Mass formulae for supersingular abelian varieties

Valentijn Karemaker

Utrecht University

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Introduction: why abelian varieties over finite fields?

Elliptic curves

Abelian varieties $\mathcal{A}_g$

Jacobians

Over finite fields:

- Explicit description of ISOGENY CLASSES.
- Amenable to computations.
- Useful stratifications of $\mathcal{A}_g$.

Finite fields

Definition

Let $\mathbb{F}_q$ be the **finite field** of cardinality $q = p^r$, where p is a prime.

Facts about finite fields:

- For every prime p and integer $r \geq 1$, there is a unique finite field $\mathbb{F}_{p^r}$. Also, the cardinality of any finite field is p^r for some prime p and integer $r \geq 1$.
- We have field extensions $\mathbb{F}_q \subseteq \mathbb{F}_{q^m}$ for any $m \geq 1$.
- All elements $x \in \mathbb{F}_q$ satisfy $x^q = x$.

Elliptic curves: definition

Definition (elliptic curve)

An **elliptic curve** is a genus 1 projective curve

$$E : y^2z + axyz + byz^2 = x^3 + cx^2z + dxz^2 + ez^3$$

(where in our case, $a, b, c, d, e \in \mathbb{F}_q$),
with a marked point $\mathcal{O}$ ("at infinity"), whose points form a group.

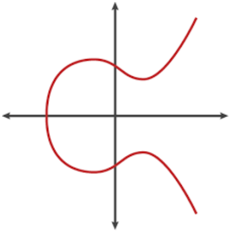

Figure: Adding points on an elliptic curve over $\mathbb{R}$.

Elliptic curves: points over finite fields

Definition ($E(\mathbb{F}_q)$)

Let $E(\mathbb{F}_{q^m}) = \{$ points $(x : y : z)$ on $E/\mathbb{F}_q$ defined over $\mathbb{F}_{q^m}\}$.

Use the Frobenius morphism ϕ of $E/\mathbb{F}_{q^m}$:

$$\phi((x : y : z)) = (x^{q^m} : y^{q^m} : z^{q^m}).$$

Then $$E(\mathbb{F}_{q^m}) = \{\text{fixed points of } \phi/\mathbb{F}_{q^m}\}.$$

Elliptic curves: zeta function

Definition (Weil polynomial)

The **Weil polynomial** $P_\phi(E/\mathbb{F}_q, T) \in \mathbb{Z}[T] = (T-\alpha)(T-\overline{\alpha})$ is the characteristic polynomial of $\phi/\mathbb{F}_q$.

1. (Riemann hypothesis) $|\alpha| = \sqrt{q}$.
2. (Weil conjectures) $|E(\mathbb{F}_{q^m})| = (1-\alpha^m)(1-\overline{\alpha}^m)$ for all $m \geq 1$
3. (Honda-Tate theory) α determines E up to isogeny.

Definition (Zeta function)

The **zeta function** of an elliptic curve $E/\mathbb{F}_q$ is

$$Z(E/\mathbb{F}_q, T) = \exp\left(\sum_{m\geq 1} |E(\mathbb{F}_{q^m})| \frac{T^m}{m}\right) = \frac{(1-\alpha T)(1-\overline{\alpha} T)}{(1-T)(1-qT)}.$$

Elliptic curves: p-torsion

Definition (p-torsion, ordinary, supersingular)

We have

$$E[p](\overline{\mathbb{F}}_q) \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } E \text{ is } \textbf{ordinary}, \\ 0 & \text{if } E \text{ is } \textbf{supersingular}. \end{cases}$$

Abelian varieties: definition and zeta function

Definition (abelian variety)

An **abelian variety** is a non-singular projective group variety.

The zeta function of an abelian variety $X/\mathbb{F}_q$ of dimension g

$$Z(X/\mathbb{F}_q, T) = \exp\left(\sum_{m\geq 1} |X(\mathbb{F}_{q^m})| \frac{T^m}{m}\right) = \frac{P_1(T)\dots P_{2g-1}(T)}{P_2(T)\dots P_{2g}(T)}$$

is determined by the Weil polynomial

$$P_\phi(X/\mathbb{F}_q, T) = T^{2g} P_1(T^{-1}) = \prod_{i=1}^{2g}(T-\alpha_i).$$

1. (Riemann hypothesis) $|\alpha_i| = \sqrt{q}$.
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Abelian varieties: p-torsion

