Electronic Journal of Qualitative Theory of Differential Equations
20XX, No. **XX**, 1–38; https://doi.org/10.14232/ejqtde.20XX.1.XX www.math.u-szeged.hu/ejqtde/

S-shaped bifurcations in a two-dimensional Hamiltonian system

André Zegeling✉[1] and **Paul Andries Zegeling**[2]

[1]Guilin University of Aerospace Technology, Jinji Road 2, Guilin, China
[2]Utrecht University, Department of Mathematics, Budapestlaan 6, De Uithof, the Netherlands

Received 3 December 2020, appeared XX June 2021
Communicated by Hans-Otto Walther

Abstract. We study the solutions to the following Dirichlet boundary problem:

$$\frac{d^2x(t)}{dt^2} + \lambda f(x(t)) = 0,$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, with boundary conditions:

$$x(0) = x(1) = A \in \mathbb{R}.$$

Especially we focus on varying the parameters λ and A in the case where the phase plane representation of the equation contains a saddle loop filled with a period annulus surrounding a center.

We introduce the concept of mixed solutions which take on values above and below $x = A$, generalizing the concept of the well-studied positive solutions.

This leads to a generalization of the so-called period function for a period annulus. We derive expansions of these functions and formulas for the derivatives of these generalized period functions.

The main result is that under generic conditions on $f(x)$ so-called S-shaped bifurcations of mixed solutions occur.

As a consequence there exists an open interval for sufficiently small A for which λ can be found such that three solutions of the same mixed type exist.

We show how these concepts relate to the simplest possible case $f(x) = x(x+1)$ where despite its simple form difficult open problems remain.

Keywords: ordinary differential equations, boundary value problem, period function.

2020 Mathematics Subject Classification: 34B15, 34C08, 34C23, 37C10.

1 Introduction

We study the existence and bifurcation of solutions to a Dirichlet boundary problem:

$$\frac{d^2x(t)}{dt^2} + \lambda f(x(t)) = 0, \tag{1.1}$$

✉Corresponding author. Email: zegela1@yahoo.com

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, with boundary conditions:

$$x(0) = x(1) = A \in \mathbb{R}. \tag{1.2}$$

The differential equation can be interpreted as the scalar motion of a particle in a conservative potential field depending on its position only. The boundary condition implies that a particle returns to its initial position after one second.

A possible interpretation of this problem is to find the initial speed $\frac{dx(t)}{dt}|_{t=0}$ such that the solution with initial conditions $x(0) = A$, $\frac{dx(t)}{dt}|_{t=0}$ returns to $x = A$ after one second.

In [5] Chicone studied a similar problem for Neumann and Dirichlet boundary problems. In this paper we generalize his analysis. The different types of mixed solutions which we study in this paper were not considered there. It turns out that these mixed solutions lead to a richer and more complex solution structure than the cases studied in [5].

Conditions on $f(x)$

The function $f(x)$ is taken to be *real analytic*. For some results this condition could be weakened but for the clarity of reading we will assume that $f(x)$ is real analytic in all the cases of this paper.

In particular we will consider the case where the corresponding system in the phase plane has a center at the origin:

$$f(0) = 0, \qquad f'(0) > 0. \tag{1.3}$$

This will ensure that a continuum of periodic orbits exists, i.e. a period annulus surrounding a singularity of center type.

Furthermore to obtain global results we will typically impose that the corresponding system in the phase plane has a saddle for $x = x_s$:

$$f(x_s) = 0, \qquad f'(x_s) < 0. \tag{1.4}$$

Finally we impose that outside these two singularities the following relation holds:

$$x(x - x_s)f(x) > 0, \qquad x \neq 0,\ x \neq x_s. \tag{1.5}$$

which ensures that no other singularities exist.

Conservative forces and applications

Boundary problems of the type (1.1), (1.2) have been studied extensively in the literature. Typical choices for the conservative force $f(x)$ are e^x (see [2, 10]), $(x - a)(x - b)(x - c)$ (see [14, 27, 29]), $e^{\frac{x}{1+\epsilon x}}$ (see [11, 30]), $\sum_{k=0}^{k=n} \frac{x^k}{k!}$ (see [32]), convex $f(x)$ (see [19, 20]), quadratic $f(x)$ (see [4]). Applications of the BVP typically appear in the steady-state solutions of diffusion equations, see [15] and [16] for an extensive discussion. Other examples of applications can be found in the theory of combustion, see e.g. [3, 11, 30]. For other interesting flavours of boundary value problems, see [1] and [25], where a constant damping term $c\frac{dx(t)}{dt}$ was added to the equation, and [9], where another type of damping was introduced. These cases with damping are out of scope for this paper and require a different kind of analysis, since in general no first integral of the differential equation is known. The analysis of the systems with constant damping can be related to the study of limit cycles in so-called Liénard systems after a Filippov transformation. The discussion of this relation is outside the scope of this paper.

Positive and negative solutions

In most of the papers on this subject $A = 0$ and only so-called positive solutions are studied, where $x(t) > 0$ for $0 < t < 1$. This generalizes to our formulation as the requirement that $x(t) > A$ for $0 < t < 1$: the solution does not return to its initial value before $t = 1$. We will refer to this as a *positive* solution to be consistent with the literature. Similarly a *negative* solution can be defined as a solution for which $x(t) < A$ for $0 < t < 1$.

In the study of positive solutions many deep results have been proved in recent years. In particular we refer to the papers [11,12,14], where upper bounds were found for the number of solutions to the boundary value problems for general classes of potential functions.

In the phase plane (x, y), where $y = \frac{dx(t)}{dt}$, positive (negative) solutions are identified by the property that the solution curve stays to the right (left) of the vertical line $x = A$ before returning to $x = A$.

Another important property of these types of solutions is that $\frac{dx(t)}{dt}|_{t=0} > 0$ (< 0) for positive (negative) solutions. Positive (negative) solutions necessarily start at a point $x = A$, $y = y_0 > 0$ $(y = y_0 < 0)$ in the phase plane.

The study of negative solutions is essentially the same as for positive solutions. Similar techniques can be applied. In this paper we typically prove results for the positive case and state the results for the negative cases if needed without giving the detailed proofs.

Periodic orbits and mixed solutions

The main novelty of the research presented in this paper is the study of mixed solutions, crossing the line $x = A$ in the phase plane before their final return to $x = A$. Formally a mixed solution is a solution such that $\exists \bar{t} \in (0, 1)$ with $x(\bar{t}) = A$, i.e. the solution will return at least once to the initial value $x(0) = A$ before $t = 1$. It implies that there exist values $t_1, t_2 \in (0, 1)$ such that $x(t_1) < A$ and $x(t_2) > A$, hence the terminology *mixed solution*. A necessary condition for this situation to be possible is that the solution lies on a periodic orbit in the phase plane of (1.1). In systems of the type (1.1), because of its conservative nature, no isolated period orbits (limit cycles) can occur and therefore necessarily we are looking at systems which have a continuum of period orbits, a so-called period annulus.

In the fundamental paper on this subject [27] mixed solutions were studied for the case $f(x) = (x - a)(x - b)(x - c)$. The case of mixed solutions has not received much attention in the literature since and we will show that new complex phenomena may occur even for the simplest cases of $f(x)$. In particular we will argue that the argument in [27] where it was stated that for sufficiently large λ no bifurcation values will occur is not necessarily true in general. Even for the simple quadratic case $f(x) = x(x + 1)$ there are values of A such that bifurcations exist no matter how large λ is chosen.

Time-to-return functions

Our approach will be to study the problem by a simple rescaling of the time parameter after which we can continue the analysis by studying the time-to-return functions of system (1.1) with $\lambda = 1$. These are functions depending on the integration constant (or energy level in terms of the mechanical interpretation of the system) representing the time it takes to return to the vertical line $x = A$ in the phase plane. Returning to the initial x-coordinate can be done in many different ways when the orbit in the phase plane is a closed curve representing a periodic solution. Part of the purpose of this paper is to categorize these different return mechanisms and to analyze the corresponding time-to-return functions.

S-shaped bifurcations

In the literature one particular bifurcation phenomenon was observed for this type of boundary value problem: the occurrence of S-shaped bifurcations for positive solutions, see [11, 13, 30]. Essentially this corresponds to the existence of two different critical λ values where solutions to the equations bifurcate under a change of λ, while there exist λ-values for which three solutions occur. We will show in this paper that S-shaped bifurcations occur for mixed solutions under generic conditions on the function $f(x)$, if the phase plane contains a period annulus which is bounded on the outside by solution containing a saddle singularity (i.e. a saddle loop) and on the inside by a singularity of center type.

Quadratic Hamiltonian

As illustration of the results for the general case we consider the simplest example by taking $f(x) = x(x+1)$. For this quadratic Hamiltonian system several results have been obtained in the past. It is well-known that for the case of positive and negative solutions at most two solutions can occur for given λ, see [4, 19, 20]. The full period function is monotonic (see e.g. [8]). The case of mixed solutions leads to more complicated situations. It will be shown that for the mixed solution types with $f(x) = x(1+x)$, there exist λ-values for which at least three mixed solutions occur and that S-shaped bifurcations occur.

Period functions

The problems addressed in this paper can be viewed as a generalization of the work on the so-called period function of a period annulus. There is a rich literature on this subject (see for example the pioneering work of [6] in the field of so-called quadratic systems and more recent work in [21–24, 31]). In a sense, problems related to the period function can be interpreted as a subset of the problems presented in this paper. We will show that in a generic setting at least two local extreme values of the time-to-return functions can occur in the case of a mixed solution, showing the increased complexity compared to the study of the period function.

Results

The main results of this paper are:

- a full classification of the solution types of system (1.1) with boundary conditions (1.2);
- analytical expressions for the corresponding time-to-return functions for each solution type and their expansions near the center singularity;
- a new recursive formula for the derivatives of the full period function;
- existence of an S-shaped bifurcation phenomenon for systems with a generic form $f(x)$ under the condition that $f''(0) \neq 0$ and that a period annulus exists with a center and saddle loop on its boundaries;
- finiteness of the number of solutions for each mixed solution type for a generic class of $f(x)$.

2 Time-to-return functions

It is more convenient to study the boundary value problem (1.1), (1.2) in its equivalent form in the phase plane, through the introduction of the auxiliary variable $y(t) \equiv \frac{dx(t)}{dt}$:

$$\begin{aligned} \frac{dx(t)}{dt} &= y(t), \\ \frac{dy(t)}{dt} &= -\lambda f(x(t)), \end{aligned} \tag{2.1}$$

$$x(0) = x(1) = A. \tag{2.2}$$

A simple scaling of the variables changes the boundary value problem (2.1), (2.2) into a more tractable and traditional form, where a straightforward time-traversal can be studied for all solutions. This is a well-known procedure, see e.g. [19].

Introducing new variables $t = \frac{\bar{t}}{\sqrt{\lambda}}$, $y(t) = \bar{y}(t)\sqrt{\lambda}$, the boundary problem (2.1), (2.2) becomes:

$$\begin{aligned} \frac{dx(\bar{t})}{d\bar{t}} &= \bar{y}(\bar{t}), \\ \frac{d\bar{y}(\bar{t})}{d\bar{t}} &= -f(x(\bar{t})), \end{aligned} \tag{2.3}$$

with boundary conditions

$$x(\bar{t} = 0) = x(\bar{t} = \sqrt{\lambda}) = A. \tag{2.4}$$

i.e. the dependency on λ has been removed from the system of differential equations and has been put into the boundary condition. In the following we will focus on this system and drop the bars for notational convenience. We will refer to trajectories of (2.3) in the (x, y) phase plane as *orbits* while we will refer to those trajectories satisfying not only (2.3), but the additional boundary condition (2.4) as well, as *solutions*. So the set of solutions to (2.3), (2.4) is contained in the set of orbits defined by (2.3) but not every orbit in the phase plane will necessarily correspond to a solution.

2.1 Reformulating the original boundary value problem

In order to find solutions to the original boundary problem (1.1), (1.2), according to (2.4) we need to find the time it takes an orbit of (2.3) starting at the line $x = A$ in the phase plane to reach the same line $x = A$ again: for given λ those orbits of (2.3) returning to the original vertical line $x = A$ in $\sqrt{\lambda}$-time correspond to solutions of the original boundary problem (1.1), (1.2). Depending on the nature of the solution curves in the phase plane, there is not necessarily a unique way (if any) to achieve this. If the solution curve returns to $x = A$, then we refer to the time it takes to traverse back to its original x-value as the *time-to-return function*. Typically for periodic orbits there will not be a unique way to return to the original x-value and therefore we will have to consider multiple time-to-return functions, each distinguished by the way the solution returns to $x = A$.

The terminology *function* is used here to indicate that the time it takes to return to the original x-value is a function of the initial starting point in the phase plane, i.e. depending on the initial velocity (the initial y-value in the phase plane, i.e. $\frac{dx(t)}{dt}|_{t=0}$ in system (2.3)).

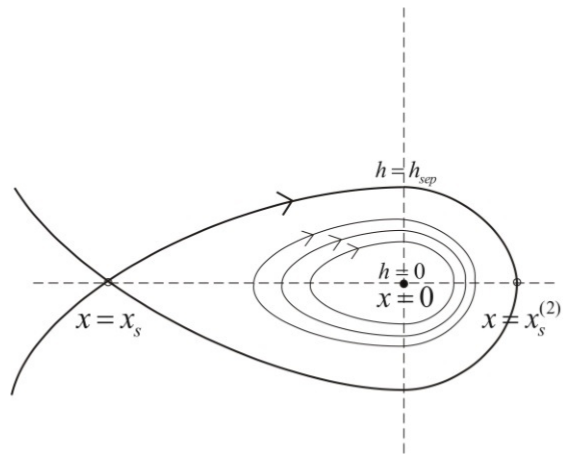

Figure 2.1: Phase portrait for system (2.3) with conditions (1.3), (1.4) and (1.5) on $f(x)$.

2.2 Phase plane interpretation of the Hamiltonian system

The orbits of the solutions of (2.3) in the phase plane can be written down explicitly:

$$h = \frac{1}{2}y^2(t) + F(x(t)), \tag{2.5}$$

where

$$F(u) \equiv \int_0^u f(x)dx.$$

Each h corresponds to an integral curve in the phase plane. We will assume that conditions (1.3), (1.4) and (1.5) hold. This implies that the phase portrait of the system contains two singularities: a saddle at $(x = x_s,\ y = 0)$ and a center at $(x = 0,\ y = 0)$. Through a change of variables $x \to -x$ (if necessary) the saddle can be positioned to the left of the center, i.e. $x_s < 0$, which we will assume to hold true in the following for convenience of discussion.

