K3 SURFACES SEMINAR

ELLIPTIC K3 SURFACES

[Huy. Ch. 11; 1,2,3
Miranda BTES]

1) Elliptic curves : recap. of facts
2) Elliptic surfaces
3) Fibres of elliptic K3 surfaces
4) Weierstrass model of an elliptic K3 surf
5) Mordell-Weil Group

1. ELLIPTIC CURVES

Let k be a field, char$(k) \neq 2,3$

Def: An elliptic curve is a pair (E, O), where E is a smooth projective curve of genus 1 over k and $O \in E(k)$

E can be described by

Weierstrass equation

$$E_W: \; y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad \text{in } \mathbb{P}^2_k.$$

$\boxed{O = \text{point at } \infty}$

char$(k) \neq 2,3$

Under a linear coordinate change ...

$$\boxed{y^2 = 4x^3 - g_2x - g_3}$$

Simplified Weierstrass Equation

equation

Remark: (g_2, g_3) unique up to $(\lambda^4 g_2, \lambda^6 g_3), \lambda \in k^*$

Fact: We have a group structure on $E(k)$ where O is the neutral element

Invariants:

$$\boxed{\Delta(E) := g_2^3 - 27 g_3^2}$$ Discriminant

$$\boxed{j(E) := 1728 \frac{g_2^3}{\Delta(E)}}$$ j-invariant

FACTS

i) E_W is smooth iff $\Delta(E_W) \neq 0$

ii) $\Delta(E_W)$ is unique up to λ^{12}, $\lambda \in k^*$

iii) $E \simeq E'$ iff $j(E) = j(E')$. k alg. closed.

Minimal W. equation for E:

Now, let k have a discrete valuation, v.

Then we say that a Weierstrass equation for E_W is minimal iff $v(a_i) \geq 0 \ \forall i$ and $v(\Delta)$ is minimal.

E_W is either:

smooth, nodal or cuspidal.

CONVENTIONS

k = algebraically closed field

char $(k) \neq 2, 3$

X = algebraic K3 surface / k
(in part. projective)

2. ELLIPTIC SURFACES

Def: An elliptic surface is a projective surface X together with a surjective morphism $\pi : X \longrightarrow C$ with

- C is an smooth projective curve
- the generic fibre is smooth integral curve of genus 1.

If X is a K3 $\Rightarrow$ $C \simeq \mathbb{P}^1$

Hence:

Def: An elliptic K3 surface is a K3 surface X together with a surjective morphism $\pi : X \longrightarrow \mathbb{P}^1$ s.t there exists a closed point $t \in \mathbb{P}^1$ with

$X_t := \pi^{-1}(t)$ is integral smooth curve of genus 1.

FACTS:

- π is a flat morphism $\Rightarrow$ the Hilbert polynomials P_{X_t} are equal $\forall X_t$ $\Rightarrow \chi(\mathcal{O}_{X_t}) = 0$ $\forall X_t$
  $\Rightarrow p_a(X_t) = 1$ $\forall X_t$.
- The set of elliptic K3's is dense in the moduli space of complex K3's. polarized K3's.
- A K3 surface admits an elliptic fibration if
  i) $\exists L$ line bundle, $L^2 = 0$.
  ii) $\rho(X) \geq 5$.

EXAMPLES

i) $A = E_1 \times E_2$, ; E_1, E_2 elliptic curves

$$\iota : E_1 \times E_2 \longrightarrow E_1 \times E_2$$
$$(P_1, P_2) \longmapsto (-P_1, -P_2)$$

$\widetilde{E_1 \times E_2}$ $\Big\{$ blow up of $E_1 \times E_2$ in the fixed points of ι

$$\boxed{X = \widetilde{E_1 \times E_2} / \tilde{\iota}}$$ Kummer surface.

