C}$ is primitive recursive, by proving that there is a primitive recursive function g such that for all sentences χ one has

$$\begin{array}{rcl} \chi \in \mathcal{C} & \Leftrightarrow & g(\ulcorner\chi\urcorner) = 1; \\ \chi \notin \mathcal{C} & \Leftrightarrow & g(\ulcorner\chi\urcorner) = 0. \end{array}$$

You may devise your own coding for these sentences.

(b) Show that there is a PA formula $\Xi(x)$ with $\mathrm{FV}(\Xi) = \{x\}$, such that

$$\begin{array}{rcl} \chi \in \mathcal{C} & \Rightarrow & \mathrm{PA} \vdash \Xi(\overline{\ulcorner\chi\urcorner}); \\ \chi \notin \mathcal{C} & \Rightarrow & \mathrm{PA} \vdash \neg\Xi(\overline{\ulcorner\chi\urcorner}). \end{array}$$

(c) Show that there is a formula $\Omega(x)$ with $\mathrm{FV}(\Omega) = \{x\}$ such that

$$\mathrm{PA} \vdash \Omega(\overline{\ulcorner\chi\urcorner}) \;\leftrightarrow\; \chi, \text{ for all } \chi \in \mathcal{C}.$$

2. Given a sentence ϕ of PA, define ϕ_n as $\Box^n(\phi)$, for $n \in \mathbb{N}$. More precisely

$$\begin{array}{rcl} \phi_0 & = & \phi, \\ \phi_{n+1} & = & \Box(\phi_n). \end{array}$$

(a) Show that there is a primitive recursive function f such that for all sentences ϕ and all $n \in \mathbb{N}$ one has

$$f(n, \ulcorner\phi\urcorner) = \ulcorner\phi_n\urcorner.$$

(b) Show that if PA is consistent, then there is no formula $\Theta(x, a)$ with $\mathrm{FV}(\Theta) = \{x, a\}$, such that for all sentences ϕ and all $n \in \mathbb{N}$ one has

$$\mathrm{PA} \vdash \Theta(\overline{n}, \overline{\ulcorner\phi\urcorner}) \;\leftrightarrow\; \phi_n.$$

[Hint. Suppose Θ exists. Define $\triangle(\phi) = \Theta(\overline{0}, \overline{\ulcorner\phi\urcorner})$. Then for all sentences ϕ one has

$$\text{PA} \vdash \triangle(\phi) \leftrightarrow \phi.$$

Immitating the liar paradox, apply the Diagonalization Lemma to get a contradiction.]

(c) Show that there is a formula $\Theta(x, a)$ with $\text{FV}(\Theta) = \{x, a\}$, such that for all sentences ϕ and all $n \in \mathbb{N}$, with $n > 0$ one has

$$\text{PA} \vdash \Theta(\overline{n}, \overline{\ulcorner\phi\urcorner}) \;\leftrightarrow\; \phi_n.$$

3. In this exercise, you may assume that PA is consistent. By the Diagonaization Lemma, let G be a sentence in the language of PA such that

$$\text{PA} \;\vdash\; G \leftrightarrow \Box\neg\Box G$$

We recall that in the course we proved the following three *derivability conditions*:

D1 $\;\text{PA} \vdash \phi \;\Rightarrow\; \text{PA} \vdash \Box\phi$
D2 $\;\text{PA} \vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
D3 $\;\text{PA} \vdash \Box\phi \to \Box\Box\phi$

(a) Prove that for any two sentences ϕ and ψ in the language of PA,

$$\text{PA} \;\vdash\; \Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$$

(b) Prove that $\text{PA} \vdash G \to \Box\bot$. Conclude that G is false in the standard model.

(c) Prove that also, $\text{PA} \vdash \Box\bot \to G$.

(d) Conclude from the previous two items that G is independent of PA.

4. Let $\mathcal{M}$ be a nonstandard model of PA.

(a) Show that there exists a nonstandard element $a \in \mathcal{M}$ such that the set $\{a \pm n \,|\, n \in \mathbb{N}\}$ contains no squares.
[Hint: take $c \in \mathcal{M}$ nonstandard; consider c^2 and $(c+1)^2$]

(b) Define the relation $\ll$ between nonstandard elements of $\mathcal{M}$ by: $a \ll b$ iff for all standard n, $na < b$. Prove that $a \ll b$ is equivalent to: there is a nonstandard element c such that $ac < b$.

(c) Prove that the relation $\ll$ is dense, that is: if $a \ll b$ then there is an element c such that $a \ll c \ll b$.