The integration constant $h \equiv h_{sep} = F(x_s)$ corresponds to a saddle loop, passing through the saddle, see Figure 2.1. The integration constant $h = 0$ corresponds to the center point. For the values $0 < h < h_{sep}$ the region between center and saddle loop is filled with closed orbits corresponding to periodic solutions, i.e. each h in this interval corresponds to one closed orbit, which is symmetrical with respect to the x-axis as the integral formula (2.5) shows. The time it takes to traverse a solution in the region $y > 0$ is the same as it takes to traverse the reflected path for $y < 0$. Therefore when we consider traversal times along orbits we can always restrict our attention to the part of the curve lying in $y > 0$.

The saddle loop intersects the x-axis in two points: through the saddle itself located at $x = x_s$ and at the regular point $x = x_s^{(2)} > 0$ as long as $\exists x_s^{(2)} > 0$ such that $F(x_s) = F(x_s^{(2)})$. We will assume that such a point exists, i.e. that the original system has a saddle loop. The arguments of this paper generalize to the situation where $x_s^{(2)} \to \infty$ but for notational convenience we will omit this case here.

Since we are interested in the behaviour of the solutions to (1.1), (1.2) related to the periodic orbits, we restrict the value of A to the interval $x_s < A < x_s^{(2)}$. For values of A outside this interval no periodic orbits can reach the vertical line $x = A$ in the phase plane.

The set of periodic orbits for $h \in (0, h_{sep})$ is referred to as a *period annulus* in the literature. The closed orbit representing a periodic orbit in the phase plane is denoted in the following

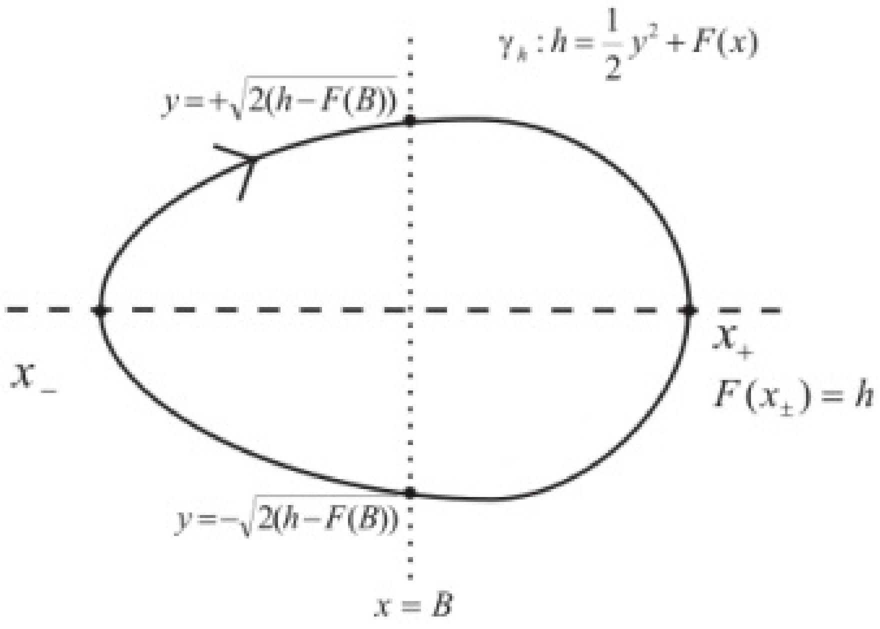

Figure 2.2: Properties of a periodic orbit of (2.3).

by γ_h.

Well-known properties of γ_h are:

- The orbit γ_h satisfies (2.5) for some integration constant $h \in (0, h_{sep})$.
- The periodic orbit γ_h is symmetrical with respect to the x-axis. The time it takes to traverse the periodic orbit for $y > 0$ is the same as for $y < 0$.
- For each $h \in (0, h_{sep})$ γ_h crosses the x-axis in exactly two points, of which the coordinates $x_-(h) < 0$ and $x_+(h) > 0$ satisfy $F(x_\pm(h)) = h$.
- A periodic orbit γ_h intersecting a line $x = B$ will do so in exactly two points $(x = B, y = \sqrt{2(h - F(B))})$, $(x = B, y = -\sqrt{2(h - F(B))})$, except when $x = B$ coincides with the crossing of the x-axis by γ_h at $x = x_-(h)$ or $x = x_+(h)$. In those latter cases there is only one intersection point: the vertical line $x = B$ is tangent to γ_h at the crossing of the x-axis at $(x_\pm(h) = B, 0)$.

These properties are summarized in Figure 2.2.

3 Categorization of solution types

3.1 Types of solutions

With the results from the previous section in mind we can categorize the different ways in which a solution to (2.3), (2.4) can start and end on the vertical line $x = A$. In Figure 3.1 the full list of possible solution types are displayed. Assume that the line $x = A$ intersects the period

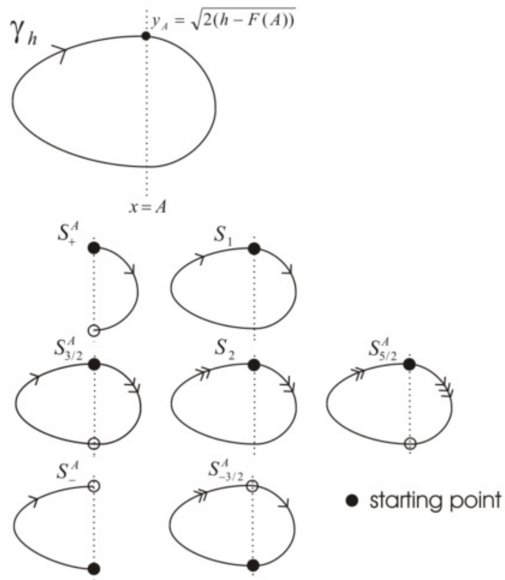

Figure 3.1: Solution types for the boundary value problem on a periodic orbit of (1.1).

orbit γ_h in two points $(x = A, y = y_A \equiv \sqrt{2(h - F(A))})$, $(x = A, y = -y_A \equiv -\sqrt{2(h - F(A))})$, see Figure 2.2. If there is no such intersection, then the orbit γ_h cannot generate solutions to (2.3), (2.4).

Positive solutions. First we discuss the case of starting at $(x = A, y = y_A > 0)$, i.e. above the x-axis. The solution starts at $(x = A, y = y_A)$ on the periodic orbit. It will cross the x-axis at $(x = x_+(h), 0)$ and return to $x = A$ for the first time by reaching the reflected point $(x = A, y = -y_A)$. We denote this part of γ_h by $S_+^A(h)$. We call it the positive part of the periodic orbit because all x-values are larger than A in correspondence with the notation in the literature. The time to reach this first point of return we refer to as $T_+^A(h)$.

Negative solutions. These solutions have the same properties as the positive solutions except that the solutions have to stay on the left of the line $x = A$. It translates into a starting point $(x = A, y = y_A < 0)$, i.e. below the x-axis, with the solution returning to its reflected point above the x-axis. We denote this part of γ_h by $S_-^A(h)$ and the time to reach the other side by $T_-^A(h)$.

Full solutions. A full solution returns to its original starting point $(x = A, y = y_A)$, i.e. a full period rotation has been made in the phase plane. We denote the time to make a full rotation by $T_{full}(h)$ (the period of γ_h) and the trajectory itself by $S_n(h)$, where $n = 1, 2, ..$ indicates the number of full rotations that were made. The corresponding time-to-return function is written as $T_n(h) \equiv nT_{full}(h)$. The function $T_{full}(h)$ is what in the literature is referred to as the so-called period function. Clearly from the definition $S_1(h) = S_+^A(h) \oplus S_-^A(h)$ and $T_{full}(h) = T_+^A(h) + T_-^A(h)$.

Mixed solutions. The argument can be continued by considering a positive solution starting at $(x = A, y = y_A > 0)$, returning to $(x = A, y = -y_A < 0)$ and then making one full rotation. This orbit type is a combination of a partial rotation $S_+^A(h)$ followed by a full rotation along $S_{full}(h)$. For notational convenience we label this trajectory by $S_{3/2}^A(h)$ to indicate that it is a union of the two trajectories $S_1(h)$ and $S_+^A(h)$. It is important to note that this trajectory contains parts where $x < A$ and $x > A$ before returning. Therefore we refer to this type of solution as a mixed solution. The full rotations are mixed as well, but these solutions we will keep referring to as $S_n(h)$.

Similarly we can define mixed solution types that start below the x-axis. For example starting at $(x = A, y = y_A < 0)$, a partial trajectory is followed by one full rotation. This is denoted by $S_{-3/2}^A(h)$.

In this way we find a countably infinite number of ways of returning to the line $x = A$, starting at $(x = A, y = y_A)$ above and below the x-axis. In Figure 3.1 the different solution types are indicated with the corresponding trajectories on the periodic orbit. We summarize the possibilities as follows (the dependency on the parameter h was dropped for convenience of reading):

- S_+^A: one partial rotation from $y > 0$ to $y < 0$, ending at the reflection in the x-axis of the starting point.
- $S_{full} \equiv S_1$: one full rotation on the period orbit returning to its original point.
- $S_{3/2}^A = S_+^A \oplus S_{full}$.
- $S_2 = S_{full}^A \oplus S_{full}$.
- $S_{5/2}^A = S_+^A \oplus S_{full} \oplus S_{full}$.
- ...
- S_-^A, similar to S_+^A but starting at $y < 0$ and ending at $y > 0$.
- $S_{-3/2}^A = S_-^A \oplus S_{full}$.
- $S_{-5/2}^A = S_-^A \oplus S_{full} \oplus S_{full}$.

The full set of solutions can be categorized by the following types:

Proposition 3.1. *Solutions to the boundary value problem* (2.3) *and* (2.4) *corresponding to a given period orbit* γ_h*, where* h *is the integration constant in* (2.5)*, can be categorized by:*

- *Positive solution:* S_+^A
- *Full solutions:* S_n*, where* $n = 1, 2, 3, \ldots$
- *Mixed solutions:* $S_{n+1/2}^A$, $S_{-n-1/2}^A$*, where* $n = 1, 2, 3, \ldots$

Remark 3.2. In the proposition we grouped all full period solutions under the same label as a full solution. For all these cases the time-to-return function to the starting point in the phase plane does not depend on A. The behaviour of the solutions solely depends on the structure of the period function of the period annulus.

Each of the solution types in Proposition 3.1 is characterized by the number of times it crosses the x-axis in the phase plane and where it crosses the x-axis, i.e. for $x < A$ or $x > A$. The way to choose the solution types was chosen to have an easy reference to these crossings. In terms of the original boundary problem (1.1) and (1.2) a crossing of the x-axis corresponds with a local minimum ($x < A$) or local maximum ($x > A$) of the solution as a function of

t. This is due to the interpretation of the variable y in the phase plane as $\frac{dx}{dt}$. Therefore the number of x-axis crossings equals the number of local extrema of the original solution. Obviously an increase in rotations along the period orbit γ_h in the phase plane increases the number of local extrema (i.e. each full rotation adds a local maximum and local minimum). The conclusion is:

Proposition 3.3. *According to the categorization of solutions in Proposition 3.1 to the original boundary value problem (1.1) and (1.2) each type of solution is characterized by the number of crossings of the x-axis by the periodic orbit γ_h in the phase plane of system (2.3):*

- *Positive solution: S_+^A: one local maximum*
- *Negative solution: S_-^A: one local minimum*
- *Full solutions: S_n, where $n = 1, 2, 3, \ldots$: n local minima and n local maxima.*
- *Mixed solutions: $S^A_{n+1/2}$, where $n = 1, 2, 3, \ldots$: $2n + 1$ local extreme points, $n + 1$ local maxima and n local minima, the first local extreme point being a local maximum.*
- *Mixed solutions: $S^A_{-n-1/2}$, where $n = 1, 2, 3, \ldots$: $2n + 1$ local extreme points, n local maxima and $n + 1$ local minima, the first local extreme point being a local minimum.*

3.2 Time-to-return functions for the different types of solutions

To each of the solution types as described in Proposition 3.1 we can associate the time it takes to follow the trajectory from start to end point. As noted before, the corresponding trajectories reflected in the x-axis are traversed in the same time span. The time-to-return functions can be written as a linear combination of three fundamental functions:

Lemma 3.4. *The time-to-return function for a positive solution of the type S_+^A is given by:*

$$T_+^A(h) = 2\int_{x=A}^{x=x_+(h)} \frac{dx}{y_h(x)}, \tag{3.1}$$

where $y_h(x) \equiv \sqrt{2(h - F(x))}$ and $F(x_+(h)) = h$, $x_+(h) > A$.

The time-to-return function for a negative solution of the type S_-^A is given by:

$$T_-^A(h) = 2\int_{x=x_-(h)}^{x=A} \frac{dx}{y_h(x)}, \tag{3.2}$$

where $F(x_-(h)) = h$, $x_-(h) < A$.

The time-to-return function for a full solution of the type S_1 is given by:

$$T_{full}(h) = 2\int_{x=x_-(h)}^{x_+(h)} \frac{dx}{y_h(x)}. \tag{3.3}$$

Proof. Consider the case of a positive solution S_+^A, i.e. $x(t) > A$. The solution starts at $x = A$, crosses the x-axis at $(x_+(h), 0)$ and then returns to $x = A$ along a trajectory which is the reflection of the trajectory above the x-axis. The time it takes to traverse the trajectory above the x-axis is the same as the time it takes to traverse the trajectory below the x-axis. Therefore the total return time is twice the time it takes to reach $(x_+(h), 0)$. The formula in the lemma follows by using the relation $\frac{dx(t)}{dt} = y_h(x(t))$ which implies $t_1 - t_0 = \int_{x(t_0)}^{x(t_1)} \frac{dx}{y_h(x)}$, where we defined $y_h(x) = \sqrt{2(h - F(x))}$ for trajectories above the x-axis. Here $x(t_0) = A$ and $x(t_1) = x_+(h)$.

The proof for the negative solution follows the same arguments.

The formula for the full period is well-known in the literature (see e.g. [4]) . □

Remark 3.5. The three functions are related by the obvious relation $T_{full}(h) = T_+^A(h) + T_-^A(h)$.