$i = 1, 2$ $\quad \pi_i : X \longrightarrow E_i / \pm \simeq \mathbb{P}^1$

↳ These are elliptic fibrations

(ii) Fermat quartic

$$X \subseteq \mathbb{P}^3 \; ; \; x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

If we rewrite $\boxed{\zeta \text{ prim } 8\text{th root}}$

$$(x_0^2 + \zeta^2 x_1^2)(x_0^2 - \zeta^2 x_1^2) + (x_2^2 + \zeta^2 x_3^2)(x_2^2 - \zeta^2 x_3^2) = 0$$

The map

$$\pi : X \longrightarrow \mathbb{P}^1$$
$$[x_0 : x_1 : x_2 : x_3] \longmapsto [x_0^2 + \zeta^2 x_1^2 : x_2^2 + \zeta^2 x_3^2]$$

↳ elliptic fibration.

3. FIBRES OF AN ELLIPTIC K3 SURFACE

Notice:

(i). If x_t is smooth, $\mathcal{O}(x_t) \simeq \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$.

so $\mathcal{O}(x_t)|_{x_t} \simeq \mathcal{O}_{x_t}$.

Using the adjunction formula

$$\boxed{\omega_{x_t}} = (\omega_X \otimes \mathcal{O}_X(x_t))|_{x_t} \simeq (\mathcal{O}_X \otimes \mathcal{O}_X(x_t))|_{x_t} \simeq$$
$$\simeq \mathcal{O}_X(x_t)|_{x_t} \simeq \boxed{\mathcal{O}_{x_t}}.$$

(ii) Not all the fibres are smooth.

If π is smooth $\Rightarrow$ $h^{1}(X, \mathbb{Z}) = 1$ Contradiction.

Proposition: $\pi : X \to \mathbb{P}^1$ elliptic K3.

(i) $\pi_* \mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}^1}$ and $R^1 \pi_* \mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$.

(ii) All the fibres X_t are connected.

(iii) No fibre is multiple.

(iv) If $X_t = \sum_{i=1}^{\ell} m_i C_i$, C_i are integral.

- $(C_i . X_t) = 0$
- $(\underbrace{\sum n_i C_i}_{C})^2 \leq 0 \quad \forall n_i \in \mathbb{Z}$ $\; (=)$ if C is a multiple of X_t.

proof (sketch).

(ii) Is a consequence $h^1(X_t, \mathcal{O}_{X_t}) = 1$

(iii) Is a consequence of Riemann-Roch.

(i) Take X_t to be smooth. From the seq.

$$0 \to \mathcal{O}(-X_t) \to \mathcal{O}_X \to \mathcal{O}_{X_t} \to 0$$

And the cohom. long exact seq:

$$0 \to \underbrace{\cancel{H^0(X, \mathcal{O}_X(-X_t))}}_{\text{Kod. Van.}\ 0} \to \underbrace{H^0(X, \mathcal{O}_X)}_{\simeq k} \to H^0(X, \mathcal{O}_{X_t}) \simeq H^0(X_t, \mathcal{O}_{X_t}) \simeq k$$

0   0 $\simeq H^0(X_t, \mathcal{O}_{X_t}) \simeq k$

$$\to \cancel{H^1(X, \mathcal{O}_X(-X_t))} \to \cancel{H^1(X, \mathcal{O}_X)} \to H^1(X, \mathcal{O}_{X_t})$$

$$\to \underset{\substack{\cong\\ k^2}}{H^2(X, \mathcal{O}_X(-X_t))} \to \underset{\substack{\cong\\ k}}{H^2(X, \mathcal{O}_X)} \to \underset{\substack{\cong \\ H^0(X_t, \mathcal{O}_{X_t}) \\ \| \\ 0}}{H^2(X, \mathcal{O}_{X_t})}.$$

$\leftarrow$ betti numbers

— we know.

— we get.

Now, $H^i(X, \mathcal{O}_X(-X_t))$ are all the same.

$$h^0(X_t, \mathcal{O}_{X_t}) = h^1(X_t, \mathcal{O}_{X_t}) = 1 \quad \forall X_t$$

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \pi_* \mathcal{O}_X$$ this is an <u>isomorphism</u>.