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is determined by the Weil polynomial

$$P_\phi(X/\mathbb{F}_q, T) = T^{2g} P_1(T^{-1}) = \prod_{i=1}^{2g}(T - \alpha_i).$$

Definition (ordinary, supersingular)

We say X is $\begin{cases} \text{ordinary} \\ \text{supersingular} \end{cases}$ if $\begin{cases} |X[p](\overline{\mathbb{F}}_q)| = p^g \\ X \sim E^g \text{ with } E \text{ supersingular} \end{cases}$

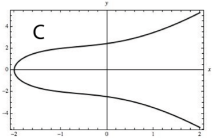
Special case: Jacobian varieties

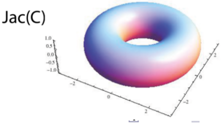
Let C be a smooth projective connected curve over $\mathbb{F}_q$ of genus g. We can construct a g-dimensional abelian variety $\mathrm{Jac}(C)$, called the JACOBIAN of C. The zeta function of C

$$Z(C/\mathbb{F}_q, T) = \exp\left(\sum_{m\geq 1} |C(\mathbb{F}_{q^m})| \frac{T^m}{m}\right) = \frac{P(T)}{(1-T)(1-qT)}$$

is determined by the Weil polynomial of $\mathrm{Jac}(C)$ through

$$P_\phi(\mathrm{Jac}(C)/\mathbb{F}_q, T) = T^{2g} P(T^{-1}) = \prod_{i=1}^{2g} (T - \alpha_i).$$





Moduli space $\mathcal{A}_g$

Let k be an algebraically closed field of characteristic p.

Definition

Let $\mathcal{A}_g$ be the moduli space over k of principally polarised g-dimensional abelian varieties.

$\mathcal{A}_g$ is irreducible of dimension $\frac{g(g+1)}{2}$. Often write $X = (X, \lambda)$.

For $X \in \mathcal{A}_g(k)$, consider its p-divisible group $X[p^\infty]$.
The isogeny class of $X[p^\infty]$ uniquely determines a Newton polygon.
$\Rightarrow$ **Newton stratification** of $\mathcal{A}_g$.

The isogeny class of $X[p^\infty]$ also determines the p-RANK f of X:
$|X[p](k)| = p^f$, so $0 \leq f \leq g$.
$\Rightarrow$ ***p*-rank stratification** of $\mathcal{A}_g$.

Moduli space $\mathcal{S}_g$

Recall: $X \in \mathcal{A}_g(k)$ is supersingular if $X \sim E^g$ with $E[p](k) = 0$.

Definition

Let $\mathcal{S}_g$ be the moduli space over k of principally polarised g-dimensional supersingular abelian varieties.

- All supersingular abelian varieties have the same Newton polygon, i.e., $\mathcal{S}_g$ is a Newton stratum of $\mathcal{A}_g$.
- A supersingular abelian variety has p-rank zero.
- Every component of $\mathcal{S}_g$ has dimension $\lfloor \frac{g^2}{4} \rfloor$.

The a-number stratification

Definition

Let $X \in \mathcal{A}_g(k)$. Its ***a*-number** is $a(X) := \dim_k \mathrm{Hom}(\alpha_p, X)$.
It depends on the isomorphism class of $X[p]$.

For $X \in \mathcal{A}_g(k)$ with p-rank f, we have $0 \leq a(X) \leq g - f$.
For $X \in \mathcal{S}_g(k)$, we have $1 \leq a(X) \leq g$.
$\Rightarrow$ ***a*-number stratification** of $\mathcal{S}_g = \coprod_{a=1}^{g} \mathcal{S}_g(a)$.

- Every component of $\mathcal{S}_g(a)$ has dimension $\lfloor \frac{g^2-a^2+1}{4} \rfloor$.
- $a(X) = g \Leftrightarrow X$ is SUPERSPECIAL, i.e., $X \simeq E^g$.
  The superspecial stratum $\mathcal{S}_g(g)$ is zero-dimensional.

The Ekedahl-Oort stratification

For $X \in \mathcal{A}_g(k)$, consider its p-torsion $X[p]$.
Its isomorphism class is classified by an element of the Weyl group W_g of Sp_{2g}, or equivalently by an ELEMENTARY SEQUENCE φ.
$\Rightarrow$ **Ekedahl-Oort stratification** of $\mathcal{A}_g = \coprod_\varphi \mathcal{S}_\varphi$.