As a direct consequence of the previous lemma and the solution structure as given in Proposition 3.1, we can write down the time-to-return functions for all solution types:

Proposition 3.6. *The time it takes to traverse the trajectories as defined by Proposition 3.1 of mixed type can be expressed in terms of the three fundamental time-to-return functions of Lemma 3.4 in the following way:*

$$T_{n+1/2}^A(h) = T_+^A(h) + nT_{full}(h), \tag{3.4}$$
$$T_{-n-1/2}^A(h) = T_-^A(h) + nT_{full}(h), \tag{3.5}$$

where $n = 1, 2, \ldots$

Remark 3.7. Obviously $T_{1/2}^A(h) < T_{full}(h)$, so there is a natural ordering of the values in the proposition:

$$T_{1/2}^A(h) < T_1(h) < T_{3/2}^A(h) < T_2(h) < \cdots$$
$$T_{-1/2}^A(h) < T_1(h) < T_{-3/2}^A(h) < T_2(h) < \cdots$$

Due to the symmetry in the formulas for the negative and positive time-to-return functions, we will focus on the functions $T_+^A(h)$, $T_n^A(h)$ and $T_{n+1/2}^A(h)$ in this paper. The results for the other two types $T_-^A(h)$, $T_{-n-1/2}^A(h)$ can be derived in a similar way and will differ only by introduction of some additional minus signs in the expressions. The simplest way to achieve this is by changing $x \to -x$ in (2.1), essentially changing $f(x)$ into $f(-x)$. Application of the formulas for $T_+^A(h)$ and $T_{n+1/2}^A(h)$ to the new system leads to the formulas for $T_-^A(h)$ and $T_{-n-1/2}^A(h)$ in the original system.

4 Positive solutions

4.1 Expansion of the positive time-to-return function for small h and $A = 0$

Proposition 4.1. *If $f(x)$ is real analytic and condition (1.3) holds (i.e. a center exists at the origin of the phase plane), then the positive time-to-return function (3.1) for $A = 0$ can be expanded for small h as:*

$$T_+^0(h) = d_0 + d_1 h^{\frac{1}{2}} + d_2 h + d_3 h^{\frac{3}{2}} + d_4 h^2 + \ldots \tag{4.1}$$

The first two terms explicitly take the following form:

$$T_+^0(h) = \frac{\pi}{\sqrt{a_0}} - \frac{2a_1}{a_0^2}\sqrt{h} + \ldots \tag{4.2}$$

where a_i are the coefficients of the expansion of the potential function: $F(x) = x^2(a_0 + a_1 x + a_2 x^2 + \ldots)$ near $x = 0$.

Proof. We write $T_+^0(h) = 2\int_0^{x_+(h)} \frac{dx}{y(x,h)}$ in the following convenient way (introduced in [27]):

$$T_+^0(h) = \tilde{T}_+^0(x_+) = \sqrt{2}\int_{x=0}^{x=x_+} \frac{dx}{\sqrt{F(x_+) - F(x)}}.$$

For convenience of reading (and because in the literature such a variable is used) we will write $x_+ \equiv \alpha$.

With this notation we can rewrite the integral using a scaling of the integration variable $x = \alpha u$. The integral becomes:

$$\tilde{T}_+^0(\alpha) = \sqrt{2}\int_{u=0}^{u=1} \frac{\alpha du}{\sqrt{F(\alpha) - F(\alpha u)}}. \tag{4.3}$$

By assumption we know that $F(x)$ has an expansion of the form $F(x) = x^2(a_0 + a_1 x + a_2 x^2 + \dots)$ Substitution of this expansion into the integral leads to:

$$\tilde{T}_+^0(\alpha) = \sqrt{2}\int_{u=0}^{u=1} \frac{du}{R(u,\alpha)\sqrt{1-u}},$$

where $R(u,\alpha) = \sqrt{Z_0(u) + Z_1(u)\alpha + Z_2(u)\alpha^2 + \dots}$, $Z_0(u) = a_0(1+u)$, $Z_1(u) = a_1(1+u+u^2)$, $\dots$, $Z_i(u) = a_i(1+u+u^2+\dots+u^i)$.

The function $\frac{1}{R(u,\alpha)}$ is analytical on the interval of integration, because the function $F(x)$ does not have any other zeroes on the interval of integration (we consider only x-values close to the isolated zero at $x = 0$), i.e. $R(u,\alpha) \neq 0$ for $0 \leq u \leq 1$.

It leads to the following expansion in α:

$$\tilde{T}_+^0(\alpha) = C_0 + C_1\alpha + C_2\alpha^2 + \dots \tag{4.4}$$

where

$$C_0 = \sqrt{2}\int_0^1 \frac{du}{\sqrt{Z_0(u)}\sqrt{1-u}} = \frac{\pi}{\sqrt{2a_0}},$$

$$C_1 = \sqrt{2}\int_0^1 \frac{-Z_1(u)du}{2Z_0(u)^{\frac{3}{2}}\sqrt{1-u}} = -\frac{\sqrt{2}a_1}{a_0^{\frac{3}{2}}},$$

$$C_2 = \sqrt{2}\int_0^1 \frac{(3Z_1(u)^2 - 4Z_0(u)Z_2(u))du}{8Z_0(u)^{\frac{5}{2}}\sqrt{1-u}} = \frac{\sqrt{2}(15\pi a_1^2 - 12\pi a_0 a_2 - 16a_1^2)}{8a_0^{\frac{5}{2}}}.$$

In order to find the expansion of the positive time-to-return function in terms of h we need to find the relation between h and α for small h. After substitution of the above expansion for $F(x)$ into the relation $\sqrt{F(\alpha)} = \sqrt{h}$ we get:

$$\alpha\sqrt{a_0 + a_1\alpha + a_2\alpha^2 + \dots} = \sqrt{h}.$$

Note that the term $\mu(\alpha) \equiv \sqrt{a_0 + a_1\alpha + a_2\alpha^2 + \dots}$ is analytical in α and $\mu(0) \neq 0$. It means that we can apply the Lagrange inversion theorem to get:

$$\alpha(u) = \sum_{n=1}^{\infty} g_n \frac{u^n}{n!}, \tag{4.5}$$

where $u = \sqrt{h}$ and

$$g_n = \lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}}\left[\frac{1}{\mu^n(z)}\right].$$

It follows that the first coefficients of the expansion are:

$$g_1 = \lim_{z\to 0} \frac{1}{\mu(z)} = \frac{1}{\sqrt{a_0}},$$

$$g_2 = \lim_{z\to 0} \frac{d}{dz}\left[\frac{1}{\mu^2(z)}\right] = -\frac{a_1}{a_0^2},$$

$$g_3 = \lim_{z\to 0} \frac{d^2}{dz^2}\left[\frac{1}{\mu^3(z)}\right] = \frac{15a_1^2 - 12a_0a_2}{4a_0^{\frac{7}{2}}}.$$

Substitution into (4.4) gives the expansion of the positive time-to-return function in terms of h.

$$T_+^0(h) = C_0 + C_1\alpha(h) + C_2\alpha(h)^2 + \dots = \frac{\pi}{\sqrt{a_0}} - \frac{2a_1}{a_0^2}\sqrt{h} + \dots \tag{4.6}$$

□

4.2 Derivative of the positive time-to-return function for $A \neq 0$

Lemma 4.2. *For $A \neq 0$ the derivative of the time-to-return functions $T_+^A(h)$ (3.1) with respect to h is given by the following equivalent expressions:*

$$h\frac{dT_+^A(h)}{dh} = \frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}} + \int_{y=0}^{y=\sqrt{h-h_A}} \omega'(x(y))dy, \tag{4.7}$$

where $h_A = F(A)$, $\omega(u) \equiv \frac{F(u)}{f(u)^2}$, $\omega'(u) = \frac{d\omega(u)}{du}$, $x(y)$ satisfies $h = \frac{1}{2}y^2 + F(x(y))$, $y > 0$.

$$\frac{dT_+^A(h)}{dh} = \int_{\bar{A}=0}^{\bar{A}=A} \frac{1}{\sqrt{2}(h-h_{\bar{A}})^{\frac{3}{2}}} d\bar{A} + \frac{dT_+^0(h)}{dh}. \tag{4.8}$$

Proof. Multiply the expression for $T_+^A(h)$ as given in (3.1) by h and use that $h = \frac{1}{2}y^2 + F(x)$ to write it in the form:

$$hT_+^A(h) = 2\int_{x=A}^{x=x_+(h)} \left(\frac{1}{2}y_h(x) + \frac{F(x)}{y_h(x)}\right) dx.$$

Integration by parts using $\frac{dy_h(x)}{dx} = \frac{-f(x)}{y_h(x)}$ leads to:

$$hT_+^A(h) = 2\left[\frac{h_A y_h(A)}{f(A)} + \int_{x=A}^{x=x_+(h)} \left(\frac{1}{2} + \left(\frac{F(x)}{f(x)}'\right)\right) y_h(x)dx\right],$$

where $y_h(A) = \sqrt{2(h-h_A)}$.

Taking the derivative of this expression with respect to h leads to:

$$h\frac{dT_+^A(h)}{dh} = \frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}} + 2\left[\int_{x=A}^{x=x_+(h)} \left(-\frac{1}{2} + \left(\frac{F(x)}{f(x)}'\right)\right) \frac{1}{y_h(x)}dx\right],$$

where the relation $\frac{\partial y_h(x)}{\partial h} = \frac{1}{y_h(x)}$ was used. Next we write $-\frac{1}{2} + \left(\frac{F(x)}{f(x)}\right)' = \frac{1}{2}f(x)\omega'(x)$ and change the integration variable from x to y to obtain the first equation of the lemma:

$$\begin{aligned} h\frac{dT_+^A(h)}{dh} &= \frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}} + \int_{x=A}^{x=x_+(h)} f(x)\omega'(x)\frac{1}{y_h(x)}dx \\ &= \frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}} - \int_{y=\sqrt{h-h_A}}^{y=0} \omega'(x)dy, \end{aligned}$$

where the last step uses the fact that $y(x)$ satisfies the differential equation $\frac{dy}{dx} = -\frac{f(x)}{y}$.

The first step in proving the second equation (4.8) in the lemma is to differentiate the expression for $T_+^A(h)$ with respect to A:

$$\frac{\partial T_+^A(h)}{\partial A} = -\frac{\sqrt{2}}{\sqrt{h - h_A}}. \tag{4.9}$$

Differentiating this expression with respect to h gives:

$$\frac{\partial^2 T_+^A(h)}{\partial h\, \partial A} = \frac{1}{\sqrt{2}(h - h_A)^{\frac{3}{2}}}.$$

The second equation (4.8) in the lemma then follows by integration over the variable A with the notation that $\frac{dT_+^0(h)}{dh}$ represents the derivative of the positive time-to-return function for $A = 0$. $\square$

4.3 Limits of the positive time-to-return function

This section contains the limits of the positive time-to-return function $T_+^A(h)$ near the boundary of its definition, i.e. $h = 0$, the center, and $h = h_{sep}$, the saddle loop.

Proposition 4.3. *The behaviour near $h = h_A$ of the positive time-to-return function $T_+^A(h)$ in (3.1) and its derivative, defined on $h \in (h_A, h_{sep})$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) \neq 0$ is as follows:*
For $A < 0$:

$$\lim_{h \downarrow h_A} T_+^A(h) = T_{full}^A > 0. \tag{4.10}$$

For $A = 0$:

$$\lim_{h \downarrow 0} T_+^0(h) = \frac{1}{2} T_0 > 0. \tag{4.11}$$

For $A > 0$:

$$\lim_{h \downarrow h_A} T_+^A(h) = 0, \tag{4.12}$$

where the period of the periodic orbit γ_{h_A} is abbreviated as T_{full}^A and is given by the expression:

$$T_{full}^A \equiv 2 \int_{x=x_-(h_A)}^{x=x_+(h_A)} \frac{dx}{y_{h_A}(x)},$$

where $h_A \equiv F(A)$. The orbit γ_{h_A} is the periodic orbit tangent to the vertical line $x = A$, passing through the point $(x = A, y = 0)$ in the phase plane.

The limiting value T_0 is given by the expression $\frac{2\pi}{f'(0)}$ and is the limiting period of the period orbits in the period annulus when approaching the center in the phase plane.
The limits of the derivative are: For $A < 0$:

$$\lim_{h \downarrow h_A} \frac{dT_+^A(h)}{dh} = -\infty. \tag{4.13}$$

For $A = 0$:

$$\lim_{h \downarrow 0} \frac{dT_+^0(h)}{dh} = -\operatorname{sign}(f''(0))\infty. \tag{4.14}$$

For $A > 0$:

$$\lim_{h \downarrow h_A} \frac{dT_+^A(h)}{dh} = \infty. \tag{4.15}$$

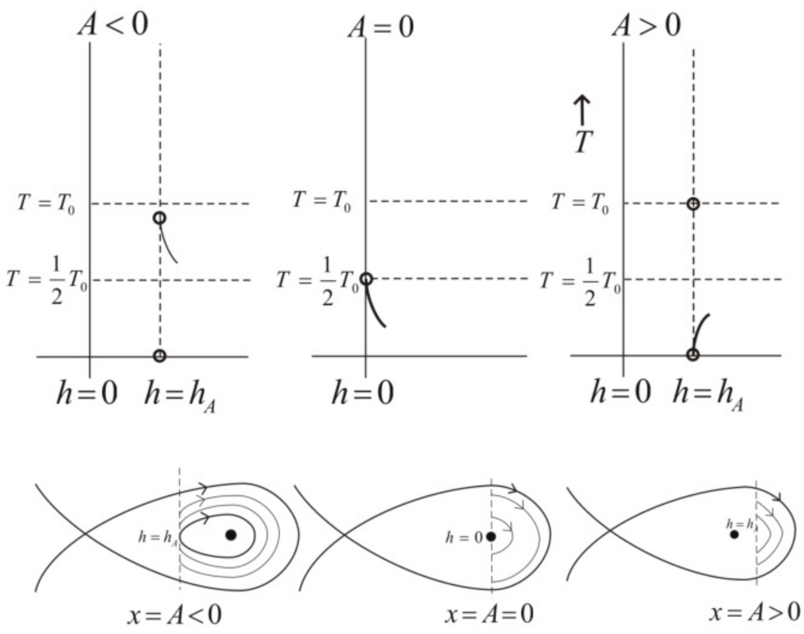

Figure 4.1: The three different cases for the limits of the function $T_+^0(h)$ depending on the sign of A. The case depicted here is for $f''(0) > 0$.

Proof. For $A \neq 0$ the limits for $h \downarrow h_A$ for $T_+^A(h)$ follow from the facts that:

For $A > 0$ the curve S_+ shrinks and approaches the point $(x = A, y = 0)$, i.e. in (3.1) the upper integral limit $x_+(h)$ approaches A and the integral approaches 0.