[Hartshorne] $\Rightarrow$ π surjecitve

$$R^1 \pi_* \mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}}(d) \quad d \in \mathbb{Z}$$

Leray. $$\underset{\substack{\cong \\ k}}{H^2(X, \mathcal{O}_X)} \cong H^1(\mathbb{P}^1, R^1\pi_* \mathcal{O}_X).$$

[stacks project]. $\dim(H^1(\mathbb{P}^1, \mathcal{O}(d))) = 1$ iff $d = -2$

$\square$.

(iv) For any $X_t = \sum m_i C_i$

We use that $\boxed{\omega_{X_t} = \mathcal{O}_{X_t}}$ and $\mathcal{O}(X_t)|_{C_i} \simeq \mathcal{O}_{C_i}$

$\Rightarrow$ to show that $(X_t \cdot C_i) = 0$

The second part is computational

$\square$.

Corollary: $X_t \xrightarrow{} \text{fibre}$ is one of the following

- X_t irreducible
- $X_t = \sum_{i=1}^{\ell} m_i C_i$ with C_i (-2)-curves $\to$ integral. and $C_i \cong \mathbb{P}^1$. and $(m_1, \dots, m_\ell) = 1$

proof: If $X_t = C + C'$

$$0 \to \mathcal{O}_{X_t}(C) \to \mathcal{O}_{X_t} \to \mathcal{O}(\cancel{X_t})|_{C'} \to 0.$$

$\dim\left(\frac{X_t^2}{2} + 2\right) = 2$

$$\Rightarrow 0 \to \boxed{H^0(X, \mathcal{O}(C))} \to H^0(X, \mathcal{O}(X_t)) \to$$

$\sim k.$ $\qquad$ $\simeq k^2$

$$\to H^0(X, \mathcal{O}(X_t)|_{C'}) \to \cancel{H^1(X, \mathcal{O}(C))}.$$

0 Ram. kod vanishing

— we know. $\mathcal{O}_{C'} \cong k$

— we get $\square$

CLASSIFICATION OF X_t : graphs

The dual graph of a curve $C = \sum n_i C_i$ consists of

vertices: $\{C_i\}_i$

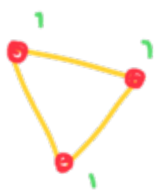
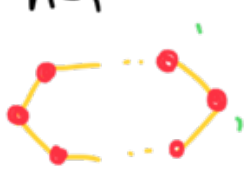
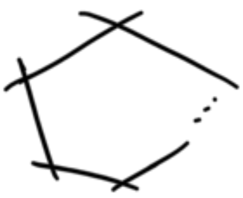
edges: C_i, C_j are connected by $(C_i . C_j)$. edges.

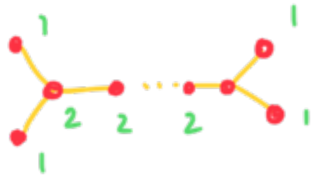
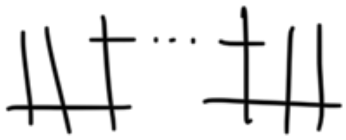
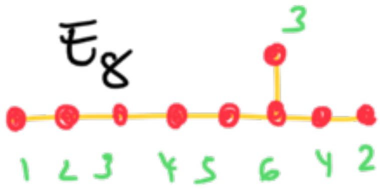
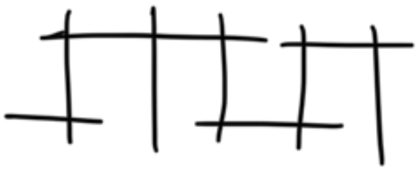
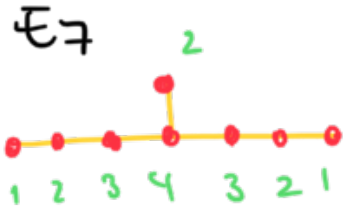
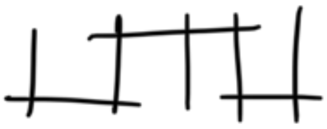
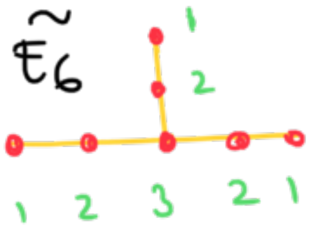
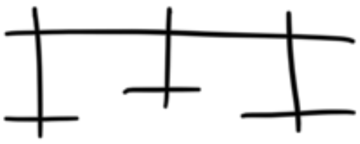
Notation: $\boxed{G(C)}$ dual graph of C.