Solution Exercise 3:

a) This could be done in a number of ways, but the point of the exercise is that you can do almost everything just making use of D1–D3. So I present the solution in this way.

PA $\vdash \phi \wedge \psi \to \phi$ by Logic, hence by D1 we have PA $\vdash \Box(\phi \wedge \psi \to \phi)$ whence by D2, PA $\vdash \Box(\phi \wedge \psi) \to \Box\phi$. Similarly, PA $\vdash \Box(\phi \wedge \psi) \to \Box\psi$, so PA $\vdash \Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$. For the converse, we observe that PA $\vdash \phi \to (\psi \to \phi \wedge \psi)$ by Logic, hence by using D1 and twice D2 we get PA $\vdash \Box\phi \to (\Box\psi \to \Box(\phi \wedge \psi))$ and therefore by Logic PA $\vdash (\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi)$ as desired. This part was worth 3 points: 1 for the first implication, 2 for the second.

b) Let's write H for $\neg\Box G$, so PA $\vdash G \leftrightarrow \Box H$. By D1 and D2, applied to $\vdash \Box H \to G$, we get $\vdash \Box\Box H \to \Box G$. By D3 we have $\vdash \Box H \to \Box\Box H$. Combining, we see that $\vdash G \to \Box G$. By another application of D3 we have $\vdash G \to \Box\Box G$. But by choice of G we also have $\vdash G \to \Box\neg\Box G$. Applying part a) we see that $\vdash G \to \Box(\Box G \wedge \neg\Box G)$. Since $\vdash \Box G \wedge \neg\Box G \to \bot$ by Logic, hence $\vdash \Box(\Box G \wedge \neg\Box G) \to \Box\bot$ by D1 and D2, we have $\vdash G \to \Box\bot$ as required.

   It follows that $G \to \Box\bot$ is true in the standard model (in fact, in any model); by assumption (that PA is consistent), $\Box\bot$ is false in the standard model. Hence G is false in the standard model.

   This part was worth 3 points: 2 for the derivation of $\vdash G \to \Box\bot$, and 1 for the conclusion that G is false in the standard model.

c) By Logic we have $\vdash \bot \to \neg\Box G$, so D1 and D2 give us $\vdash \Box\bot \to \Box\neg\Box G$; so by choice of G, $\vdash \Box\bot \to G$. This part was worth 2 points.

d) By the Second Incompleteness Theorem, $\neg\Box\bot$ is independent of PA so its negation, $\Box\bot$ is also independent of PA. In parts b) and c) we have seen that PA $\vdash G \leftrightarrow \Box\bot$. It follows that also G is independent of PA. This part was worth 2 points.

Solution Exercise 4:

a) Take $c \in \mathcal{M}$ nonstandard. Then $(c+1)^2 = c^2 + 2c + 1 > c^2 + n$ for all standard n, so $(c+1)^2$ lies in a different copy of $\mathbb{Z}$ than the one c^2 lies in. Since the ordering of copies of $\mathbb{Z}$ is dense, there is a copy of $\mathbb{Z}$ lying in between. That copy cannot contain any squares, because the sentence $\forall x(x^2 \leq c^2 \vee (c+1)^2 \leq x^2)$ is true in $\mathcal{M}$ (it is a theorem of PA). So if a is an element of that copy, a satisfies the statement. This part was worth 4 points.

b) If $ac < b$ for some nonstandard c then certainly $an < b$ for all standard n, since $n < c$ and multiplication is monotone. For the converse, suppose $an < b$ for all standard n. Then by Overspill there must be a nonstandard element c such that $ac < b$. To spell it out: suppose $ac < b$ does not hold for any nonstandard c. Then we have $\mathcal{M} \models a0 < b$ (since b is nonstandard) and $\mathcal{M} \models \forall y(ay < b \to a(y+1) < b)$ so by Induction we would have $\mathcal{M} \models \forall y(ay < b)$ which is absurd. This part was worth 3 points.

c) Suppose a, b nonstandard and $a \ll b$. Pick (by b)) a nonstandard c such that $ac < b$. Let d be the least element such that $c \leq (d+1)^2$. This exists because the function $F(y) = \mu z < y.y \leq (z+1)^2$ is primitive recursive, hence representable in PA, hence a function in $\mathcal{M}$. Then d is nonstandard, and $d^2 < c$. Alternatively one can say: for all standard n, $\mathcal{M} \models n^2 < c$ hence by overspill there is a nonstandard d such that $d^2 < c$.

We see that $a(d-1) < ad$ so $a \ll ad$, and $(ad)d = ad^2 < ac < b$ so $ad \ll b$. We conclude that $\ll$ is dense. This part was worth 3 points.