For $A < 0$, the curve S_+ approaches the periodic orbit tangent to $x = A$ if $h \downarrow h_A$ and therefore the value of the positive time-to-return function approaches the full period of this periodic orbit.

For $A \neq 0$ the limits for $h \downarrow h_A$ for the derivative $\frac{dT_+^A(h)}{dh}$ follow from the expression (4.7) in Lemma 4.2. The integral expression is bounded (and actually approaches 0 in the limit) because of the continuity and boundedness of the function $\omega'(x)$ in the integrand and therefore the behaviour of the derivative is dominated by the first term $\frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}}$ which approaches $\pm\infty$ with the sign depending on the sign of $f(A)$ which is positive (negative) for $A > 0$ ($A < 0$).

For $A = 0$ we can use the expansion of Proposition 4.1, i.e. expansion (4.2). The equations of the lemma follow taking into account that the sign of a_1 is determined by $f''(0)$. If $f''(0) = 0$ higher order contributions of the expansion need to be taken into account, which can be achieved by a straightforward procedure which is outside the scope of the paper. □

The limits for the different cases are summarized in Figure 4.1.

Note 4.4. The crucial observation in Proposition 4.3 is that the limits in (4.10), (4.11) and (4.12) are not continuous as a function of A. The value in (4.10) approaches T_0 when $A \uparrow 0$, while the value is equal to $\frac{1}{2}T_0$ for $A = 0$ and is identically equal to 0 for $A > 0$. The change in the sign of the derivatives (4.13), (4.14), (4.15) while crossing $A = 0$ is exactly the cause of the occurrence of S-shaped bifurcations in the mixed solution cases of this paper.

At the end point of the interval for h, i.e. $h = h_{sep}$, we can use the position of the saddle loop to arrive at:

Lemma 4.5. *The limiting behaviour of the positive time-to-return function in $T_+^A(h)$ (3.1) and its derivative, defined on $h \in (h_A, h_{sep})$ for system (2.3) near $h = h_{sep}$, is as follows:*

$$\lim_{h \downarrow h_{sep}} T_+^A(h) = C(A) > 0, \tag{4.16}$$

$$\lim_{h \downarrow h_{sep}} \frac{dT_+^A(h)}{dh} = C_2(A). \tag{4.17}$$

Proof. These limits follow from the fact that the part of the saddle loop surrounding the period annulus for $x > A$ is traversed in a finite positive time, because the saddle is positioned at $x = x_s < A$. Note that the sign of $C_2(A)$ is undetermined, which is of no further importance for the discussion in this paper. □

5 Full solutions

The full period solutions as defined in (3.3) correspond to the traditional period function of the period annulus. First we derive a new iterative procedure for determining the derivatives of all order for the period function.

5.1 Derivatives of the period function

Proposition 5.1. *The n-th derivative $\frac{d^n T_{full}(h)}{dh^n} \equiv T_{full}^{(n)}(h)$, $n \geq 0$ of (3.3) can be expressed in the form (with $n = 0$ referring to the function $T_{full}(h)$ itself):*

$$h^n T_{full}^{(n)}(h) = c_n \int_{x_-(h)}^{x_+(h)} y_h(x)^{2n-1} \psi_n(x) dx, \tag{5.1}$$

where

$$c_n = \frac{1}{2^{n-1}} \frac{1}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)},$$

$$\psi_n(x) = \mathcal{L}[\mathcal{I}]_{\omega(x)}^{(n)}(x),$$

$$\mathcal{L}[g]_{\omega(x)}^{(n)}(x) \equiv \mathcal{L}[\mathcal{L}[\ldots[\mathcal{L}[g]]\ldots]](x),$$

$$\mathcal{L}[g]_{\omega(x)}(x) \equiv [(\omega(x)g(x))' + \omega(x)g'(x)]',$$

$$\omega(x) \equiv \frac{F(x)}{f(x)^2},$$

and the identity function $\mathcal{I}$ is defined by:

$$\mathcal{I}(x) \equiv 1.$$

The initial values for the iterations are:

$$c_0 = 2,$$
$$\psi_0(x) = 1.$$

Proof. The proof is by induction. The formula is true for $n = 0$, because of (3.3). It implies that:

$$c_0 = 2,$$
$$\psi_0(x) = 1.$$

Next we show that it will hold true for $n+1$ if the formula is true for n. For notational simplicity we write $T_{full} = T$ and suppress the dependency of x_- and x_+ on h.

Multiply (5.1) with respect to h on both sides to obtain:

$$h^{n+1}T^{(n)}(h) = c_n \int_{x_-}^{x_+} \left[\frac{1}{2}y_h(x)^{2n+1}\psi_n(x) + y_h(x)^{2n-1}F(x)\psi_n(x)\right] dx,$$

where we used (2.5), the expression relating h, x, y on an integral curve.

To the second term on the right hand side we apply integration by parts using $\frac{dy}{dx} = -\frac{f(x)}{y_h(x)}$, which is allowed since $F(x)$ has a double zero at $x = 0$ compensating for the zero of $f(x)$ at $x = 0$ introduced in the denominator:

$$c_n \int_{x_-}^{x_+} y_h(x)^{2n-1}F(x)\psi_n(x)dx = -c_n \int_{x_-}^{x_+} \frac{F(x)\psi_n(x)}{(2n+1)f(x)} dy_h(x)^{2n+1},$$

which leads to (since the boundary terms vanish):

$$h^{n+1}T^{(n)}(h) = c_n \int_{x_-}^{x_+} y_h(x)^{2n+1} \left[\frac{1}{2}\psi_n(x) + \frac{1}{(2n+1)}\left(\frac{F(x)\psi_n(x)}{f(x)}\right)'\right] dx.$$

Differentiating this expression with respect to h gives:

$$h^{n+1}T^{(n+1)}(h) = c_n \int_{x_-}^{x_+} y_h(x)^{2n-1} \left[\left(\frac{F(x)\psi_n(x)}{f(x)}\right)' - \frac{1}{2}\psi_n(x)\right] dx.$$

In this expression the integrand can be rewritten in the following convenient form:

$$\left(\frac{F(x)\psi_n(x)}{f(x)}\right)' - \frac{1}{2}\psi_n(x) = \frac{1}{2}\left[f(x)\left(\frac{F(x)\psi_n(x)}{f(x)^2}\right)' + \frac{F(x)\psi_n'(x)}{f(x)}\right].$$

It follows that:

$$h^{n+1}T^{(n+1)}(h) = \frac{1}{2}c_n \int_{x_-}^{x_+} y_h(x)^{2n-1} \left[f(x)\left(\frac{F(x)\psi_n(x)}{f(x)^2}\right)' + \frac{F(x)\psi_n'(x)}{f(x)}\right] dx.$$

Note that $F(x)$ has a double zero at $x = 0$ and therefore the expression $\frac{F(x)}{f(x)^2}$ should be well-behaved near $x = 0$.

Again with the use of the relation $\frac{dy}{dx} = -\frac{f(x)}{y_h(x)}$, another integration by parts leads to:

$$h^{n+1}T^{(n+1)}(h) = \frac{c_n}{2(2n+1)} \int_{x_-}^{x_+} y_h(x)^{2n+1} \left[\left(\frac{F(x)\psi_n(x)}{f(x)^2}\right)' + \frac{F(x)\psi_n'(x)}{f(x)^2}\right]' dx.$$

This confirms the general form for the *nth* derivative of the period function as indicated in equation (5.1):

$$h^{n+1}T^{(n+1)}(h) = c_{n+1} \int_{x_-}^{x_+} y_h(x)^{2n+1}\psi_{n+1}(x)dx,$$

where

$$c_{n+1} = \frac{1}{2(2n+1)} c_n,$$

$$\psi_{n+1}(x) = \left(\frac{F(x)\psi_n(x)}{f(x)^2}\right)'' + \left(\frac{F(x)\psi_n'(x)}{f(x)^2}\right)'. \qquad \square$$

According to this iterative procedure the first couple of derivatives take the following form:

$$hT_{full}'(h) = c_1 \int_{x_-(h)}^{x_+(h)} y_h(x)\psi_1(x)dx,$$

$$h^2 T_{full}''(h) = c_2 \int_{x_-(h)}^{x_+(h)} [y_h(x)]^3 \psi_2(x)dx,$$

$$h^3 T_{full}'''(h) = c_3 \int_{x_-(h)}^{x_+(h)} [y_h(x)]^5 \psi_3(x)dx,$$

$$\begin{aligned} \psi_1(x) &= \omega''(x), \\ \psi_2(x) &= (\omega''(x))^2 + 3\omega'(x)\omega'''(x) + 2\omega(x)\omega^{iv}(x), \\ \psi_3(x) &= (\omega''(x))^3 + 22\omega(x)\omega''(x)\omega^{iv}(x) + 18\omega'(x)\omega''(x)\omega'''(x) + 15(\omega'(x))^2\omega^{iv}(x) \\ &\quad + 10\omega(x)(\omega'''(x))^2 + 20\omega(x)\omega'(x)\omega^{v}(x) + 4(\omega(x))^2\omega^{vi}(x), \end{aligned} \tag{5.2}$$

where $\omega(x) \equiv \frac{F(x)}{f(x)^2}$.

The first derivative corresponds with the well-known expression used in the literature, e.g. see [8]. The expressions for the higher order derivatives seem to be new.

5.2 Properties of the full period function

Proposition 5.2. *The behaviour of the full time-to-return function $T_{full}(h)$in (3.3) and its derivative, defined on $h \in (h_A, h_{sep})$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5), near the boundaries of its domain is as follows:*

For $A \neq 0$:

$$\begin{aligned} \lim_{h \downarrow h_A} T_{full}(h) &= T_{full}^A > 0, \\ \lim_{h \downarrow h_A} \frac{dT_{full}(h)}{dh} &= C_3(A) < \infty. \end{aligned} \tag{5.3}$$

For $A = 0$:

$$\begin{aligned} \lim_{h \downarrow 0} T_{full}(h) &= T_0 > 0, \\ \lim_{h \downarrow 0} \frac{dT_{full}(h)}{dh} &= C_3(0) < \infty, \end{aligned} \tag{5.4}$$

where the period of the periodic orbit $\gamma_{h_{eff}}$ tangent to $x = A$ is abbreviated as T_{full}^A and is given by the expression:

$$T_{full}^A \equiv 2\int_{x=x_-(h_{eff})}^{x=x_+(h_{eff})} \frac{dx}{y_{h_{eff}}(x)},$$

where $h_{eff} \equiv F(A)$. Notice that for $A > 0$ ($A < 0$), we have the relation $x_+(h_{eff}) = A$ ($x_-(h_{eff}) = A$). The limiting value T_0 is given by the expression $\frac{2\pi}{f'(0)}$.

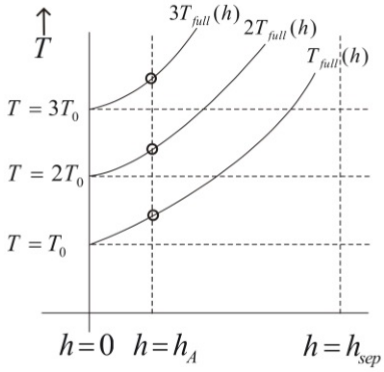

Figure 5.1: The limits of the full period functions $nT_{full}(h)$ with $n = 1, 2, 3$.

Near the outer boundary of the period annulus enclosed by a saddle loop we have the straightforward result:

$$\begin{aligned} \lim_{h\uparrow h_{sep}} T_{full}(h) &= \infty, \\ \lim_{h\uparrow h_{sep}} \frac{dT_{full}(h)}{dh} &= \infty. \end{aligned} \tag{5.5}$$

Proof. The results for $\lim_{h\downarrow h_A}$ and $A \neq 0$ follow from the definition of the full period function. $T_{full}(h)$ does not depend on A and will therefore assume the value of the function at h_A due to continuity. The limit for $A = 0$ is a classical result for the period of a periodic solution near an elementary center, see e.g. [4]. In particular for the results of this paper it is important to notice that the derivative remains bounded when approaching the center point at $h = 0$.

The outer boundary $h = h_{sep}$ is the saddle loop. The integrand inside the integral defining the period function $T_{full}(h)$ approaches an essential singularity for $\lim_{h\uparrow h_{sep}}$ and the value of the integral goes to ∞. The intuition behind this is that the periodic orbits near the outer boundary of the period annulus approach the saddle singularity where the solutions of the ODE become slower and the passage time approaches ∞. A similar argument can be used for the derivatives. □

The limits for the full period functions are shown in Figure 5.1.

6 Mixed solutions

6.1 Properties of the mixed time-to-return functions

Using the limits for $T_+^A(h)$ and $T_{full}^A(h)$ we can immediately write down the limits for $T_{n+1/2}^A(h)$.

Proposition 6.1. *The behaviour of the mixed time-to-return function $T_{n+1/2}^A(h)$ in (3.4) and its derivative, defined on $h \in (h_A, h_{sep})$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5), near $h = h_A$ and $h = h_{sep}$, is as follows:*

For $A < 0$:

$$\lim_{h\downarrow h_A} T_{n+1/2}^A(h) = (n+1)T_{full}^A > 0. \tag{6.1}$$

For $A = 0$:

$$\lim_{h\downarrow 0} T_{n+1/2}^A(h) = (n + \frac{1}{2})T_0. \tag{6.2}$$

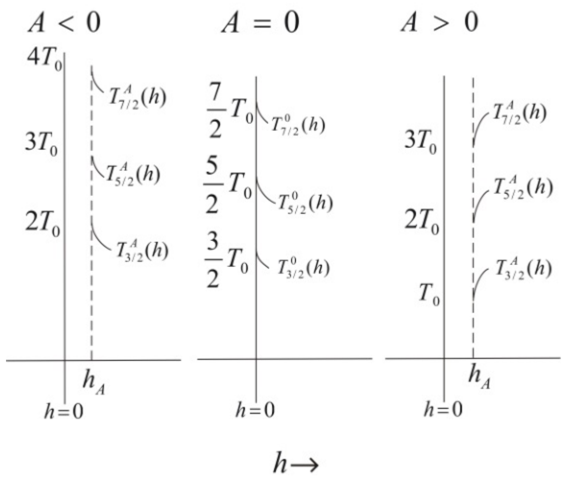

Figure 6.1: The limits of the mixed period functions $T^A_{n+1/2}(h)$ with $n = 1, 2, 3$ for the case $f''(0) > 0$.