KODAIRA'S CLASSIFICATION THEOREM.

Corollary: $G(X_t)$ is one of $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ (extended Dynkin diagrams).

Theorem: The fibres X_t are classified by the table below.

$G(X_t)$	X_t	e	Type Name
$\tilde{A}_0$ 	 ell. curve	0	I_0
	 node	1	I_1
	 cusp	2	II
$\tilde{A}_1$ 		2	I_2
		3	III
$\tilde{A}_2$ 		4	IV
$\tilde{A}_{n-1}$ 		n	$I_{n\geq 3}$

$\tilde{D}_{n+4}$ 		$n+6$	$I^*_{n\geq 0}$
$\tilde{E}_8$ 		10	IV^*
$\tilde{E}_7$ 		9	III^*
$\tilde{E}_6$ 		8	II^*

The green numbers are the multiplicities of each component

<u>Proof</u> (Idea)

we compute $G(X_t)$ (all the possibilities)

- we use X_t are connected
- $X_t = \sum m_i C_i$

$$(X_t \cdot C_i) = 0$$

$$(C_i)^2 = -2 \qquad C_i \simeq \mathbb{P}^1$$

$\square$.

Remark: We can use the euler number $e(X)$

$$24 = e(X) = \sum_{\substack{X_t \\ \text{sing.}}} e(X_t)$$

In particular we have 24 singular fibres. $9 = e(III^*)$
(with multiplicities). How many fibres of type III* are in an ell. K3?

4. WEIERSTRASS MODELS OF K3

As for elliptic curves, we want now to find a Weierstrass equation for an elliptic K3
↳ "Family of elliptic curves"

Remark: Needed for W. equation:

Weierstrass curves	v.s.	elliptic K3 surf.
(1) • E_W irreducible curve of genus 1		• Not all the fibres X_t of X are irreducible
(2) • We have a "given" point $O \in E(k)$		• We don't have a specified point on X_t.

How do we fix this?

(2) We assume the existence of a section on X

Def: Let $\pi : X \to \mathbb{P}^1$ be an elliptic K3.

A section of π is a curve $C_0 \subseteq X$ such that $\pi|_{C_0} : C_0 \to \mathbb{P}^1$ is an isomorphism.

$$\pi|_{C_0}(C_0) \simeq \mathbb{P}^1 \Rightarrow (X_t \cdot C_0) = 1$$

- If $X_t = \sum m_i C_i$

  C_0 intersects only one C_j and $m_j = 1$.

Remark: Natural question: When do we have a section on X?

If $E := X_\eta = \pi^{-1}(\eta)$ is the generic fibre.

then we know that E is a smooth curve of genus 1 over $K := k(\eta)$

we have a bijection

$$\{C \text{ section of } \pi\} \longleftrightarrow E(K).$$
$$C \longmapsto E \cap C = p.$$

We have a section iff $E(K) \neq \emptyset$.

(1) Contracting fibres X_t into irreducible.

$\boxed{X_t = \sum m_i C_i}$, fix a section C_0;

C_j is the component of X_t cutting C_0

Now, $C = \sum_{i \neq j} m_i C_i \rightsquigarrow G(C)$ A_n, D_n, E_6, E_7, E_8 Dynkin diagram.