- Ekedahl-Oort stratification refines the p-rank stratification.
- Also consider Ekedahl-Oort stratification $\coprod_\varphi (\mathcal{S}_\varphi \cap \mathcal{S}_g)$ of $\mathcal{S}_g$.
  Combinatorial criterion determines when $\mathcal{S}_\varphi \subseteq \mathcal{S}_g$.
  These strata are reducible; all other strata are irreducible.
- The a-number is constant on Ekedahl-Oort strata.
  $\Rightarrow \mathcal{S}_g(a) = \coprod_\varphi (\mathcal{S}_\varphi \cap \mathcal{S}_g)$.

A foliation of $\mathcal{S}_g$

Want to consider *p-divisible groups* up to *isomorphism*

Definition

For $x = (X_0, \lambda_0) \in \mathcal{S}_g(k)$, define the **central leaf**

$$\Lambda_x = \{(X, \lambda) \in \mathcal{S}_g(k) : (X, \lambda)[p^\infty] \simeq (X_0, \lambda_0)[p^\infty]\}.$$

- Each Λ_x is finite, but determining its size is very hard.
- Let $G_x/\mathbb{Z}$ be the automorphism group scheme, such that

$$G_x(R) = \{h \in (\mathrm{End}(X_0) \otimes_{\mathbb{Z}} R)^\times : h'h = 1\}$$

for any commutative ring R. Then there is a bijection

$$\Lambda_x \simeq G_x(\mathbb{Q}) \backslash G_x(\mathbb{A}_f) / G_x(\widehat{\mathbb{Z}}).$$

A finer stratification?

$$\Lambda_x = \{(X,\lambda) \in \mathcal{S}_g(k) : (X,\lambda)[p^\infty] \simeq (X_0,\lambda_0)[p^\infty]\}.$$

Goal

For any $x \in \mathcal{S}_g$, compute the **mass**

$$\mathrm{Mass}(\Lambda_x) = \sum_{x' \in \Lambda_x} |\mathrm{Aut}(x')|^{-1}.$$

N.B. $\mathrm{Mass}(\Lambda_x) = \mathrm{vol}(G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)) = \mathrm{Mass}(G_x, G_x(\widehat{\mathbb{Z}}))$.

$\Rightarrow$ **"Mass stratification"** of $\mathcal{S}_g$.
Expected to refine the a-number and Ekedahl-Oort stratifications.

How do we describe $\mathcal{S}_3$?

We now focus on the case where $g = 3$.
Let $E/\mathbb{F}_{p^2}$ be a supersingular elliptic curve with $\pi_E = -p$.
Let μ be any principal polarisation of E^3.

Definition

A **polarised flag type quotient (PFTQ)** with respect to μ is a chain

$$(E^3, p\mu) =: (Y_2, \lambda_2) \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0)$$

such that $\ker(\rho_1) \simeq \alpha_p$, $\ker(\rho_2) \simeq \alpha_p^2$, and $\ker(\lambda_i) \subseteq \ker(V^j \circ F^{i-j})$ for $0 \leq i \leq 2$ and $0 \leq j \leq \lfloor i/2 \rfloor$.

Let $\mathcal{P}_\mu$ be the moduli space of PFTQ's.
It is a two-dimensional geometrically irreducible scheme over $\mathbb{F}_{p^2}$.

How do we describe $\mathcal{S}_3$?

An PFTQ w.r.t. μ is $(E^3, p\mu) =: (Y_2, \lambda_2) \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0)$.
It follows that $(Y_0, \lambda_0) \in \mathcal{S}_3$, so there is a projection map

$$\mathrm{pr}_0 : \mathcal{P}_\mu \to \mathcal{S}_3$$
$$(Y_2 \to Y_1 \to Y_0) \mapsto (Y_0, \lambda_0)$$

such that $\coprod_\mu \mathcal{P}_\mu \to \mathcal{S}_3$ is surjective and generically finite.
Let $C : t_1^{p+1} + t_2^{p+1} + t_3^{p+1} = 0$ be a Fermat curve in $\mathbb{P}^2$.
It has genus $p(p-1)/2$ and admits a left action by $U_3(\mathbb{F}_p)$.
Then $\pi : \mathcal{P}_\mu \simeq \mathbb{P}_C(\mathcal{O}(-1) \oplus \mathcal{O}(1)) \to C$ is a $\mathbb{P}^1$-bundle.
There is a section $s : C \to T \subseteq \mathcal{P}_\mu$.

