Mass formula for supersingular abelian threefolds

Valentijn Karemaker
(Utrecht University & Stockholm University)

Joint work with F. Yobuko and C.-F. Yu

WAGON

April 19, 2020

What is a mass (formula)?

Definition

Let S be a finite set of objects with finite automorphism groups. The MASS of S is the weighted sum

$$\mathrm{Mass}(S) = \sum_{s \in S} \frac{1}{|\mathrm{Aut}(s)|}.$$

A mass formula computes an expression for the mass.

What mass formula are we looking for?

Let k be an algebraically closed field of characteristic p.
Let A/k be a three-dimensional abelian variety.
A/k is SUPERSINGULAR (resp. SUPERSPECIAL) if it is *isogenous* (resp. *isomorphic*) to a product of supersingular elliptic curves.
Let $\mathcal{S}_{3,1}$ be the moduli space of principally polarised supersingular abelian threefolds (X, λ).

Definition

For $x = (X_0, \lambda_0) \in \mathcal{S}_{3,1}(k)$, let

$$\Lambda_x = \{(X, \lambda) \in \mathcal{S}_{3,1}(k) : (X, \lambda)[p^\infty] \simeq (X_0, \lambda_0)[p^\infty]\}.$$

It is known that Λ_x is finite.

Goal

Compute $\mathrm{Mass}(\Lambda_x) = \sum_{x' \in \Lambda_x} |\mathrm{Aut}(x')|^{-1}$ for any $x \in \mathcal{S}_{3,1}$.

How do we describe $\mathcal{S}_{3,1}$?

Let $E/\mathbb{F}_{p^2}$ be a supersingular elliptic curve with $\pi_E = -p$.
Let μ be any principal polarisation of E^3.

Definition

A POLARISED FLAG TYPE QUOTIENT (PFTQ) WITH RESPECT TO μ is a chain

$$(E^3, p\mu) =: (Y_2, \lambda_2) \xrightarrow{\rho_2} (Y_1, \lambda_1) \xrightarrow{\rho_1} (Y_0, \lambda_0)$$

such that $\ker(\rho_1) \simeq \alpha_p$, $\ker(\rho_2) \simeq \alpha_p^2$, and $\ker(\lambda_i) \subseteq \ker(V^j \circ F^{i-j})$ for $0 \leq i \leq 2$ and $0 \leq j \leq \lfloor i/2 \rfloor$.
Let $\mathcal{P}_\mu$ be the moduli space of PFTQ's.

It follows that $(Y_0, \lambda_0) \in \mathcal{S}_{3,1}$, so there is a projection map

$$\mathrm{pr}_0 : \mathcal{P}_\mu \to \mathcal{S}_{3,1}$$
$$(Y_2 \to Y_1 \to Y_0) \mapsto (Y_0, \lambda_0).$$

How do we describe $\mathcal{P}_\mu$?

Let $C : t_1^{p+1} + t_2^{p+1} + t_3^{p+1} = 0$ be a Fermat curve in $\mathbb{P}^2$.

Then $\pi : \mathcal{P}_\mu \simeq \mathbb{P}_C(\mathcal{O}(-1) \oplus \mathcal{O}(1)) \to C$ is a $\mathbb{P}^1$-bundle.
There is a section $s : C \to T \subseteq \mathcal{P}_\mu$,

Upshot

For each (X, λ) there exist a μ and a $y \in \mathcal{P}_\mu$ such that $\mathrm{pr}_0(y) = [(X, \lambda)]$.
This y is uniquely characterised by a pair (t, u) with $t = (t_1 : t_2 : t_3) \in C(k)$ and $u = (u_1 : u_2) \in \pi^{-1}(t) \simeq \mathbb{P}^1_t(k)$.

The structure of $\mathcal{P}_\mu$

$\pi : \mathcal{P}_\mu \simeq \mathbb{P}_C(\mathcal{O}(-1) \oplus \mathcal{O}(1)) \to C$ has section $s : C \to T \subseteq \mathcal{P}_\mu$

Definition

Let X/k be an abelian variety. Its a-NUMBER is

$$a(X) := \dim_k \mathrm{Hom}(\alpha_p, X).$$

For a PFTQ $y = (Y_2 \to Y_1 \to Y_0)$, we say $a(y) = a(Y_0)$.