(the ones on the

table - 1 vertex of
mult 1).

Thm. du Val.

→ C can be contracted into a node or a cusp.

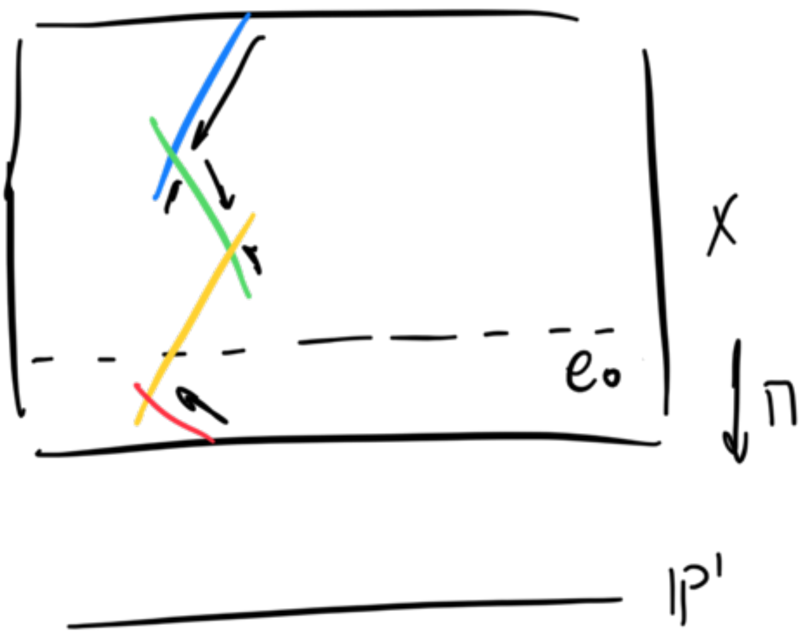
$\boxed{\overline{X} \text{ is } X \text{ after all such contractions}}$ Weierstrass model of X

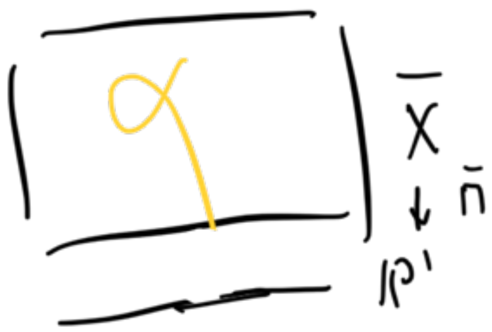
$\overline{\pi} : \overline{X} \longrightarrow \mathbb{P}^1$ elliptic fibration

↳ All the fibres are irreducible curves.

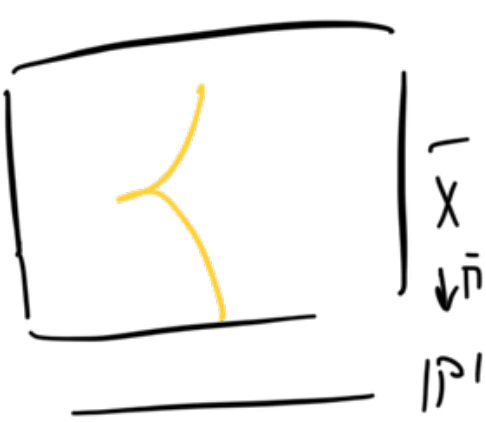
<u>Def</u>: such a fibration is called a Weierstrass fibration.

So, esentially we have we get either





or



With conditions (1)+(2) fixed, we now construct the global W. equation for X. (for $\bar{X}$)

Taking $\varphi : X \twoheadrightarrow \bar{X}$ contraction morphism.

with $\bar{\pi} : \bar{X} \longrightarrow \mathbb{P}^1$

and $\pi = \bar{\pi} \circ \varphi$

- $\bar{C}_0 = \varphi(C_0)$ is a section for $\bar{X}$

How do we find the W. equation?

