Exam Topos Theory
January 21, 2019; 10:00–13:00
With solutions

Exercise 1. Let $\mathcal{E}$ be a topos, and $j : \Omega \to \Omega$ a Lawvere-Tierney topology in $\mathcal{E}$. By $\mathrm{Sh}_j(\mathcal{E})$ we denote the category of j-sheaves.

a) (5) Let X be a j-sheaf, and let $X \stackrel{\nu_X}{\to} \tilde{X}$ be a partial map classifier in $\mathrm{Sh}_j(\mathcal{E})$. Show that for any diagram

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "f"'] & Y \\ X & \end{tikzcd}$$

in $\mathcal{E}$ with m mono, there exists an arrow $\tilde{f} : Y \to \tilde{X}$ such that the square

$$(*) \quad \begin{tikzcd} M \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "\tilde{f}"] \\ X \arrow[r, "\nu_X"'] & \tilde{X} \end{tikzcd}$$

commutes.

b) (5) Now suppose the mono m represents a j-closed subobject of Y. Show that there is a *unique* $\tilde{f} : Y \to \tilde{X}$ making the square $(*)$ a pullback.

Solution: a): consider the sheafification functor L and the natural transformation $\eta : \mathrm{id} \Rightarrow L$. We know that the functor L preserves finite limits; in particular it preserves monos. Since X is a j-sheaf, the partial map diagram transposes to a diagram

$$\begin{tikzcd} L(M) \arrow[r, "L(m)"] \arrow[d, "\bar{f}"'] & L(Y) \\ X & \end{tikzcd}$$

which is a partial map $L(Y) \rightharpoonup X$ in $\mathrm{Sh}_j(\mathcal{E})$. By the property of the partial map classifier $\tilde{X}$, we have a unique arrow $\tilde{\bar{f}} : L(Y) \to \tilde{X}$ making the diagram

$$\begin{tikzcd} L(M) \arrow[r, "L(m)"] \arrow[d, "\bar{f}"'] & L(Y) \arrow[d, "\tilde{\bar{f}}"] \\ X \arrow[r, "\nu_X"'] & \tilde{X} \end{tikzcd}$$

a pullback. If we compose this with the naturality square

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "\eta_M"'] & Y \arrow[d, "\eta_Y"] \\ L(M) \arrow[r, "L(m)"'] & L(Y) \end{tikzcd}$$

and define $\tilde{f}$ to be the composite $\tilde{\bar{f}}\eta_Y$, we have our commuting square.

b): here one has to see that the map $M \stackrel{m}{\to} Y$ is j-closed precisely when the naturality square is a pullback. This is so because the universal closure operation corresponding to the Lawvere-Tierney topology j sends a mono $m : M \to Y$ to the pullback of $L(m)$ along η_Y. Clearly now, if m is j-closed then the construction given for part a) yields a pullback square.

For uniqueness, suppose the square $(*)$ is a pullback. Since $X \stackrel{\nu_X}{\to} \tilde{X}$ is a diagram of sheaves, $(*)$ transposes to a diagram

$$\begin{tikzcd} L(M) \arrow[r, "L(m)"] \arrow[d, "\bar{f}"'] & L(Y) \arrow[d, "\tilde{\bar{f}}"] \\ X \arrow[r, "\nu_X"'] & \tilde{X} \end{tikzcd}$$

which, modulo the isomorphisms $X \simeq L(X)$ and $\tilde{X} \simeq L(\tilde{X})$, is just the L-image of the diagram $(*)$, and hence a pullback; we see that $\tilde{\bar{f}}$ represents the partial map $\bar{f}$ in $\mathrm{Sh}_j(\mathcal{E})$, and is therefore uniquely determined by m and $\bar{f}$. Hence its transpose $\tilde{f}$ is uniquely determined by m and f.

Exercise 2. Again, $\mathcal{E}$ is a topos and j is a Lawvere-Tierney topology in $\mathcal{E}$. Let $\mathrm{Sep}_j(\mathcal{E})$ be the full subcategory of $\mathcal{E}$ on the j-separated objects, and let $M : \mathcal{E} \to \mathrm{Sep}_j(\mathcal{E})$ be left adjoint to the inclusion functor $\mathrm{Sep}_j(\mathcal{E}) \to \mathcal{E}$.

a) (4) Prove that if X is j-separated then so is X^Y, for any Y.

b) (4) Prove that the functor M preserves finite products.

c) (2) Does M preserve equalizers in general? Motivate your answer.