For $A > 0$:

$$\lim_{h \downarrow h_A} T^A_{n+1/2}(h) = 0. \tag{6.3}$$

The limits of the derivative are:

For $A < 0$:

$$\lim_{h \downarrow h_A} \frac{dT^A_{n+1/2}(h)}{dh} = -\infty. \tag{6.4}$$

For $A = 0$:

$$\lim_{h \downarrow 0} \frac{dT^0_{n+1/2}(h)}{dh} = -\operatorname{sign}(f''(0))\infty. \tag{6.5}$$

For $A > 0$:

$$\lim_{h \downarrow h_A} \frac{dT^A_{n+1/2}(h)}{dh} = \infty. \tag{6.6}$$

Near the outer boundary the functions and their derivatives approach infinity:

$$\begin{aligned} \lim_{h \uparrow h_{sep}} T^A_{n+1/2}(h) &= \infty, \\ \lim_{h \uparrow h_{sep}} \frac{dT^A_{n+1/2}(h)}{dh} &= \infty. \end{aligned} \tag{6.7}$$

Proof. Since the mixed time-to-return functions are linear combinations of the positive time-to-return function and the full period function through:

$$T^A_{n+1/2}(h) = T^A_+(h) + nT_{full}(h),$$

the limiting behaviour near the boundaries follows in a straightforward way from the previously derived limits for $T^A_{1/2}(h)$ (Proposition 4.3) and $T_{full}(h)$ (Proposition 5.2). The critical quantity is the limit in (6.5) which is determined by the sign of $f''(0)$. □

The limits for the mixed period functions are shown in Figure 6.1 for the case $f''(0) > 0$.

Note 6.2. As was indicated in Note 4.4, the functions display a discontinuity while changing A, i.e. while crossing $A = 0$, as indicated in the limits (6.1), (6.2) and (6.3). Moreover, the derivatives (6.4), (6.5) and (6.6) will change sign while crossing $A = 0$. This is the cause for the occurrence of an additional local extreme value of the mixed period functions while changing A. The discontinuity requires an accurate analysis and is the reason for the technical proofs in the next sections.

6.2 Existence of a local minimum for the mixed time-to-return functions $T^A_{n+1/2}(h)$ for sufficiently small A and $f''(0) > 0$

For convenience of the exposition we will discuss the case $f''(0) > 0$. The case $f''(0) < 0$ can be analysed in a similar fashion. The function $T^A_{n+1/2}(h)$ for $A = 0$ tends to $+\infty$ for $h \uparrow h_{sep}$ according to (6.7). Since at $h = 0$ the derivative is $-\infty$ according to (6.5), there must exist a local minimum for some $h \in (0, h_{sep})$.

Proposition 6.3. *For each $n > 0$ and with $A = 0$ the mixed time-to-return function $T^A_{n+1/2}(h)$ in (3.4) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, defined on $h \in (0, h_{sep})$ for system (2.3) has at least one local minimum for $h = h^0_1$.*

Proof. The proof follows easily from the limiting behaviour of the mixed time-to-return functions as given in Proposition 6.1. For each n we have $\lim_{h\downarrow 0} \frac{dT^0_{n+1/2}(h)}{dh} = -\infty$, $\lim_{h\uparrow h_{sep}} \frac{dT^0_{n+1/2}(h)}{dh} = \infty$.

It follows from the continuity of $T^0_{n+1/2}(h)$ that there exists at least one value $0 < h_1(n) < h_{sep}$ such that $\frac{dT^0_{n+1/2}(h)}{dh}|_{h=h_1(n)} = 0$ and $(h - h_1(n))dT^0_{n+1/2}(h) > 0$ in a sufficiently small neighborhood of $h = h_1(n)$, i.e. $T^0_{n+1/2}(h)$ has a local minimum at $h = h^0_1(n)$. □

The visualization of the proof is shown in Figure 6.2. Next we prove that the minimum of the previous proposition persists when A is perturbed. For the following we also need estimates on the location of this minimum.

Proposition 6.4. *For each $n > 0$ there exists $A = \epsilon(n)$ such that the mixed time-to-return function $T^A_{n+1/2}(h)$ in (3.4) with $A \in (0, \epsilon(n))$ defined on $h \in (h_A, h_{sep})$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, has at least one local minimum for $h = h^{\epsilon(n)}_1(n)$ with $|h^{\epsilon(n)}_1(n) - h^0_1(n)| < \delta(\epsilon(n))$, where $h^0_1(n)$ corresponds to the local minimum of $T^0_{n+1/2}(h)$ in Proposition 6.3.*

Proof. From the representation (4.8) in Lemma 4.2 of the derivative of the positive time-to-return function it follows immediately that:

$$\frac{dT^\epsilon_+(h)}{dh} - \frac{dT^0_+(h)}{dh} = \int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}(h - h_{\bar{A}})^{\frac{3}{2}}} d\bar{A}.$$

Since $T^\epsilon_{n+1/2}(h) = T^\epsilon_+(h) + nT_{full}(h)$ where $T_{full}(h)$ does not depend on ϵ, the above relationship implies that:

$$\frac{dT^\epsilon_{n+1/2}(h)}{dh} - \frac{dT^0_{n+1/2}(h)}{dh} = \int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}(h - h_{\bar{A}})^{\frac{3}{2}}} d\bar{A}. \quad (6.8)$$

Consider an interval $h_{\delta_1} < h^0_1(n) < h_{\delta_2}$ such that $\frac{dT^0_{n+1/2}(h)}{dh} < 0$ (> 0) for $h_{\delta_1} < h^0_1(n)$ $(h^0_1(n) < h_{\delta_2})$. According to Proposition 6.3 this is possible. For given n choose $h^* \in (h_{\delta_1}, h^0_1(n))$. Since

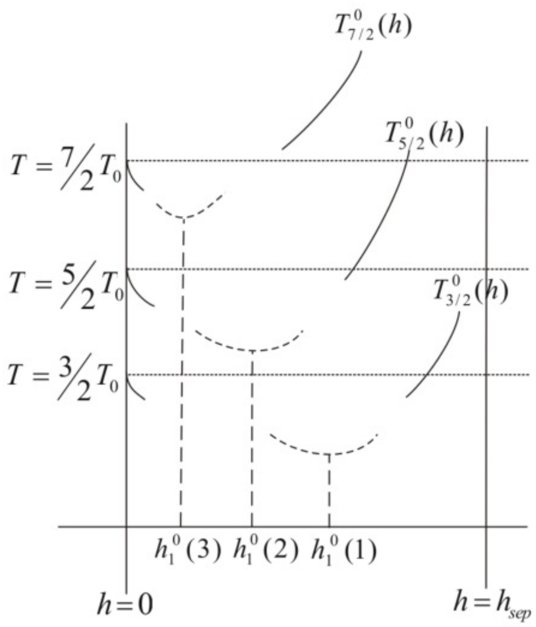

Figure 6.2: The existence of a minimum $h_1^0(n)$ for the mixed time-to-return functions $T^A_{n+1/2}(h)$ with $n = 1,2,3$ for $A = 0$ in system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$.

in (6.8) the integrand on the right hand side is bounded for fixed $h^* > h_{\bar{A}}$, for all $0 < \bar{A} < \epsilon$, we can choose ϵ small such that:

$$\frac{dT^{\epsilon}_{n+1/2}(h)}{dh}|_{h=h^*} = \frac{dT^{0}_{n+1/2}(h)}{dh}|_{h=h^*} + \int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}(h^* - h_{\bar{A}})^{\frac{3}{2}}} d\bar{A} < 0.$$

Moreover we have by construction that:

$$\frac{dT^{\epsilon}_{(n+1/2)}(h)}{dh}|_{h=h^{**}} = \frac{dT^{0}_{(n+1/2)}(h)}{dh}|_{h=h^{**}} + \int_{\bar{A}=0}^{\bar{A}=\epsilon} \frac{1}{\sqrt{2}(h^{**} - h_{\bar{A}})^{\frac{3}{2}}} d\bar{A} > 0,$$

for $h^{**} \in (h_1^0(n), h_{\delta_2})$.

From these two equations we conclude that for each n we can find a value $h = h_1^{\epsilon(n)}(n)$ such that $\frac{dT^A_{n+1/2}(h)}{dh}$ has a zero where the sign changes from minus to plus for increasing h, i.e. $T^A_{n+1/2}(h)$ has a local minimum for a value of h close to the local minimum $h_1^0(n)$ of $T^0_{n+1/2}(h)$. □

The persistence of the minimum of the mixed period functions for small ϵ is shown in Figure 6.3.

Note 6.5. This proposition basically states that for sufficiently small $A > 0$ the local minimum of the mixed time-to-return function $T^A_{n+1/2}(h)$ persists as would be expected from continuity.

6.3 Existence of a local maximum for the mixed time-to-return functions $T^A_{n+1/2}(h)$ for sufficiently small A and $f''(0) > 0$

The result of the previous section showed that a local minimum exists for $T^A_{n+1/2}(h)$ when A is sufficiently small. However, the results of the limits for the derivatives in Figure 6.1 show

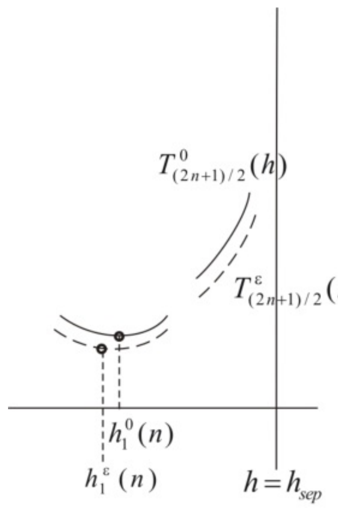

Figure 6.3: The persistence of a minimum $h_1^\epsilon(n)$ for the mixed time-to-return functions $T^A_{n+1/2}(h)$ for $A = \epsilon > 0$ in system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$.

that crossing $A = 0$ the derivative changes sign. This can only be explained by the creation of a local maximum on the function $T^A_{n+1/2}(h)$.

Proposition 6.6. *For each $n > 0$ there exists $A = \epsilon(n)$ such that the mixed time-to-return function $T^A_{n+1/2}(h)$ in (3.4) with $A \in (0, \epsilon(n))$ defined on $h \in (h_A, h_{sep})$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, has at least one local maximum for $h = h_2^{\epsilon(n)}(n)$ with $h_A < h_2^{\epsilon(n)}(n) < \delta_2(\epsilon(n))$.*

Proof. In the proof of Proposition 6.4 it was shown that for sufficiently small A, there will be a value $h = h^*$ depending on n for which the derivative of the mixed time-to-return function $T^A_{n+1/2}(h)$ is negative. From Proposition 6.1 we know that for $A > 0$:

$$\lim_{h \downarrow h_A} \frac{dT^A_{n+1/2}(h)}{dh} = \infty,$$

implying that there exists h^{**} close enough to h_A such that $\frac{dT^A_{n+1/2}(h)}{dh}|_{h^{**}} > 0$. Therefore there is (at least one) a value $h_2^{\epsilon(n)}(n) \in (h^{**}, h^*)$ such that $\frac{dT^A_{n+1/2}(h)}{dh}|_{h_2^{\epsilon(n)}(n)} = 0$. Moreover the derivative changes sign from minus to plus around this zero, showing that it represents a local maximum of the function $T^A_{n+1/2}(h)$ as we set out to prove. $\square$

The creation of a local maximum of the mixed period functions for small ϵ is shown in Figure 6.4.

Note 6.7. This proposition basically states that for sufficiently small $A > 0$ a local maximum of the mixed time-to-return function $T^A_{n+1/2}(h)$ is created from the center point at $(x = 0, y = 0)$ in the phase plane by changing the parameter A. The condition $f''(0) > 0$ ensures that a local maximum is created. This is a critical ingredient for the S-shaped bifurcation of the next section.

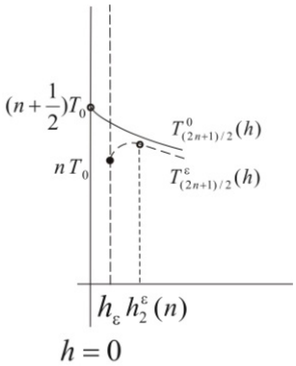

Figure 6.4: The creation of a maximum $h_2^\epsilon(n)$ for the mixed time-to-return functions $T^A_{n+1/2}(h)$ for $A = \epsilon > 0$ in system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$.

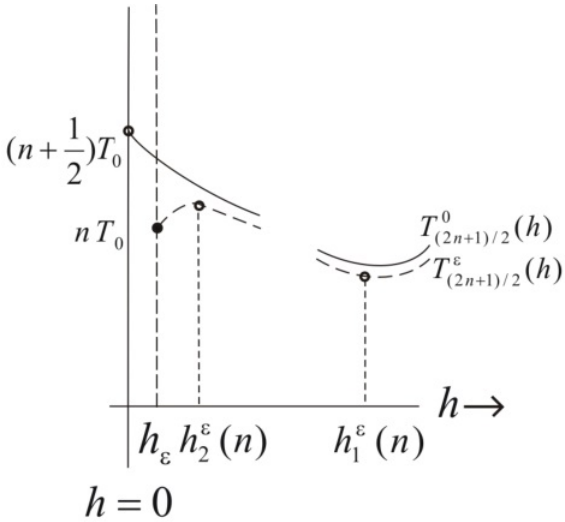

Figure 6.5: The S-shaped mixed time-to-return functions $T^A_{n+1/2}(h)$ with $n = 1, 2, \ldots$ for $A = \epsilon > 0$ and $f''(0) > 0$.

6.4 Co-existence of a local maximum and a local minimum for the mixed time-to-return functions $T^A_{n+1/2}(h)$ for sufficiently small A and $f''(0) > 0$

The results of the previous two sections showed the existence of a local minimum and maximum for the function $T^A_{n+1/2}(h)$. Combining these results we immediately get our main result.

Theorem 6.8. *For each $n > 0$ there exists $A = \epsilon(n)$ such that the mixed time-to-return function $T^A_{n+1/2}(h)$ in (3.4) with $A \in (0, \epsilon(n))$ defined on $h \in (h_A, h_{sep})$ for system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, has at least one local minimum $h = h_1^{\epsilon(n)}(n)$ and one local maximum for $h = h_2^{\epsilon(n)}(n)$ for small enough $\epsilon(n)$.*

Proof. The theorem is a direct consequence of the statements in Proposition 6.4 and Proposition 6.6. □

The co-existence of a local maximum and local minimum of the mixed period functions for small ϵ is shown in Figure 6.5.