We use the sequences

$$0 \to \mathcal{O}_{\bar{X}}((n-1)\bar{C}_0) \to \mathcal{O}_{\bar{X}}(n\bar{C}_0) \to \mathcal{O}_{\bar{C}_0}(-2n) \to 0$$

and apply $\bar{\pi}_*$

$$0 \to \bar{\pi}_* \mathcal{O}_X((n-1)\bar{C}_0) \to \bar{\pi}_* \mathcal{O}_{\bar{X}}(n\bar{C}_0) \to \bar{\pi}_* \mathcal{O}_{\bar{C}_0}(-2n) \to \cancel{R^1\pi_* \mathcal{O}_{\bar{X}}((n-1)\bar{C}_0)}\ 0$$

SPLIT.

It follows (using $n=1,2,3$)

$$\boxed{\bar{\pi}_* \mathcal{O}_{\bar{X}}(3\bar{C}_0) \simeq \underset{1}{\mathcal{O}_{\mathbb{P}^1}} \oplus \underset{x}{\mathcal{O}_{\mathbb{P}^1}(-4)} \oplus \underset{y}{\mathcal{O}_{\mathbb{P}^1}(-6)} =: F}$$

Idea: $\bar{X} \subseteq \mathbb{P}(F^*)$, If we compute $\mathcal{O}(\bar{X})$

$$f \in H^0(\mathbb{P}(F^*), \mathcal{O}(\bar{X})) \simeq H^0(\mathbb{P}^1, \underline{S^3(F)} \otimes \mathcal{O}_{\mathbb{P}^1}(-12)).$$

(f is equivalent to a homogeneous

$\hookrightarrow f$ is ... polynomial of degree 3 on x, y

locally, take $\mathcal{U}_i \subseteq \overline{X}$ s.t. $\underline{\mathcal{O}_{\overline{C_0}}(-2)}$ is trivial.
$\hookrightarrow$ normal bundle of $\overline{X}$

1 : local coord. for $\mathcal{O}_{\mathbb{P}^1}|_{\overline{\pi}(\mathcal{U}_i)}$

x_i : local coord for $\mathcal{O}_{\mathbb{P}^1}(-4)|_{\overline{\pi}(\mathcal{U}_i)}$

y_i : local coord for $\mathcal{O}_{\mathbb{P}^1}(-6)|_{\overline{\pi}(\mathcal{U}_i)}$

we get $\{1, x_i, y_i\}$. basis of $\overline{\pi}_* \mathcal{O}_{\overline{X}}(3\overline{C_0})|_{\overline{\pi}(\mathcal{U}_i)}$.

Previous sequence + Riemann-Roch $\Rightarrow$

$\{1, x_i, x_i^2, x_i^3, x_i y_i, y_i\}$ basis of $\overline{\pi}_* \mathcal{O}_{\overline{X}}(6\overline{C_0})|_{\overline{\pi}(\mathcal{U}_i)}$

In particular

Now, let $f_i \in \mathcal{O}_{\overline{X}}(2\overline{C_0})|_{\mathcal{U}_i} \dashrightarrow x_i$

$h_i \in \mathcal{O}_{\overline{X}}(3\overline{C_0})|_{\mathcal{U}_i} \dashrightarrow y_i$

Then $h_i^2 \in \mathcal{O}_{\overline{X}}(6\overline{C_0})|_{\mathcal{U}_i}$

and

$$h_i^2 = a_{6,i} f_i^3 + a_{5,i} f_i h_i + a_{4,i} f_i^2 + a_{3,i} h_i + a_{2,i} f_i + a_{1,i}$$

$$\boxed{a_{j,i} \in \mathcal{O}_{\overline{X}}((6-j)\overline{C_0})|_{\mathcal{U}_i}}$$

Take $\{U_i\}_i$ open cover of $\bar{X}$ and glue together.
we obtain:

$$\boxed{h^2 = a_6 f^3 + a_5 f h + a_4 f^2 + a_3 h + a_2 f + a_1 \qquad a_j \in \mathcal{O}_{\bar{X}}((6-j)\bar{C}_0)}$$

All the points of $\bar{X}$ are described by such an equation

char$(k) \neq 2,3$.

