K3 seminar talk 2: Complex geometry & Hodge structures.

References: Ch.3 of the K3 book, Complex geometry } Huybrechts.

Def A kähler form on a complex manifold M is real 2-form ω which is non degenerate, closed and $g(u,v) = \omega(u, iv)$ is a symmetric, positive definite form.

↳ Gives $[\omega] \in H^2(M, \mathbb{R})$ via de Rham cohomology.

A manifold is Kähler if it admits a kähler form.

Examples

- $\mathbb{CP}^n$ is kähler
- Closed submanifolds of kähler mflds are kähler

} Proj. mflds are kähler.

- All K3 surfaces are kähler (nontrivial)

Rmk if W is a vector space of the form $V \otimes \mathbb{C}$, then there is a complex conjugation operation $v \otimes z \mapsto v \otimes \bar{z}$.

Thm (Hodge decomposition). If X is compact kähler, then there is a decomposition

$$H^n(X, \mathbb{R}) \otimes \mathbb{C} = H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^q(X, \Omega_X^p)$$

in which $\overline{H^q(X, \Omega_X^p)} = H^p(X, \Omega_X^q)$.

Recall: $b_i(X) = \dim H^n(X, \mathbb{C})$

$h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$.

Then: $b_n(X) = \sum_{p+q=n} h^{p,q}(X)$

For a K3 surface: $b_2(X) = h^{0,2} + h^{1,1} + h^{0,2} = 1+20+1 = 22$.

<u>Def</u> A rational / integral / real Hodge structure of weight n is a free $\mathbb{Q}/\mathbb{Z}/\mathbb{R}$-module V together with a decomposition

$$V_{\mathbb{C}} = V \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Example: $H^n(X, \mathbb{Q})$ and $H^n(X, \mathbb{R})$, where X is compact Kähler.
$H^n(X, \mathbb{Z})$ if this is free (for example, when X is K3).

The Tate Hodge structure $\mathbb{Z}(1)$ of weight -2 is defined by setting

$$(\mathbb{Z}(1))^{-1,-1} = \mathbb{C}$$

$$(\mathbb{Z}(1))^{p,q} = 0 \quad \text{otherwise.}$$

A <u>morphism</u> of Hodge str. is a linear map $f: V \longrightarrow V'$ such that

$$(f \otimes \mathbb{C})(V^{p,q}) \subseteq V'^{p,q}.$$

(So, V and V' have the same weight.)

Some operations.

- Direct sum. If V, V' are of weight n, then $V \oplus V'$ has Hodge str.

$$(V \oplus V')^{p,q} = V^{p,q} \oplus V'^{p,q}.$$

- Tensor product If V is of weight n, V' of weight m, then $V \otimes V'$ is of weight $n+m$ and $(V \otimes V')^{p,q} = \bigoplus V^{r,s} \otimes V'^{t,u}$

$r+t = p$
$s+u = q$

$r+s = n$
$t+u = m$

If $x \in V^{r,s}$ and $y \in V'^{t,u}$

then $x \otimes y$ is in $(V \otimes V')^{r+t,\, s+u}$.

- Exterior product defined in the same way.

$$x \in V^{r,s}, \quad y \in V'^{t,u}$$
$$x \wedge y \in (V \wedge V')^{r+t,s+u}.$$

- Duals: if V is of weight n, then V^* is of weight $-n$ and
$$(V^*)^{p,q} = \mathrm{Hom}(V^{-p,-q}, \mathbb{C}) = (V^{-p,-q})^*$$

- Homs: $\mathrm{Hom}(V, V') = V^* \otimes V'$.

If V is any vector space, can make V into a Hodge str of weight $2k$ by setting
$$V^{k,k} = V \otimes \mathbb{C}$$
$$V^{p,q} = 0 \text{ otherwise.}$$

Def If V is of weight n and V' is of weight $n+2k$, then a morphism of weight k is a linear map $f: V \to V'$ such that
$$f(V^{p,q}) \subseteq V'^{p+k,q+k}.$$

Equivalently, it is a morphism $V \to V'(k) = V' \otimes \mathbb{Z}(1)^{\otimes k}$

Let X be smooth projective var $/\mathbb{C}$. If $Z \subseteq X$ is a subvariety, we get a fundamental class $[Z] \in H^{2k}(X, \mathbb{Z})$, where $k = \operatorname{codim} Z$.

In fact, $[Z] \in H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X)$.

Conj. (Hodge) $\langle [Z] \mid Z \subseteq X \text{ of codim } k \rangle_{\mathbb{Q}} = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$.

This is known when $k=1$.

Def The Weil operator C is defined on $V_{\mathbb{C}}$ by acting on $V^{p,q}$ by multiplication by i^{p-q}.

Rmk if $v \in V_{\mathbb{R}}$, then $Cv \in V_{\mathbb{R}}$. If $v = w + \bar{w}$ for $w \in V^{p,q}$, then

$$Cv = C(w+\bar{w}) = i^{p-q} w + i^{q-p} \bar{w} = i^{p-q} w + \overline{i^{p-q} w}.$$

In general, $v = \sum w_i + \bar{w}_i$.

Def V of weight n. A polarisation on V is a morphism

$$\psi : V \otimes V \to \mathbb{Z}(-n)$$

such that $\psi(u, Cv)$ is a positive def. symmetric form on $V_{\mathbb{R}}$.

Let X be compact, connected Kähler. Then $H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$ and is generated by $[\{p\}]$.
$$\lambda \mapsto \int_X \lambda.$$

Try to build a polarisation on $H^k(X, \mathbb{R})$ for $k \leq n$.

$$Q: H^k(X, \mathbb{R}) \otimes H^k(X, \mathbb{R}) \to H^{2k}(X, \mathbb{R}) \xrightarrow{\cdot \omega^{n-k}} H^{2n}(X, \mathbb{R}) \xrightarrow{\int_X} \mathbb{R}.$$

Def The primitive cohomology $H^k(X, \mathbb{R})_p$ is the kernel of multiplication by ω^{n-k+1}.