Upshot

For each (X, λ) there exist a μ and a $y \in \mathcal{P}_\mu$ such that $\mathrm{pr}_0(y) = [(X, \lambda)]$.
This y is uniquely characterised by a pair (t, u) with $t = (t_1 : t_2 : t_3) \in C(k)$ and $u = (u_1 : u_2) \in \pi^{-1}(t) \simeq \mathbb{P}^1_t(k)$.

The structure of $\mathcal{P}_\mu$

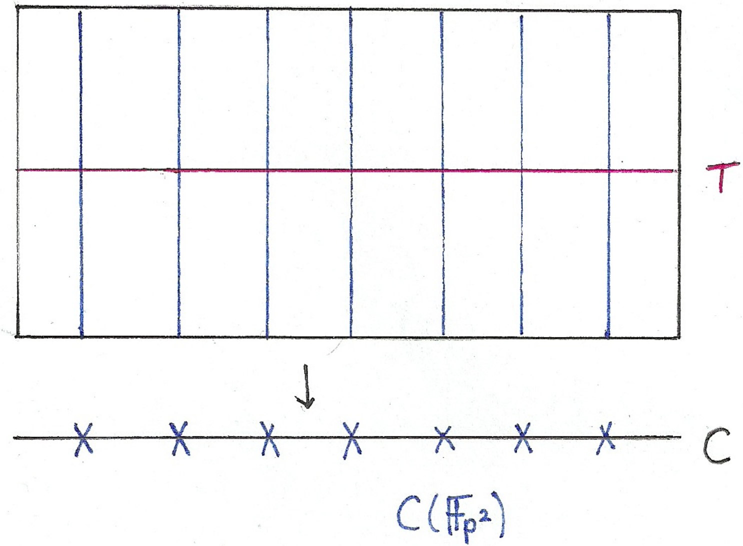
$\pi : \mathcal{P}_\mu \simeq \mathbb{P}_C(\mathcal{O}(-1) \oplus \mathcal{O}(1)) \to C$ has section $s : C \to T \subseteq \mathcal{P}_\mu$

Definition

Recall that X/k has a-number $a(X) = \dim_k \mathrm{Hom}(\alpha_p, X)$.
For a PFTQ $y = (Y_2 \to Y_1 \to Y_0)$, we say $a(y) = a(Y_0)$.

- For a supersingular threefold X we have $a(X) \in \{1, 2, 3\}$, and $a(X) = 3 \Leftrightarrow X$ is superspecial.
- If $y \in T$, then $a(y) = 3$.
- For $t \in C(k)$, we have $t \in C(\mathbb{F}_{p^2}) \Leftrightarrow a(y) \geq 2$ for any $y \in \pi^{-1}(t)$.
- For $y \in \mathcal{P}_\mu$, we have $a(y) = 1 \Leftrightarrow y \notin T$ and $\pi(y) \notin C(\mathbb{F}_{p^2})$.

The structure of $\mathcal{P}_\mu$: a picture



Using PFTQ's to construct minimal isogenies

Any supersingular abelian variety X admits a MINIMAL ISOGENY

$$\varphi : Y \to X$$

from a *superspecial* abelian variety $Y \simeq E^g$.

Idea

Construct the minimal isogeny for X from its corresponding PFTQ

$$Y_2 \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0 = X.$$

(If $Y_2 \to Y_1 \to Y_0$ is a PFTQ, then Y_2 is superspecial!)

- If $a(X) = 3$ then X is superspecial and $\varphi = \mathrm{id}$.
- If $a(X) = 2$, then $a(Y_1) = 3$ and $\varphi = \rho_1$ of degree p.
- If $a(X) = 1$, then $\varphi = \rho_1 \circ \rho_2$ of degree p^3.

From minimal isogenies to masses

Let $x = (X, \lambda)$ be supersingular and $\varphi : Y \to X$ a minimal isogeny. Write $\tilde{x} = (Y, \varphi^*\lambda)$. Recall automorphism group scheme G_x.