- For a threefold X we have $a(X) \in \{1, 2, 3\}$, and $a(X) = 3 \Leftrightarrow X$ is superspecial.
- If $y \in T$, then $a(y) = 3$.
- For $t \in C(k)$, we have $t \in C(\mathbb{F}_{p^2}) \Leftrightarrow a(y) \geq 2$ for any $y \in \pi^{-1}(t)$.
- For $y \in \mathcal{P}_\mu$, we have $a(y) = 1 \Leftrightarrow y \notin T$ and $\pi(y) \notin C(\mathbb{F}_{p^2})$.

Using PFTQ's to construct minimal isogenies

Any supersingular abelian variety X admits a MINIMAL ISOGENY

$$\varphi : Y \to X$$

from a *superspecial* abelian variety Y.

Idea

Construct the minimal isogeny for X from its corresponding PFTQ

$$Y_2 \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0 = X.$$

(If $Y_2 \to Y_1 \to Y_0$ is a PFTQ, then Y_2 is superspecial!)

- If $a(X) = 3$ then X is superspecial and $\varphi = \mathrm{id}$.
- If $a(X) = 2$, then $a(Y_1) = 3$ and $\varphi = \rho_1$ of degree p.
- If $a(X) = 1$, then $\varphi = \rho_1 \circ \rho_2$ of degree p^3.

From minimal isogenies to masses

Let $x = (X, \lambda)$ be supersingular and $\varphi : Y \to X$ a minimal isogeny. Write $\tilde{x} = (Y, \varphi^* \lambda)$.

Lemma

$\mathrm{Mass}(\Lambda_x) = [\mathrm{Aut}((Y, \phi^* \lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] \cdot \mathrm{Mass}(\Lambda_{\tilde{x}})$.

Moreover, the superspecial masses are known in any dimension!

Lemma [Ekedahl, Harashita, Hashimoto, Ibukiyama, Yu]

Let $\tilde{x} = (Y, \lambda)$ be a superspecial abelian threefold.

- If λ is a principal polarisation, then
$$\mathrm{Mass}(\Lambda_{\tilde{x}}) = \frac{(p-1)(p^2+1)(p^3-1)}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$
- If $\ker(\lambda) \simeq \alpha_p \times \alpha_p$, then
$$\mathrm{Mass}(\Lambda_{\tilde{x}}) = \frac{(p-1)(p^3+1)(p^3-1)}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$

It remains to compute $[\mathrm{Aut}((Y, \phi^* \lambda)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

The case $a(X) = 2$

Let $x = (X, \lambda) \in \mathcal{S}_{3,1}$ such that $a(X) = 2$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2})$.
We need to compute $[\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

There are reduction maps

$$\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_{p^2})$$
$$\mathrm{Aut}((X, \lambda)[p^\infty]) \twoheadrightarrow \mathrm{SL}_2(\mathbb{F}_{p^2}) \cap \mathrm{End}(u)^\times,$$

where

$$\mathrm{End}(u) = \{g \in M_2(\mathbb{F}_{p^2}) : g{\cdot}u \subseteq k{\cdot}u\} \simeq \begin{cases} \mathbb{F}_{p^4} \text{ if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ \mathbb{F}_{p^2} \text{ if } u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

The case $a(X) = 2$

Let $x = (X, \lambda) \in \mathcal{S}_{3,1}$ such that $a(X) = 2$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2})$.

So $[\mathrm{Aut}((Y_1, \lambda_1)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] =$

$$[\mathrm{SL}_2(\mathbb{F}_{p^2}) : \mathrm{SL}_2(\mathbb{F}_{p^2}) \cap \mathrm{End}(u)^\times] = \begin{cases} p^2(p^2-1) & \text{if } u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ |\mathrm{PSL}_2(F_{p^2})| & \text{if } u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

Theorem (K.-Yobuko-Yu)

$$\mathrm{Mass}(\Lambda_x) = \frac{1}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$
$$\begin{cases} (p-1)(p^3+1)(p^3-1)(p^4-p^2) & : u \in \mathbb{P}^1_t(\mathbb{F}_{p^4}) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^2}); \\ 2^{-e(p)}(p-1)(p^3+1)(p^3-1)p^2(p^4-1) & : u \in \mathbb{P}^1_t(k) \setminus \mathbb{P}^1_t(\mathbb{F}_{p^4}). \end{cases}$$

The case $a(X) = 1$

Let $x = (X, \lambda) \in \mathcal{S}_{3,1}$ such that $a(X) = 1$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C^0(k) := C(k) \setminus C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k)$.
We need to compute $[\mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])]$.