Note 6.9. This proposition basically states that for sufficiently small $A > 0$ an S-shaped bifurcation occurs for each type of mixed time-to-return function $T^A_{n+1/2}(h)$ with $n = 1, 2, \ldots$ This S-shape does not occur if $f''(0) < 0$. In that case it is not difficult to verify that the other type of mixed time-to-return function $T^A_{-n-1/2}(h)$ will exhibit an S-shaped bifurcation. The case $f''(0) = 0$ is more difficult because it would require taking higher order contributions into account. We believe that even in these cases S-shaped bifurcations will take place.

7 Example of a quadratic Hamiltonian system

In this section we provide an application of the previous sections to the simplest possible nonlinear case $f(x) = x(x+1)$. The conditions (1.3), (1.4) and (1.5) and $f''(0) > 0$ are satisfied, since $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2 > 0$ and $x_s = -1 < 0$, $f(-1) = 0$, $f'(-1) = -1$ and $f(x)x(x - x_s) = x^2(x+1)^2$. The integral of this system is given by:

$$h = \frac{1}{2}y^2(t) + \frac{1}{2}x(t)^2 + \frac{1}{3}x(t)^3. \tag{7.1}$$

with the saddle loop represented by $h = h_{sep} = \frac{1}{6}$. The saddle loop passes through the saddle singularity at $x = -1$ and the regular point $x = \frac{1}{2}$, i.e. we consider $-1 < A < \frac{1}{2}$.

The derivatives of the full period function satisfy the following relations.

Lemma 7.1. *On the interval $h \in \left(0, \frac{1}{6}\right)$ the period function $T_{full}(h)$ for the quadratic Hamiltonian system (2.3) with $f(x) = x(1+x)$ satisfies:*

$$\begin{aligned} \frac{dT_{full}(h)}{dh} &> 0, \\ \frac{d^2T_{full}(h)}{dh^2} &> 0, \\ \frac{d^3T_{full}(h)}{dh^3} &> 0. \end{aligned}$$

Proof. In Proposition (5.1) a recurrence relation was derived for the full period function of orbits in (2.3). The first three derivatives are given by:

$$\begin{aligned} h\frac{dT_{full}(h)}{dh} &= c_1 \int_{x_-}^{x_+} y_h(x)\psi_1(x)dx, \\ h^2\frac{d^2T_{full}(h)}{dh^2} &= c_2 \int_{x_-}^{x_+} y_h^3(x)\psi_2(x)dx, \\ h^3\frac{d^3T_{full}(h)}{dh^3} &= c_3 \int_{x_-}^{x_+} y_h^5(x)\psi_3(x)dx, \end{aligned}$$

where $y_h(x) = \sqrt{2(h - F(x))}$, c_1, c_2, c_3 positive constants,

$$\begin{aligned} \psi_1(x) &= \left(\frac{F(x)}{f(x)^2}\right)'', \\ \psi_2(x) &= \left(\frac{F(x)\psi_1(x)}{f(x)^2}\right)'' + \left(\frac{F(x)\psi_1'(x)}{f(x)^2}\right)', \\ \psi_3(x) &= \left(\frac{F(x)\psi_2(x)}{f(x)^2}\right)'' + \left(\frac{F(x)\psi_2'(x)}{f(x)^2}\right)'. \end{aligned}$$

The result of the lemma follows by proving that the three functions $\psi_1(x)$, $\psi_1(x)$, $\psi_1(x)$ are positive on the interval of interest. In our case $x \in \big(-1, \frac{1}{2}\big)$ and $\frac{F(x)}{f(x)^2} = \frac{\frac{1}{2}x^2 - \frac{1}{3}x^3}{x^2(x+1)^2} = \frac{\frac{1}{2} - \frac{1}{3}x}{(x+1)^2}$. With this the three functions can be written out to become:

$$\psi_1(x) = \frac{1}{3}\frac{(5+2x)}{(1+x)^4},$$

$$\psi_2(x) = \frac{35}{9}\frac{(11+10x+2x^2)}{(1+x)^8},$$

$$\psi_3(x) = \frac{35}{27}\frac{(2431+3486x+1560x^2+208x^3)}{(1+x)^{12}}.$$

These expressions are easily seen to be positive on the interval $x \in \big(-1, \frac{1}{2}\big)$. □

In particular this lemma proves that the full period function is convex and monotonically increasing as a function of h. The convexity property seems to be new result. The positive time-to-return function is more difficult to analyze, even for this simple case. The following results were already established in the literature.

Lemma 7.2. *The positive time-to-return function $T_+^A(h)$ in (3.1) for $A = 0$ is monotonically decreasing for (2.3) with $f(x) = x(1+x)$.*

Proof. As before we first write:

$$T_+^0(h) = \tilde{T}_+^0(x_+) = \sqrt{2}\int_{x=0}^{x=x_+} \frac{dx}{\sqrt{F(x_+) - F(x)}}.$$

Again we will write $x_+ \equiv \alpha$.

With this notation we can rewrite the integral using a scaling of the integration variable $x = \alpha u$. The integral becomes:

$$\tilde{T}_+^0(\alpha) = \sqrt{2}\int_{u=0}^{u=1} \frac{\alpha du}{\sqrt{F(\alpha) - F(\alpha u)}}.$$

We formally differentiate with respect to α to get:

$$\frac{d\tilde{T}_+^0(\alpha)}{d\alpha} = \sqrt{2}\int_{u=0}^{u=1} \frac{\sqrt{F(\alpha) - F(\alpha u)} - \alpha\frac{f(\alpha) - uf(\alpha u)}{2\sqrt{F(\alpha) - F(\alpha u)}}}{F(\alpha) - F(\alpha u)} du.$$

Rewriting the integrand we get

$$\frac{d\tilde{T}_+^0(\alpha)}{d\alpha} = \frac{\sqrt{2}}{2}\int_{u=0}^{u=1} \frac{2(F(\alpha) - F(\alpha u)) - (\alpha f(\alpha) - \alpha u f(\alpha u))}{(F(\alpha) - F(\alpha u))^{\frac{3}{2}}} du.$$

This can be rewritten in the following compact form (after changing back to the integration variable $x = \alpha u$):

$$\frac{d\tilde{T}_+^0(\alpha)}{d\alpha} = \frac{\sqrt{2}}{2}\int_{x=0}^{x=\alpha} \frac{\Delta[\theta(x)]}{\Delta[F(x)]^{\frac{3}{2}}}\frac{dx}{\alpha}, \tag{7.2}$$

where $\theta(x) \equiv 2F(x) - xf(x)$ and $\Delta[Z(x)] \equiv Z(\alpha) - Z(x)$.

In the case of the quadratic Hamiltonian $f(x) = x(1+x)$, $F(x) = \frac{1}{2}x^2 + \frac{1}{3}x^3$ which leads to: $\theta(x) = x^2 + \frac{2}{3}x^3 - x^2(1+x) = -\frac{1}{3}x^3 < 0$ and $\theta'(x) < 0$ for $x > 0$.

This implies that the integrand of (7.2) will be negative for $0 < x < \alpha$. Therefore $\frac{d\tilde{T}_+^0(\alpha)}{d\alpha} < 0$ implying $\frac{dT_+^0(h)}{dh} < 0$ as we set out to prove. □

Note 7.3. It is straightforward by using the same type of argument to prove that the positive time-to-return function $T_+^0(h)$ is monotonically decreasing for $-1 < A < 0$ as well.

If $A > 0$ the situation is slightly more complicated, because the function $T_+^A(h)$ will have a local maximum for some A. For our discussion of mixed time-to-return functions it is important to establish that there exists a value $0 < A^* < \frac{1}{2}$ such that $T_+^0(h)$ is monotonically increasing for $0 < A^* \leq A < \frac{1}{2}$.

Lemma 7.4. $\exists A^* \in (0, \frac{1}{2})$ *such that the positive time-to-return function* $T_+^A(h)$ *in* (3.1) *is monotonically increasing for a quadratic Hamiltonian system* (2.3) *with* $f(x) = x(1+x)$.

Proof. The proof is similar to the case $A = 0$, except that we cannot use the formula introduced above which was only valid for $A = 0$. Formally, the same procedure leads to the following expression for the derivative $\frac{d\tilde{T}_+^A(\alpha)}{d\alpha}$:

$$\frac{d\tilde{T}_+^A(\alpha)}{d\alpha} = \frac{\sqrt{2}}{2} \int_{x=0}^{x=\alpha} \frac{\Delta[\theta^A(x)]}{\Delta[F(x)]^{\frac{3}{2}}} \frac{dx}{\alpha}, \tag{7.3}$$

where $\theta^A(x) \equiv 2F(x) - (x - A)f(x)$.

Substitution of $f(x) = x(1+x)$ gives: $\theta^A(x) = \frac{1}{3}x(-x^2 + 3Ax + 3A)$ and $\frac{d\theta^A(x)}{dx} = -x^2 + 2Ax + A$. A straightforward calculation shows that $\theta^A(x)$ is monotonically increasing on the interval $x \in (A, \frac{1}{2})$ if $\frac{1}{8} < A < \frac{1}{2}$. Therefore for this range of A, $\Delta[\theta^A(x)]$ is positive and $\frac{d\tilde{T}_+^A(\alpha)}{d\alpha} > 0$ proving that the derivative of $T_+^A(h)$ is positive. The lemma follows with $A^* < \frac{1}{8}$. $\square$

Note 7.5. It is not so straightforward to prove that the positive time-to-return function $T_+^0(h)$ has a unique local maximum for $0 < A < A^*$ with this technique. It follows from other results in the literature, i.e. see [4, 19].

The application of Theorem 6.8 to the case $f(x) = x(x+1)$ shows that the case of the mixed solutions is much more complicated. A full proof of the exact number of local maxima and minima seems to be difficult even for this case.

Theorem 7.6. *For each* $n > 0$ *there exists* $A = \epsilon(n)$ *such that the mixed time-to-return function* $T_{n+1/2}^A(h)$ *in* (3.4) *with* $A \in (0, \epsilon(n))$ *defined on* $h \in (h_A, h_{sep})$ *for system* (2.3) *with* $f(x) = x(x+1)$ *has at least one local minimum* $h = h_1^{\epsilon(n)}(n)$ *and one local maximum for* $h = h_2^{\epsilon(n)}(n)$ *for sufficiently small* $\epsilon(n)$.

8 Number of solutions for fixed λ

The previous sections showed the existence of an S-shaped bifurcation for mixed time-to-return functions. This section is aimed at investigating how λ affects the number of solutions to the original boundary value problem (1.1), (1.2) with conditions (1.3), (1.4) and (1.5). For this we need to consider the intersection of a horizontal line $T = \lambda^2 = constant$ with the different time-to-return functions $T_+^A(h)$, $T_n(h)$, $T_{n+1/2}^A(h)$. Each intersection will correspond to a solution to the original problem for such a value of λ. Each tangency of the horizontal line with such a function (i.e. tangent to a local minimum or maximum of the graph of the function) corresponds to a bifurcation value of λ.

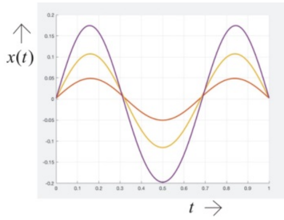

Figure 8.1: Numerical example of the co-existence of three solutions of type $S^A_{3/2}$ to the boundary value problem for $\lambda = 86.23908$, $A = 0.002$ for the quadratic Hamiltonian case $f(x) = x(1+x)$. The initial conditions for the three solutions are $\frac{dx(t)}{dt}|_{t=0} = 0.46$, $\frac{dx(t)}{dt}|_{t=0} = 1.03$, $\frac{dx(t)}{dt}|_{t=0} = 1.71$.

8.1 Number of solutions for fixed λ, fixed solution type

First we consider each type separately and find an estimate on the number of possible solutions as a function of λ.

According to the results of the previous sections we know that at least three solutions of the type $S^A_{n+1/2}$ (for each n) exist for a proper choice of the parameter λ according to Theorem 6.8. Since for small $A > 0$ the function has at least one local maximum and local minimum, there must exist a horizontal line which crosses the graph of the function in at least three points, i.e. $T^A_{n+1/2}(h) = \lambda^2$ has at least three solutions for an appropriate choice of λ.

Proposition 8.1. *Boundary value problem (1.1) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, for sufficiently small positive A, has at least three solutions of mixed type $S^A_{n+1/2}$ (for each n) by choosing λ appropriately.*

In Figure 8.1 a numerical example is shown for this situation, i.e. an example of system (1.1) with $f(x) = x(1+x)$ and boundary condition (1.2) having three solutions of the same type. Figure 8.2 displays the mixed period function for the case $T^A_{3/2}$, i.e. an example of system (1.1) with $f(x) = x(1+x)$ and boundary condition (1.2) while varying the parameter A. It is clearly visible that for the parameters $A = 0.001$ and $A = 0.002$, a local maximum and local minimum occur. Both disappear by increasing A further as shown for the value $A = 0.0035$.

8.2 Number of simultaneous solutions for fixed λ

The first step in estimating the number of simultaneous solutions for the different solution types is to determine the range of the functions. From the properties of the previous sections, the following results are straightforward.

Lemma 8.2. *For system (2.3) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, the ranges of the time-to-return functions are:*

- $T^A_+(h) \in (C_1(A), C_2(A))$ *with* $0 < C_2(A) < \infty$; $0 < C_1(A) < \infty$, *for* $x_s < A \leq 0$, *and* $C_1(A) = 0$ *for* $0 < A < x_s^{(2)}$,
- $T_n(h) \in (nT_{full}(h_{min}), \infty)$, *where* h_{min} *corresponds to the global minimum* $T_{full}(h_{min}) > 0$ *of* $T_{full}(h)$.
- $T^A_{n+1/2}(h) \in (C_3(A, n), \infty)$, *with* $C_3(A, n) > 0$.