Solution: a): the simplest was to remark that we know this for sheaves: if X is a sheaf then X^Y is a sheaf. Now if X is j-separated, X is a subobject of a sheaf, say we have a mono $m : X \to Z$ with Z a sheaf. Then $m^Y : X^Y \to Z^Y$ is a monomorphism (since the functor $(-)^Y$, being a right adjoint, preserves monos) of X^Y into the sheaf Z^Y; so X^Y is separated.

Alternatively, suppose $M' \stackrel{m}{\to} M$ is j-dense and $M' \stackrel{f}{\to} X^Y$ is a map. By the exponential adjunction, there is a natural bijection between commutative triangles

$$\begin{tikzcd} M' \arrow[r, "m"] \arrow[d, "f"'] & M \arrow[dl, "g"] \\ X^Y & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M' \times Y \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\tilde{f}"'] & M \times Y \arrow[dl, "\tilde{g}"] \\ X & \end{tikzcd}$$

and, by stability of the closure operation, the map $m \times \mathrm{id}$ is dense if m is. Since X is separated, there is at most one $\tilde{g}$ making the right hand triangle commute. So there is at most one g making the left hand triangle commute; that is, X^Y is separated.

b): this is similar to the proof for L in the lecture notes. For the binary case one proves, for arbitrary separated objects X, that there is a natural bijective correspondence $\mathrm{Sep}_j(MY \times MZ, X) \simeq \mathrm{Sep}_j(M(Y \times Z), X)$, and applies the Yoneda lemma. It is trivial that $M(1) \simeq 1$.

c): you got full points if you remarked that if M preserved equalizers then M would preserve all finite limits, and therefore would define a subtopos of $\mathcal{E}$. Not every category of separated objects is itself a topos. For a simple example, take the poset $\mathbf{2}$, the linear order with 2 elements, and consider the $\neg\neg$-topology on $\mathrm{Set}^{\mathbf{2}}$. Note that objects of this category are arrows in Set; the category of $\neg\neg$-separated objects is the full subcategory on the injective functions. This is not a topos.

Exercise 3. Let P be the poset of finite 01-sequences ordered by extension: $\sigma \leq \tau$ if and only if σ is an initial segment of τ. We consider the toposes Set/P (the slice topos, where P is regarded as just a set) and $\widehat{P}$, the category of presheaves on the poset P.

Show that there exist both a surjection and an embedding from Set/P to $\widehat{P}$.

Solution: consider P as a category and let P_{dis} be the discrete category on the objects of P. It is an easy observation that Set/P is equivalent to $\widehat{P_{\mathrm{dis}}}$.

We know from the lectures that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ which is a surjection if F is surjective on objects, and an embedding if F is full and faithful.

So it suffices to find functors $F, G : P_{\mathrm{dis}} \to P$ such that F is surjective on objects and G is full and faithful. Note that any map from P to itself gives a functor $P_{\mathrm{dis}} \to P$.

For F we can take the identity function; this is surjective on objects (this induces the geometric morphism that most of you found). For G, we need a function from P to P such that *different* elements p and q are sent to *incomparable* elements $G(p), G(q)$. Now P is a countably infinite set; choose an enumeration $(e_n)_{n\in\mathbb{N}}$ of P. Define G such that $G(e_n)$ is the sequence $\underbrace{0\cdots0}_{n}1$. Then G is a full and faithful functor, and induces an embedding $\mathrm{Set}/P \simeq \widehat{P_{\mathrm{dis}}} \to \widehat{P}$.

Exercise 4. Recall that in a topos $\mathcal{E}$ an object X is *internally projective* if the functor $(-)^X$ preserves epimorphisms; $\mathcal{E}$ is said to *satisfy the internal axiom of choice* (IC) if every object of $\mathcal{E}$ is internally projective.

Show that the following two assertions are equivalent:

i) $\mathcal{E}$ satisfies IC and 1 is projective in $\mathcal{E}$.

ii) Every object of $\mathcal{E}$ is projective.

Solution: i)$\Rightarrow$ii) is most elegantly proven by observing that the functor $\mathcal{E}(X,-)$ is isomorphic to $\mathcal{E}(1,(-)^X)$, which is the composition of $(-)^X : \mathcal{E} \to \mathcal{E}$ and $\mathcal{E}(1,-) : \mathcal{E} \to \mathrm{Set}$. The first of these preserves epis because X is internally projective, and the second one does because 1 is projective. Hence $\mathcal{E}(X,-)$ preserves epis; that is, X is projective.
ii)$\Rightarrow$i): Let $f : A \to B$ be epi. Then f is split epi by ii), and split epis are preserved by any functor, in particular by the functor $(-)^X$. So X is internally projective.