Thm (Hodge-Riemann bilinear relations) For $\alpha \in H^{2k}(X, \mathbb{R}_p)$ of type (p,q) we have

$$i^{p-q} (-1)^{n(n-1)/2} Q(\alpha, \bar{\alpha}) > 0.$$

Prop $\psi : V \otimes V \to \mathbb{R}(-n)$ is a polarisation iff

(1) $\psi(u,v) = (-1)^n \psi(v,u)$

(2) $i^{p-q} \psi(\alpha, \bar{\alpha}) > 0$ for $\alpha \in V^{p,q}$.

$(-1)^{n(n-1)/2}\, Q$ defines a polarisation on $H^{2k}(X, \mathbb{R})_p$.

For K3 surfaces.

$\rightarrow$ Get a polarisation $-Q$ on $H^2(X, \mathbb{R})_p$.

Hard Lefschetz thm: $H^2(X, \mathbb{R}) = \underbrace{H^2(X, \mathbb{R})_p \oplus \omega \cdot \mathbb{R}}$.

$\hookrightarrow$ this is orthogonal.

If $\alpha \in H^2(X, \mathbb{R})_p$, then $-Q(\alpha, \omega) = -\int_X \alpha \cdot \omega = \int_X 0 = 0$.

Also, $-Q(\omega, \omega) = -\int_X \omega^2 < 0$.

Solution: change Q by a sign on $\omega \cdot \mathbb{R}$. $\rightarrow$ Get polarisation.

This is on $H^2(X, \mathbb{R})$. Does it extend to $H^2(X, \mathbb{Q})$?

It does when $\omega \in H^2(X, \mathbb{Q})$.

But: X admits a rational Kähler class iff it is projective.

(Kodaira embedding thm.)

(iff it is algebraic).

So for algebraic K3 we can do it.

But there are non-algebraic K3 surfaces admitting a rational polarisation.

Thm There is an equivalence

$\{$ integral Hodge str. of weight 1 $\}$ $\leftrightarrow$ $\{$ Complex tori $\}$

$$H^1(A,\mathbb{Z}) \longleftarrow\!\shortmid\ A$$

$$V \longmapsto \operatorname{coker}(V \hookrightarrow V_{\mathbb{C}} \to V^{1,0})$$

V is polarisable iff A is projective/algebraic.

<u>Thm</u> Two curves C, C' are isomorphic iff $H^1(C,\mathbb{Z}) \cong H^1(C',\mathbb{Z})$ in away which respects the intersection pairing.

<u>Thm</u> Two K3 surfaces X, X' are isomorphic iff $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$.

<u>Thm</u> $H^2(X,\mathbb{Q}) \cong H^2(X',\mathbb{Q})$ iff $\mathcal{D}(X) \cong \mathcal{D}(X')$

Fun fact: for a fixed X, there are finitely many possibilities for X'.

<u>Def</u> V of weight 2 is of <u>K3 type</u> if $V^{2,0}$ and $V^{0,2}$ are one-dim. and $V^{p,q}=0$ if $p<0$ or $q<0$.

(So, $V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$)

Ex: - $H^2(X,\mathbb{Z})$ for X K3.

- $H^2(A,\mathbb{Z})$ for A an abelian surface.

$$\begin{aligned} &\,\,\parallel \\ &\Lambda^2 H^1(A,\mathbb{Z}). \end{aligned}$$

then $H^{2,0}(A,\mathbb{Z}) = \Lambda^2 H^{1,0}(A,\mathbb{Z})$.

$\cong \Lambda^2 \mathbb{C}^2 \cong \mathbb{C}$.

If V is of K3 type, the minimal (primitive) sub Hodge str. of K3 type is called the trancendental lattice T.

Thm $H^2(X,\mathbb{Q}) \cong H^2(X',\mathbb{Q})$ iff $T(X) \cong T(X')$.

(iff $D(X) \cong D(X')$).

Take A an abelian surface. $\iota : A \to A$ $x \mapsto -x$.

Define $\tilde{A}$ to be the blowup of A in the fixed points and $X = \tilde{A}/\iota$

We know $H^2(\tilde{A},\mathbb{Z}) = H^2(A,\mathbb{Z}) \oplus \bigoplus_{i=1}^{16} \mathbb{Z}\cdot[E_i]$.

ι acts on $H^2(\tilde{A},\mathbb{Z})$, but the action is trivial.

Fact: $\pi_* H^2(A,\mathbb{Z}) \subseteq H^2(X,\mathbb{Z})$ is primitive (because it is the orthogonal complement of the images of the $[E_i]$. (Beauville)).

So, $T(A) \cong T(X)$.

But the intersection pairing does not agree.

If α, α' are in $H^2(A,\mathbb{Z})$, write $\alpha = \pi^*\beta$, $\alpha' = \pi^*\beta'$.

$$(\pi_*\alpha \cdot \pi_*\alpha') = (\pi_*\pi^*\beta, \pi_*\pi^*\beta')$$
$$= (2\beta \cdot 2\beta')$$
$$= 4(\beta\cdot\beta').$$

$$(\alpha\cdot\alpha') = (\pi^*\beta \cdot \pi^*\beta') = 2\cdot(\beta\cdot\beta').$$

Problem from last week:

$$\iota : A \to A \rightsquigarrow \iota : A^t \to A^t$$

Then the action of ι on the tangent space at $0 \in A^t$ is also given by -1.

But $H^1(A, \mathcal{O}_A) \cong T_0 A^t$.