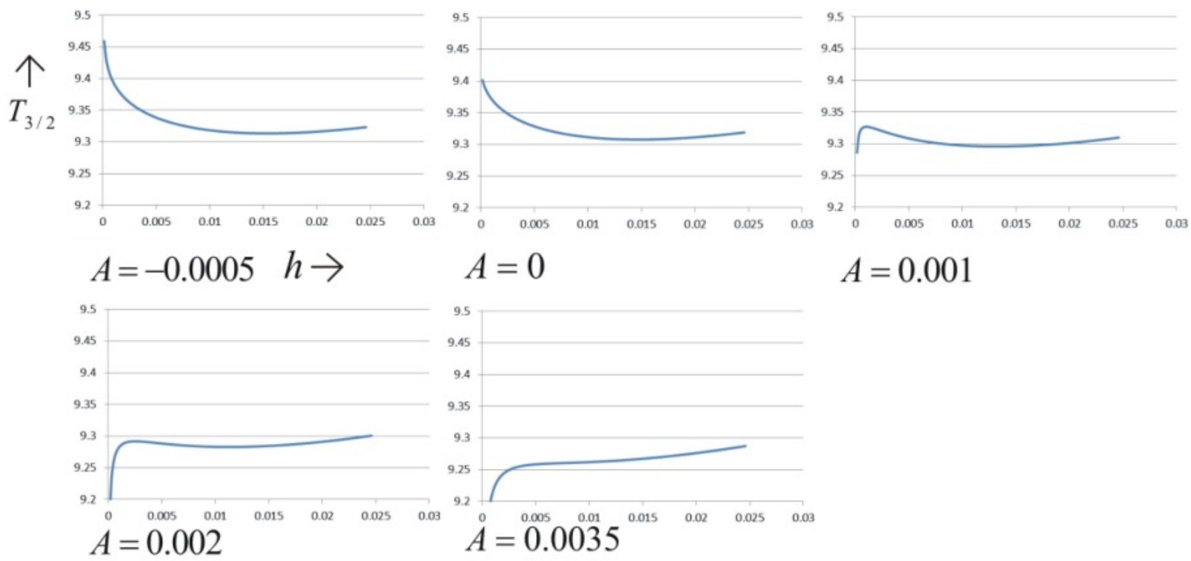

Figure 8.2: Numerical example for the mixed period function $T^A_{3/2}$ for the quadratic Hamiltonian case $f(x) = x(1+x)$ while varying the parameter A. Five cases are shown for A. For $A = -0.0005$ and $A = 0$ only one local minimum occurs. For small positive A, i.e. $A = 0.001$ and $A = 0.002$ an additional local maximum occurs. For larger A, i.e. $A = 0.0035$, the function is monotonically increasing and both local extreme points have disappeared.

Proof. The function $T_{full}(h)$ tends to ∞ for $h \rightarrow h_{sep}$ according to Lemma 5.2. Since it is positive on a bounded interval and $T_{full}(0) > 0$, a global minimum of the function must exist, denoted by h_{min}.

This establishes the results for $T_n(h)$ and $T^A_{n+1/2}(h)$ due to the continuity of the functions on the bounded open interval for h. In the latter case the minimum value $C_3(A,n)$ is not trivial to find explicitly, except when $A > 0$: we have $T^A_{n+1/2}(h) > T_n(h) = nT_{full}(h)$ and $\lim_{h \downarrow h_A} T^A_{n+1/2}(h_A) = nT_{full}(h_A)$. See Figures 8.5, 8.6 for the case of $f(x) = x(x+1)$ where the function $T_{full}(h)$ is monotonically increasing according to Lemma 7.1.

The result for the remaining case $T^A_+(h)$ follows from the fact that $T^A_+(h)$ approaches 0 for $h \downarrow h_A$ for $0 < A < \frac{1}{2}$, while it approaches a positive constant when $-1 < A \leq 0$ according to Proposition 4.3. □

Note 8.3. The important feature of the lemma is that the range of each of the countably many functions $T_n(h)$ and $T^A_{n+1/2}(h)$ extends to $+\infty$. This is due to the fact that the period annulus is bounded on the exterior by a saddle loop. The other important feature is that the lower bounds of the functions $T_n(h)$ and $T^A_{n+1/2}(h)$ grow with increasing n as we will show below.

The implication of this lemma is that for each fixed sufficiently large λ the original boundary value problem has at least one solution. In the case $x_s < A \leq 0$ there is an open interval $(0, C_1(A))$ such that for λ in this interval no solutions exist for the boundary value problem. For $A = 0$ there is a second interval $(C_2(A), T_{full}(0))$ such that no solutions exist for λ in this range. The different possibilities for the relative positions of the functions are shown in Figures 8.3, 8.4, 8.5, 8.6 for the case of $f(x) = x(x+1)$ where the function $T_{full}(h)$ is monotonically increasing according to Lemma 7.1. The figures assume that the maximum number of local extreme values on each time-to-return functions is three. Therefore the figures are

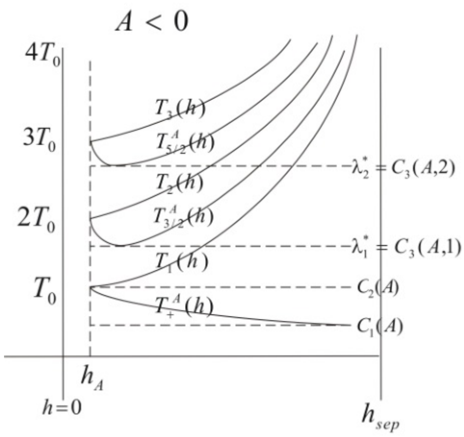

Figure 8.3: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x) = x(1+x)$, $x_s < A \leq 0$.

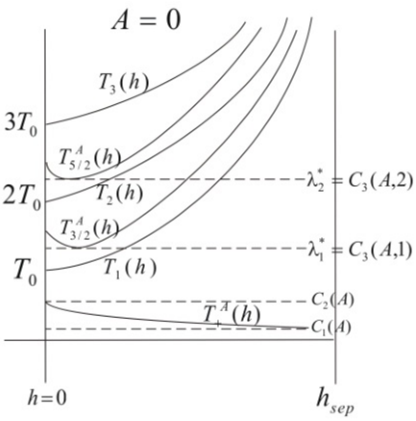

Figure 8.4: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x) = x(1+x)$, $A = 0$.

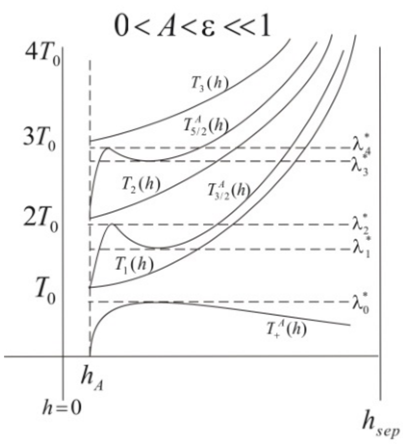

Figure 8.5: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x) = x(1+x)$, $0 < A = \epsilon \ll 1$.

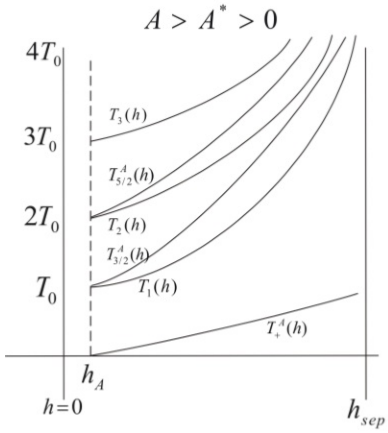

Figure 8.6: Conjectured time-to-return functions for simultaneous solutions to the boundary value problem for (2.3) with $f(x) = x(1+x)$, $0 < A^* < A < \frac{1}{2}$.

labelled as conjectured and have not been verified numerically.

The next proposition follows from the fact that in Lemma 8.2 the range of the countably infinite functions $T_n(h)$ and $T^A_{n+1/2}(h)$ is bounded below by a number which is monotonically increasing as a function of n. This is obvious for the functions $T_n(h)$ which are bounded below by $nT_{full}(h_{min})$. For the function $T^A_{n+1/2}(h)$ we have the trivial estimate $T^A_{n+1/2}(h) = T_n(h) + T^A_+(h) > T_n(h) > nT_{full}(h_{min})$. The consequence of these lower bounds is that for given λ, there are only finitely many functions which have a lower bound below λ. It implies the finiteness of solutions of the boundary problem:

Proposition 8.4. *Boundary value problem (1.1) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, has finitely many solutions for each $\lambda > 0$ if $0 < A < x_s^{(2)}$ and for each $\lambda \in (C_1(A), \infty)$ (with $C_1(A)$ defined in Lemma 8.2) if $x_s < A \leq 0$.*

The exact number is not easy to verify since we do not have an upper bound on the number of local maxima and minima of the mixed time-to-return functions. Figures 8.3, 8.4, 8.5, 8.6 show for the case $f(x) = x(x+1)$ that for increasing λ the number of solutions will grow with discrete jumps even though the exact number has not been proved, or verified numerically. The number of solutions will jump when λ will cross a value of $T^A_{n+1/2}(h)$ corresponding to a local minimum or maximum. For $-x_s < A \leq 0$ the functions each have (at least) a minimum value $C_3(A, n)$ which increases without an upper bound as a function of n. It shows that for any chosen λ_c countably infinite bifurcation values $\lambda^*_n > \lambda_c$ can be found. This contradicts the statement in the paper [27] where it was stated that only for small λ bifurcations would occur for mixed solutions.

8.3 Systems with an infinite number of solutions

The previous section showed that boundary value problem (1.1) with real analytic $f(x)$ satisfying conditions (1.3), (1.4), (1.5) and $f''(0) > 0$, has finitely many solutions for given λ. It is not difficult to point out the reason why this number is finite. The period annulus is bounded on the outside by a saddle loop. It causes the full time-to-return functions $T_n(h)$ to become unbounded when $h \uparrow h_{sep}$. The functions will have a discrete set of distinct values when $h \downarrow h_A$. These two properties combined with the continuity of the functions causes the

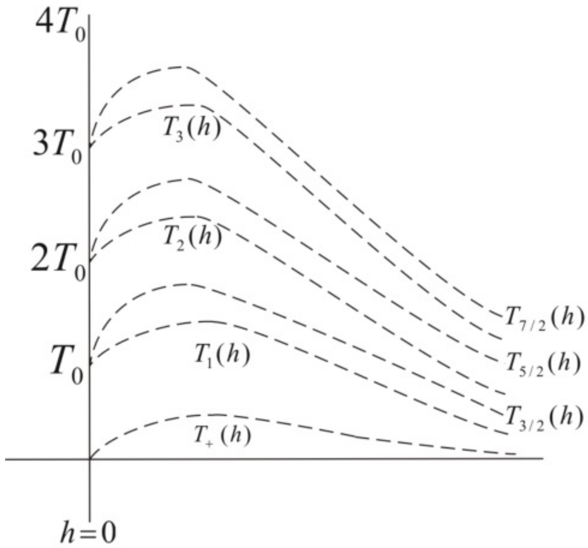

Figure 8.7: Simultaneous solutions to a boundary value problem with $f(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3$, $A = 0$ where the period annulus is unbounded.

finiteness of solutions, i.e. a finite number of intersections for any horizontal line with the collection of graphs of $T^A_{n+1/2}(h)$ and $T_n(h)$.

It leaves the problem to determine in which situations this conclusion cannot be drawn. This could happen in the case when the period annulus is not bounded by a finite solution curve. A typical example is the case of an unbounded period annulus with the property that $f(x) \to \pm\infty$ as $\mathcal{O}(x^{1+\alpha})$ when $x \to \pm\infty$ with $\alpha > 0$. In such a case the time-to-return function will approach 0 for large h instead of ∞ (as was the case for a saddle loop). If $T_{full}(h)$ tends to 0 instead of ∞, then each of the functions $T^A_{n+1/2}(h)$, $T_n(h)$ will approach 0. It implies that for each λ there will be an infinite number of intersections with the graphs of the functions $T^A_{n+1/2}(h)$, $T_n(h)$. Therefore the original boundary value problem has an infinite number of solutions. It is outside the scope of this paper to give a full classification of all the different structure types for the simultaneous solutions of equation (1.1) with boundary condition (1.2) for arbitrary $f(x)$, but the above argument can be extended to achieve this. Moreover, the existence of an S-shaped bifurcation can be generalized as well to any case of $f(x)$ such that a period annulus occurs with a center singularity on the inside and a finite loop formed by the separatrices of two saddles.

In Figure 8.7 we sketch an example of a case with infinitely many solutions for $f(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3$, $A = 0$ as numerically discussed in [32].

9 Representation of bifurcations in phase plane

It is possible to represent the bifurcations of the different mixed solution types in a transparent way as a function of A by making the following observation:

Lemma 9.1. *Suppose system (2.3) has a period annulus in the phase plane surrounding a singularity of center type, represented by the integral curve given in (2.5) on some interval $h \in (h_{min}, h_{max})$. Then for given h on the orbit γ_h in the phase plane there exists for each of the functions $T^A_{n+1/2}(h)$ exactly one point $\left(x = A^{bif}_{(2n+1)/2}, y = y(A^{bif}_{(2n+1)/2}, h)\right) \equiv (x^{bif}, y^{bif})$, such that the boundary value problem (1.1)* *with boundary condition* (1.2) *where* $A = x^{bif}$ *and* $\frac{dx(t)}{dt}|_{t=0} = y^{bif}$ *has a bifurcation value* $\lambda = \lambda^{bif}$*. The bifurcation points of the boundary value problem can be represented by a curve* $\mu^{n+1/2}(h)$ *in the phase plane intersecting the period annulus transversally. The case* $n = 0$ *is included representing the positive time-to-return function* $T_+^A(h)$*.*

Proof. For a given periodic orbit, i.e. fixed h, the domain of A-values is given by (A_-, A_+) where $F(A_\pm) = 0$. The periodic orbit in a period annulus needs to intersect the x-axis in exactly two points defined by $F(x) = 0$ and we indicate those two x-values by A_- and A_+. See Figure 9.1. Equation (4.7) in Lemma 4.2 shows an expression for $\frac{dT_+^A(h)}{dh}$. At the end points necessarily $F(A) = h_A$ and in the expression $\frac{dT_+^A(h)}{dh} = \frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}} + \int_{y=0}^{y=\sqrt{h-h_A}} \omega'(x(y))dy$ the term $\frac{\sqrt{2}h_A}{f(A)\sqrt{h-h_A}}$ will blow up when A approaches the boundary values. For a periodic orbit in a period annulus necessarily $f(A_-) < 0$ and $f(A_+) > 0$ and we conclude that $\lim_{A\to A_-,A+} \frac{dT_+^A(h)}{dh} = \mathrm{sign}(f(A_\pm))\infty = \mp\infty$. According to (4.9) $T_+^A(h)$ is monotonically decreasing as a function of A, showing that there exists exactly one value A such that $\frac{dT_+^A(h)}{dh} = 0$. This argument holds true if $f(x)$ has a unique zero on the relevant x-interval. If $f(x)$ has multiple zeroes, the period annulus is bounded on the inside by a solution curve consisting of separatrices from one or more saddle(s). In that case the conclusion will be more complicated, which is outside the scope of this paper.