Through φ, we may view both $G_{\tilde{x}}(\widehat{\mathbb{Z}})$ and $\varphi^* G_x(\widehat{\mathbb{Z}})$ as open compact subgroups of $G_{\tilde{x}}(\mathbb{A}_f)$, which differ only at p. Hence:

Lemma

$$\begin{aligned}\mathrm{Mass}(\Lambda_x) &= \frac{[G_{\tilde{x}}(\widehat{\mathbb{Z}}) : G_{\tilde{x}}(\widehat{\mathbb{Z}}) \cap \varphi^* G_x(\widehat{\mathbb{Z}})]}{[\varphi^* G_x(\widehat{\mathbb{Z}}) : G_{\tilde{x}}(\widehat{\mathbb{Z}}) \cap \varphi^* G_x(\widehat{\mathbb{Z}})]} \cdot \mathrm{Mass}(\Lambda_{\tilde{x}}) \\ &= [\mathrm{Aut}((Y, \phi^*\lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] \cdot \mathrm{Mass}(\Lambda_{\tilde{x}}).\end{aligned}$$

So we can compare any supersingular mass to a superspecial mass.

From minimal isogenies to masses

Moreover, the superspecial masses are known in any dimension!

Lemma [Ekedahl, Harashita, Hashimoto, Ibukiyama, Yu]

Let $\tilde{x} = (Y, \lambda)$ be a superspecial abelian threefold.

- If λ is a principal polarisation, then
$$\mathrm{Mass}(\Lambda_{\tilde{x}}) = \frac{(p-1)(p^2+1)(p^3-1)}{2^{10}\cdot 3^4\cdot 5\cdot 7}.$$
- If $\ker(\lambda) \simeq \alpha_p \times \alpha_p$, then
$$\mathrm{Mass}(\Lambda_{\tilde{x}}) = \frac{(p-1)(p^3+1)(p^3-1)}{2^{10}\cdot 3^4\cdot 5\cdot 7}.$$

It remains to compute $[\mathrm{Aut}((Y, \phi^*\lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

The case $a(X) = 2$

Let $x = (X, \lambda) \in \mathcal{S}_3$ such that $a(X) = 2$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2})$.
We need to compute $[\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

There are reduction maps

$$\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_{p^2})$$
$$\mathrm{Aut}((X, \lambda)[p^\infty]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_{p^2}) \cap \mathrm{End}(u)^\times,$$

where

$$\mathrm{End}(u) = \{g \in M_2(\mathbb{F}_{p^2}) : g \cdot u \subseteq k \cdot u\} \simeq \begin{cases} \mathbb{F}_{p^4} \text{ if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ \mathbb{F}_{p^2} \text{ if } u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

The case $a(X) = 2$

Let $x = (X, \lambda) \in \mathcal{S}_3$ such that $a(X) = 2$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2})$.

So $[\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] =$

$$[\mathrm{SL}_2(\mathbb{F}_{p^2}) : \mathrm{SL}_2(\mathbb{F}_{p^2}) \cap \mathrm{End}(u)^\times] = \begin{cases} p^2(p^2-1) & \text{if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ |\mathrm{PSL}_2(\mathbb{F}_{p^2})| & \text{if } u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

Theorem (K.-Yobuko-Yu)

There are two mass strata in $\mathcal{S}_3(2)$:

$$\mathrm{Mass}(\Lambda_x) = \frac{1}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$

$$\begin{cases} (p-1)(p^3+1)(p^3-1)(p^4-p^2) & : u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ 2^{-e(p)}(p-1)(p^3+1)(p^3-1)p^2(p^4-1) & : u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

The case $a(X) = 1$

Let $x = (X, \lambda) \in \mathcal{S}_3$ such that $a(X) = 1$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C^0(k) := C(k) \setminus C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k)$.
We need to compute $[\mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

Theorem (K.-Yobuko-Yu)

There are three mass strata in $\mathcal{S}_3(1)$, determined by the fibres D_t of a divisor $D \subseteq C^0 \times \mathbb{P}^1$:

$$\mathrm{Mass}(\Lambda_x) = \frac{p^3}{2^{10} \cdot 3^4 \cdot 5 \cdot 7} \cdot \begin{cases} 2^{-e(p)} p^{2d(t)} (p^2 - 1)(p^4 - 1)(p^6 - 1) & : u \notin D_t; \\ p^{2d(t)} (p - 1)(p^4 - 1)(p^6 - 1) & : u \in D_t, t \notin C(\mathbb{F}_{p^6}); \\ p^6 (p^2 - 1)(p^3 - 1)(p^4 - 1) & : u \in D_t, t \in C(\mathbb{F}_{p^6}). \end{cases}$$

What else can we use all these computations for?