Applying lin. transform.

$$\boxed{h^2 = 4f^3 + G_2 f + G_3}$$

where

$G_2 \in \mathcal{O}_{\bar{X}}(4\bar{C}_0) \dashrightarrow g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$

$G_3 \in \mathcal{O}_{\bar{X}}(6\bar{C}_0)$. $\dashrightarrow g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12$

Def: The discriminant of $\bar{X}$ is the non-trivial section of $\mathbb{P}^1$

$$\boxed{\Delta := g_2^3 - 27 g_3^2} \in \mathcal{O}_{\mathbb{P}^1}(24).$$

Δ has 24 zeroes,

$\bar{X}$ has 24 singularities.

Remark: (g_2, g_3) are unique up to $(\lambda^4 g_2, \lambda^6 g_3)$

$\lambda \in k$.

Remark: The construction can be reversed.

Given $g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(8))$, $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(12))$ $\rightsquigarrow$ $\boxed{Y^2 = X^3 + g_2 X + g_3}$

it determines a surface $\overline{X} \subseteq \mathbb{P}(F^*)$.

X minimal desingularization $\overline{X}$

↳ This is a K3 surface.

5. Mordell-Weil group

Let $E: X_\eta = \pi^{-1}(\eta)$ the generic fibre of X.

It's an ell. curve over $K = k(\eta) \simeq k(t)$ on $\mathbb{P}^1$.

↳ we are assuming the existence of a section.

Recall:

$$\{C \text{ section of } \pi\} \longleftrightarrow E(K).$$
$$C \longmapsto C \cap E = P_C$$
$$\overline{P} \longleftarrow P$$

- We can endow $\{C$ section of $\pi\}$ with the group structure coming from $E(K)$

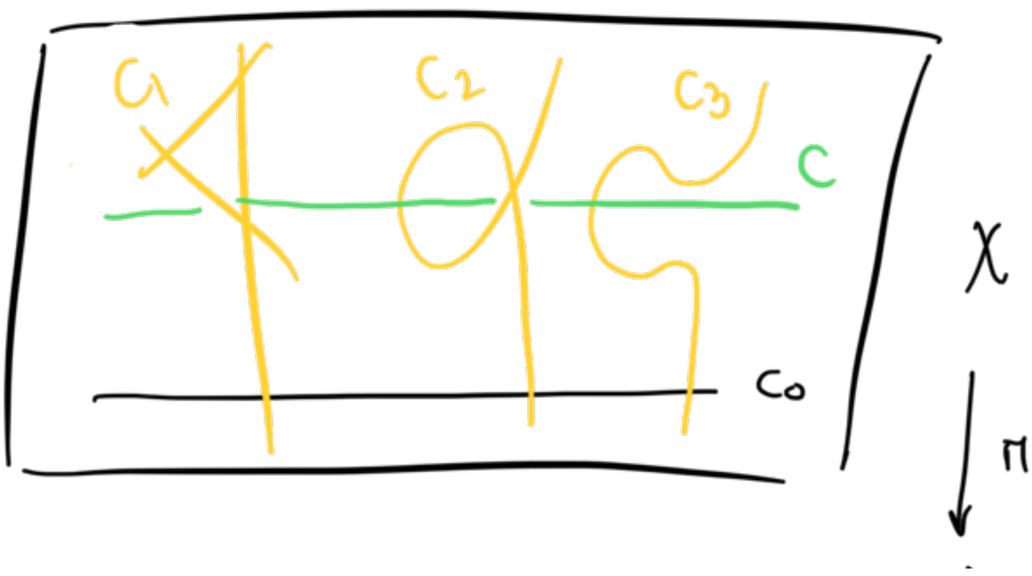
- If we chose a section C_0, s.t.

$$C_0 \cap E = O_{C_0} = O_E \in E_K,$$

C_0 is the neutral element.