The same argument applies to the function $T_{n+1/2}^A(h) = T_+^A(h) + nT_{full}(h)$ because $T_{full}(h)$ does not depend on A and does not influence the behaviour near the end points A_- and A_+ and the monotonicity of the derivative with respect to A. It follows that for each $n = 1, 2, 3...$, fixed h, there is a unique $A_{(2n+1)/2}^{bif}$ such that $\frac{dT_{n+1/2}^A(h)}{dh} = 0$, i.e. a bifurcation value for the original boundary value problem. □

There are many other properties of the bifurcation curves $\mu^{n+1/2}(h)$ mentioned in Lemma 9.1 that can be derived, but they are out of scope for this paper. We briefly indicate some results which are not difficult to prove using the formulas in Lemma 4.2:

- If $\frac{dT_{full}(h)}{dh} > 0$ (< 0) then the bifurcation points $(x = A_{(2n+1)/2}^{bif}, y = y(A_{(2n+1)/2}^{bif}, h))$ are ordered counter-clockwise (clockwise) on the periodic orbit for increasing n. If $\frac{dT_{full}(h)}{dh} = 0$ the points $(x = A_{(2n+1)/2}^{bif}, y = y(A_{(2n+1)/2}^{bif}, h))$ collapse into a single point on the periodic orbit. Figure 9.1 shows the three situations.

- If the period annulus has a singularity of center type as its inner boundary, then the curve $\mu^{n+1/2}(h)$ approaches the center in the phase plane along a vertical tangent direction. Figure 9.2 shows a sketch of this for the case $f(x) = x(1 + x)$.

- If the curve $\mu^{n+1/2}(h)$ moves to the right (left) for increasing h, then the bifurcation point corresponds to a local maximum (minimum) of the function $T_{n+1/2}^A(h)$. If the curve $\mu^{n+1/2}(h)$ has a vertical tangent line (i.e. it is changing direction in the phase plane), then a local maximum and minimum coincide to form a inflection point on $T_{n+1/2}^A(h)$.

- If a vertical line $x = A$ in the phase plane intersects $\mu^{n+1/2}(h)$ in two points, then an S-shaped bifurcation takes place. See Figure 9.2 where the situation is sketched for the case of $f(x) = x(1 + x)$. The results in the figure have been confirmed numerically. It is clearly visible how for $A = \epsilon > 0$ the situation occurs as was discussed in the previous sections.

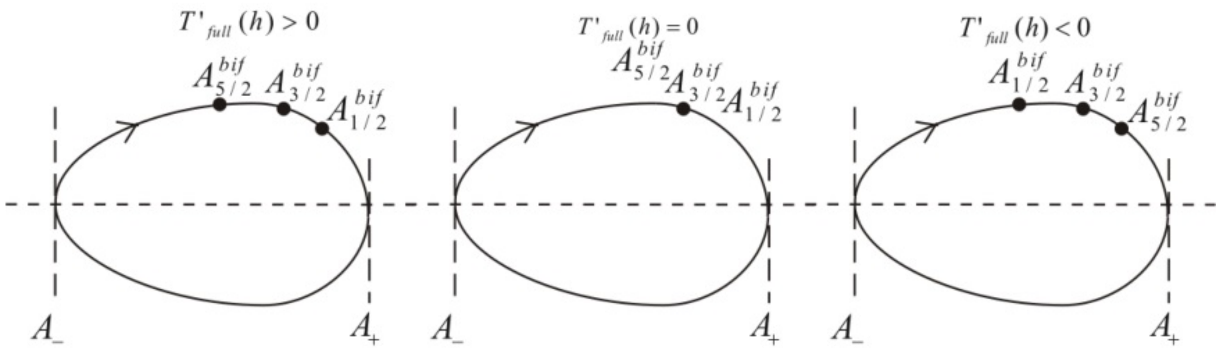

Figure 9.1: Ordered bifurcation points on a periodic orbit in a period annulus corresponding to the different types of time-to-return functions.

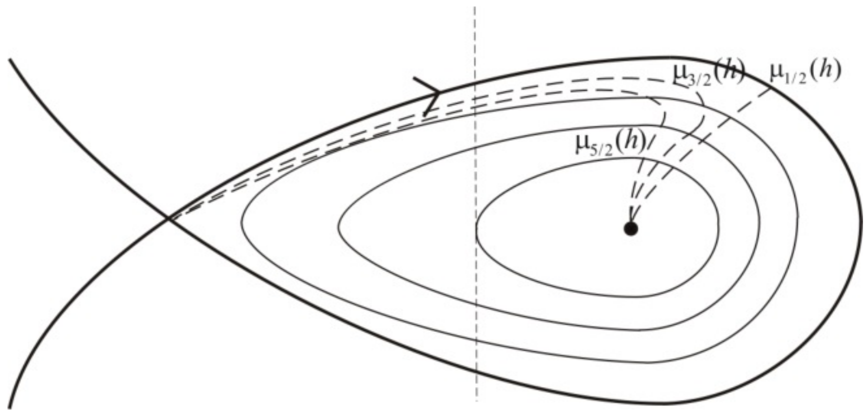

Figure 9.2: Schematic display of the different types of bifurcation curves shown in the phase plane for the quadratic Hamiltonian case $f(x) = x(1+x)$.

10 Discussion

In this paper we studied mixed solutions of a nonlinear ordinary differential equation with Dirichlet boundary conditions. The purpose was to show that generically complex bifurcation phenomena occur, even for the most simple nonlinear choice i.e. $f(x) = x(1+x)$. The obvious question remains how these results extend to more complex cases. The following topics for further study come to our mind.

1) *Generalizations*

The results of this paper do not only apply to the case of a saddle loop surrounding the period annulus with a center inside. A full categorization for all solution types in the case of the general structure of $f(x)$ is feasible and should lead to similar results as in this paper. In particular we would like to point out the condition $f''(0) > 0$ which is necessary for the mixed solutions to have an S-shaped bifurcation near the center singularity. If $f''(0) < 0$, then it is not difficult to show that an S-shaped bifurcation will occur for the negative mixed solutions $S^A_{-(n+1/2)}$, where $n = 1, 2, 3, \ldots$ It implies that if $f''(0) \neq 0$ near a center singularity then always an S-shaped bifurcation can be found among the mixed solutions.

2) *Relation between the different time-to-return functions*

There is a relation between the positive, negative and full time-to-return functions: $T_{full}(h) = T_+^A(h) + T_-^A(h)$. This indicates that even though for all solution types different phenomena occur there is still some intrinsic relation between them. For example, the expansion of the functions near the center singularity, i.e. $h \downarrow 0$ has an interesting structure caused by this relationship. The full period function is analytical in h, while the positive time-to-return function $T_+^A(h)$ is analytical in the variable $\sqrt{h}$ (see the expansion in Proposition 4.1). For the negative time-to-return function a similar result holds. The structure becomes:

$$T_{full}(h) = T_0 + c_1 h + c_2 h^2 + \dots$$

$$T_+^A(h) = \frac{1}{2}(T_0 + c_1 h + c_2 h^2 + \dots) + \sqrt{h}(d_0 + d_1 h + d2_h^2 + \dots)$$

$$T_-^A(h) = \frac{1}{2}(T_0 + c_1 h + c_2 h^2 + \dots) - \sqrt{h}(d_0 + d_1 h + d2_h^2 + \dots)$$

It would be interesting to extend the analysis for the local bifurcation of small-amplitude critical periods for the full period function (for which an extensive literature exists) to the cases of the positive and negative time-to-return functions and the different types of mixed time-to-return functions.

3) *Proving upper bounds*

This paper mainly addressed the existence of solutions without considering the upper bounds on the number of solutions. For example in the case of the quadratic Hamiltonian $x(1+x)$ the conjecture is that at most three solutions can occur for each type of mixed solution. The difficulty in proving this lies in the fact that the function contains the full period function for which the depending parameter is h and the positive time-to-return function for which the natural depending parameter is $x_+(h)$ (see the proof of Proposition 4.1). In order to study the mixed functions a way must be found to combine the different techniques for these two functions.

Acknowledgements

The authors would like to thank the referee for his useful comments which helped improve this paper.

References

[1] T. Bakri, Y. A. Kuznetsov, F. Verhulst, E. Doedel, Multiple solutions of a generalized singularly perturbed Bratu problem, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22**(2012), No. 4, 1250095, 10 pp. https://doi.org/10.1142/S0218127412500952; MR2926072

[2] G. Bratu, Sur les équations intégrales non linéaires (in French), *Bull. Soc. Math. France* **42**(1914), 113–142. MR1504727

[3] J. Bebernes, Solid fuel combustion-some mathematical problems, *Rocky Mount. J. Math.* **16**(1986), No. 3, 417–433. https://doi.org/10.1216/RMJ-1986-16-3-417; MR862274

[4] C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, *J. Differential Equations* **69**(1987), No. 3, 310–321. https://doi.org/10.1016/0022-0396(87)90122-7; MR903390

[5] C. Chicone, Geometric methods for two-point nonlinear boundary value problems, *J. Differential Equations* **72**(1988), 360–407. https://doi.org/10.1016/0022-0396(88)90160-X; MR932371

[6] C. Chicone, M. Jacobs, Bifurcation of critical periods for plane vector fields, *Trans. Amer. Math. Soc.* **312**(1989), No. 2, 433–486. https://doi.org/10.2307/2000999; MR930075

[7] C. V. Coffman, On the positive solutions of boundary-value problems for a class of nonlinear differential equations, *J. Differential Equations* **3**(1967), 92–111. https://doi.org/10.1016/0022-0396(67)90009-5; MR0204755

[8] W. A. Coppel, L. Gavrilov, The period function of a Hamiltonian quadratic system, *Differential Integral Equations* **6**(1993), No. 6, 1357–1365. MR1235199

[9] F. Dumortier, B. Smits, Transition time analysis in singularly perturbed boundary value problems, *Trans. of Amer. Math. Soc.* **347**(1995), No. 10, 4129–4145. https://doi.org/10.2307/2155217; MR1308009

[10] I. M. Gel'fand, Some problems in the theory of quasi-linear equations (in Russian), *Uspehi Mat. Nauk* **14(86)**1959, No. 2, 87–158, translated in *Amer. Math. Soc. Transl. Ser. 2* **29**(1963), 295–381. MR0110868

[11] S.-Y. Huang, S.-H. Wang, Proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory, *Arch. Ration. Mech. Anal.* **222**(2016), No. 2, 769–825. https://doi.org/10.1007/s00205-016-1011-1; MR3544317

[12] S.-Y. Huang, K.-C. Hung, S.-H. Wang, A global bifurcation theorem for a multiparameter positone problem and its application to the one-dimensional perturbed Gelfand problem, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 99, 1–25. https://doi.org/10.14232/ejqtde.2019.1.99; MR4049574

[13] K.-C. Hung, S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex–concave nonlinearity and its applications to the perturbed Gelfand problem, *J. Differential Equations* **251**(2011), No. 2, 223–237. https://doi.org/10.1016/j.jde.2011.03.017; MR2800152

[14] K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, *Trans. Amer. Math. Soc.* **365**(2013), No. 4, 1933–1956. https://doi.org/10.1090/S0002-9947-2012-05670-4; MR3009649

[15] D. D. Joseph, Non-linear heat generation and stability of the temperature distribution in conducting solids, *Int. J. Heat Mass Transf.* **8**(1965), 281–288. https://doi.org/10.1016/0017-9310(65)90115-8

[16] H. B. Keller, D. S. Cohen, Some positone problems suggested by nonlinear heat generation, *J. Math. Mech.* **16**(1967), 1361–1376. MR0213694

[17] M. A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff, Ltd, Groningen, the Netherlands, 1964. MR0181881

[18] M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Macmillam, New York, 1964. MR0159197

[19] T. Laetsch, On the number of solutions of boundary value problems with convex nonlinearities, *J. Math. Anal. Appl.* **35**(1971), 389–404. https://doi.org/10.1016/0022-247X(71)90226-5; MR0280869

[20] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.* **20**(1970/1971), No. 1, 1–13. https://doi.org/10.1512/iumj.1970.20.20001; MR0269922

[21] C. Li, K. Lu, The period function of hyperelliptic Hamiltonian of degree 5 with real critical points, *Nonlinearity* **21**(2008), No. 3, 465–483. https://doi.org/10.1088/0951-7715/21/3/006; MR2396613

[22] P. De Maesschalck, F. Dumortier, The period function of classical Liénard equations, *J. Differential Equations* **233**(2007), No. 2, 380–403. https://doi.org/10.1016/j.jde.2006.09.015; MR2292512

[23] F. Mañosas, J. Villadelprat, Criteria to bound the number of critical periods, *J. Differential Equations* **246**(2009), No. 6, 2415–2433. https://doi.org/10.1016/j.jde.2008.07.002; MR2498846

[24] P. Mardešić, D. Marin, J. Villadelprat, *The period function of reversible quadratic centers*, *J. Differential Equations* **224**2006, No. 1, 120–171. https://doi.org/10.1016/j.jde.2005.07.024; MR2220066

[25] J. B. Mcleod, S. Sadhu, Existence of solutions and asymptotic analysis of a class of singularly perturbed odes with boundary conditions, *Adv. Differential Equations* **18**(2013), No. 9–10, 825–848. MR3100053

[26] G. H. Pimbley, H. George, A sublinear Sturm–Liouville problem, *J. Math. Mech.* **11**(1962), 121–138. MR0138820

[27] J. Smoller, A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations* **39**(1981), No. 2, 269–290. https://doi.org/10.1016/0022-0396(81)90077-2; MR607786

[28] C.-C. Tzeng, K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity, *J. Differential Equations* **252**(2012), No. 12, 6250–6274. https://doi.org/10.1016/j.jde.2012.02.020; MR2911833

[29] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, *J. Differential Equations* **77**(1989), No. 1, 199–202. https://doi.org/10.1016/0022-0396(89)90162-9; MR980548

[30] S.-H. Wang, On S-shaped bifurcation curves, *Nonlinear Anal.* **22**(1994), No. 12, 1475–1485. https://doi.org/10.1016/0362-546X(94)90183-X; MR1285087

[31] L. Yang, X. Zeng, The period function of potential systems of polynomials with real zeros, *Bull. Sci. Math.* **133**(2009), No. 6, 555–577. https://doi.org/10.1016/j.bulsci.2009.05.002; MR2561363

[32] P. A. Zegeling, S. Iqbal, Nonstandard finite differences for a truncated Bratu–Picard model, *Appl. Math. Comput.* **324**(2018), 266–284. MR3743672; https://doi.org/10.1016/j.amc.2017.12.005

[33] Ya. B. Zeldovich, G. I. Barenblatt, V. B. Librovich, G. M. Makhviladze, *The mathematical theory of combustion and explosions*, Consultants Bureau [Plenum], New York, 1985. MR781350