Application: Oort's conjecture

Oort's conjecture

Every generic g-dimensional principally polarised supersingular abelian variety (X, λ) over k of characteristic p has automorphism group $C_2 \simeq \{\pm 1\}$.

This fails in general: counterexamples for $(g, p) = (2, 2)$ and $(3, 2)$.

Theorem (K.-Yobuko-Yu)

When $g = 3$, Oort's conjecture holds precisely when $p \neq 2$.

- A *generic* threefold X has $a(X) = 1$.
  Its PFTQ is characterised by $t \in C^0(k)$ and $u \notin D_t$.
- Our computations show for such (X, λ) that

$$\mathrm{Aut}((X, \lambda)) \simeq \begin{cases} C_2^3 & \text{for } p = 2; \\ C_2 & \text{for } p \neq 2. \end{cases}$$

Gauss problem

Recall the central leaf for $x = (X_0, \lambda_0) \in \mathcal{S}_g(k)$ is defined as

$$\Lambda_x = \{(X, \lambda) \in \mathcal{S}_g(k) : (X, \lambda)[p^\infty] \simeq (X_0, \lambda_0)[p^\infty]\}.$$

Gauss problem

Determine precisely for which $x \in \mathcal{S}_g(k)$ we have that

$$|\Lambda_x| = 1.$$

We can define Λ_x for any $x \in \mathcal{A}_g(k)$.
Chai proved $|\Lambda_x|$ is finite if and only if $x \in \mathcal{S}_g(k)$ is supersingular.

Main result

Theorem (in progress, Ibukiyama-K.-Yu)

Let $x \in \mathcal{S}_g$. Then $|\Lambda_x| = 1$ if and only if one of the following three cases holds:

(i) $g = 1$ and $p \in \{2, 3, 5, 7, 13\}$.

(ii) $g = 2$ and $p = 2, 3$.

(iii) $g = 3$, $p = 2$ and $a(x) \geq 2$.

The result for $g = 1$ was known before and follows from work of Vignéras on class numbers of quaternion algebras.
In this case, Λ_x is the whole supersingular locus.

The result for $g = 2$ was recently proven by Ibukiyama by studying quaternion hermitian groups.

The proof for $g \geq 5$

Let Λ_{g,p^c} denote the set of isomorphism classes of g-dimensional polarised superspecial abelian varieties (X, λ) whose polarisation λ satisfies $\ker(\lambda) \simeq \alpha_p^{2c}$.

1. If $x \in \Lambda_{g,p^c}$, then $\Lambda_x = \Lambda_{g,p^c}$.
2. For every $x \in \mathcal{S}_g(k)$ there exists a surjection $\pi : \Lambda_x \twoheadrightarrow \Lambda_{g,p^c}$ for some $0 \leq c \leq \lfloor g/2 \rfloor$.
3. We know $\mathrm{Mass}(\Lambda_{g,p^c})$ for all $g \geq 1$ and $0 \leq c \leq \lfloor \frac{g}{2} \rfloor$.

For $g \geq 5$, this yields enough information: using (3), we prove that $|\Lambda_{g,p^c}| > 1$ for all p and all $0 \leq c \leq \lfloor \frac{g}{2} \rfloor$, which by (2) implies that $|\Lambda_x| > 1$ always.

Ideas for the proof for $g = 3, 4$

When $g = 3$, we use our mass formula! Together with computations of automorphism groups, this gives the result, since

$$\mathrm{Mass}(\Lambda_x) := \sum_{x' \in \Lambda_x} |\mathrm{Aut}(x')|^{-1}.$$

When $g = 4$, and $x \in \mathcal{S}_4(k)$, the surjection $\pi : \Lambda_x \twoheadrightarrow \Lambda_{g,p^c}$ is induced from the minimal isogeny of x.
This allows us to compare $\mathrm{Mass}(\Lambda_x)$ with the appropriate superspecial mass $\mathrm{Mass}(\Lambda_{4,p^c})$, and $|\Lambda_x|$ with $|\Lambda_{4,p^c}|$.
We prove the theorem for one Ekedahl-Oort stratum at a time.

Thank you for your attention!