Def: let $D = \sum n_i C_i$ divisor on X

- D is vertical if all components are are supported on some fibre(s) X_t. Caracterization $(X_t \cdot D) = 0$
- D is horizontal if $D = C$ irreducible and $\pi|_C : C \to \mathbb{P}^1$ is a surjection



$\mathbb{P}^1$

Consider the group homomorphism

$$\begin{aligned} \phi : \mathrm{Div}(X) &\longrightarrow \mathrm{Div}(E) \\ D &\longmapsto D|_E \end{aligned}$$

- $\ker(\phi) = \{ D : (D . X_t) = 0 \} = \{$ vertical divi-sors $\}$.

Since E is the generic fibre of X $k(E) = k(X)$

so ϕ gives

$$\begin{aligned} \tilde{\phi} : \mathrm{Pic}(X) &\longrightarrow \mathrm{Pic}(E). \\ \bar{D} &\longmapsto \overline{D|_E} \end{aligned}$$

$$\ker(\tilde{\phi}) = \{ \bar{D} : D \sim \text{vertical divisor} \}.$$

- Given $L \in \mathrm{Pic}(X)$ we define the degree fibre $d_L = (X_t . L)$.

Consider the line bundle

$$L|_E \otimes \mathcal{O}(-d_L O_E) \cong \mathcal{O}(P_L - O_E)$$

line bundle Theorem's $P_L \in E(k)$ unique point.

with degree ... Abel

- If C is a section, $L = \mathcal{O}(C)$.

$$\boxed{P_L = E \cap C.}$$

Def: The Mordell-Weil group of X is the group of sections of $\pi: X \to \mathbb{P}^1$.

we have $MW(X) \simeq E(K)$

Prop: There is a short exact sequence

$$0 \to A \to NS(X) \to MW(X) \to 0$$

where

$$A = \langle \{D \text{ vertical divisors}\} \cup \{\overline{C_0}\}. \rangle$$

In particular $MW(X)$ is a finitely generated abelian group.

proof: Since X K3, we have $NS(X) \simeq Pic(X)$.

And the map

$$NS(X) \simeq Pic(X) \xrightarrow{F} E(K) \simeq MW(X)$$
$$L \longmapsto P_L$$

is:

surjective: $p \in E(K)$, $L = \mathcal{O}(\overline{p})$, $P_L = p$.

kernel: $\underset{\neq C_0}{L} \in Pic(X) \in kernel(F)$

$$P_L = O_E \longleftrightarrow (L \cdot X_t) = 0$$

$\iff L$ is a vertical divisor

Remark: k does not need to be finitely gen.

Corollary: (Shioda-Tate)

$r_t = \#$ (irred components of X_t).

$$\rho(X) = \mathrm{rank}(NS(X)) = 2 + \sum(r_t - 1) + \mathrm{rank}(MW(X)).$$

proof:

$$\mathrm{rank}(A) = 2 + \sum(r_t - 1).$$

$\square$.

WHAT CAN WE SAY ABOUT MW(X)?

TORSION. (char $k = 0$)

$$MW(X) \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

$m, n \in \mathbb{Z}$.

FREE PART We can define the mordell-weil lattice of X.

$MW(X)/MW(X)_{tor}$ together with $\langle \cdot \rangle = -(\cdot)$.

$\Uparrow$

$\mathbb{Q}$.

⚠ Multiplying by a constant ... $\langle \cdot \rangle \in \mathbb{Z}$.

the Mordell-Weil lattice is a positive defined lattice.

$$0 \leq \operatorname{rank}(MW(X)) \leq \operatorname{rank}(NS(X)) - 2 \begin{cases} \leq 18 & \text{char } 0 \\ \leq 20 & \text{char} > 0 \end{cases}$$