Let $D_p = \mathbb{Q}_{p^2}[\Pi]$ be the division quaternion algebra over $\mathbb{Q}_p$, and let $\mathcal{O}_{D_p}$ its maximal order. (We have $\Pi^2 = -p$.)

- $G_2 := \mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) \simeq \{A \in \mathrm{GL}_3(\mathcal{O}_{D_p}) : A^* A = \mathbb{I}_3\}$.
- $G := \mathrm{Aut}((X, \lambda)[p^\infty])] = \{g \in G_2 : g(X[p^\infty]) = X[p^\infty]\}$.

Reducing modulo p we obtain $\overline{G}_2$ and $\overline{G}$, where:

- $\overline{G}_2 = \{A + B\Pi \in \mathrm{GL}_3(\mathbb{F}_{p^2}[\Pi]) : A^* A = \mathbb{I}_3, B^T A^* = A^{*T} B\}$, so $|\overline{G}_2| = |U_3(\mathbb{F}_p)| \cdot |S_3(\mathbb{F}_{p^2})| = p^{15}(p+1)(p^2-1)(p^3+1)$;
- $\overline{G} = \{g \in \overline{G}_2 : g(\overline{X[p^\infty]}) \subseteq \overline{X[p^\infty]}\}$.

Moreover,

- $[\mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] = [G_2 : G] = [\overline{G}_2 : \overline{G}]$.

The case $a(X) = 1$

Let $x = (X, \lambda) \in \mathcal{S}_{3,1}$ such that $a(X) = 1$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C^0(k) := C(k) \setminus C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k)$.
We need $[\mathrm{Aut}((Y_2, \lambda_2)[p^\infty]) : \mathrm{Aut}((X, \lambda)[p^\infty])] = [G_2 : G] = [\overline{G}_2 : \overline{G}]$.

- $$\overline{G} \simeq \left\{ \left(\begin{smallmatrix} A & 0 \\ SA & A^{(p)} \end{smallmatrix} \right) : A \in U_3(\mathbb{F}_p), A \cdot t = \alpha \cdot t, \right. \\ \left. S \in S_3(\mathbb{F}_{p^2}), \psi_t(S) = u_2 u_1^{-1}(1 - \alpha^{p^3 - 1}) \right\},$$
  where $\psi_t : S_3(\mathbb{F}_{p^2}) \to k$ is a homomorphism depending on t.

The images of ψ_t for varying t define a divisor $D \subseteq C^0 \times \mathbb{P}^1$.
For $t \in C^0(k)$, let $d(t) = \dim_{\mathbb{F}_{p^2}}(\mathrm{Im}(\psi_t))$ and $D_t = \pi^{-1}(t) \cap D$.
Then $u = (u_1 : u_2) \in D_t \Leftrightarrow u_2 u_1^{-1} \in \mathrm{Im}(\psi_t)$.

- $$|\overline{G}| = \begin{cases} 2^{e(p)} p^{2(6-d(t))} & \text{if } u \notin D_t; \\ (p+1) p^{2(6-d(t))} & \text{if } u \in D_t \text{ and } t \notin C(\mathbb{F}_{p^6}); \\ (p^3+1) p^6 & \text{if } u \in D_t \text{ and } t \in C(\mathbb{F}_{p^6}). \end{cases}$$

The case $a(X) = 1$

Let $x = (X, \lambda) \in \mathcal{S}_{3,1}$ such that $a(X) = 1$.
Its PFTQ $(Y_2, \lambda_2) \to (Y_1, \lambda_1) \to (X, \lambda)$ is characterised by a pair $t \in C^0(k) := C(k) \setminus C(\mathbb{F}_{p^2})$ and $u \in \mathbb{P}^1_t(k)$.

Theorem (K.-Yobuko-Yu)

$$\mathrm{Mass}(\Lambda_x) = \frac{p^3}{2^{10} \cdot 3^4 \cdot 5 \cdot 7}.$$

$$\begin{cases} 2^{-e(p)} p^{2d(t)} (p^2 - 1)(p^4 - 1)(p^6 - 1) & : u \notin D_t; \\ p^{2d(t)} (p - 1)(p^4 - 1)(p^6 - 1) & : u \in D_t, t \notin C(\mathbb{F}_{p^6}); \\ p^6 (p^2 - 1)(p^3 - 1)(p^4 - 1) & : u \in D_t, t \in C(\mathbb{F}_{p^6}). \end{cases}$$

Question

What else can we use all these computations for?

Application: Oort's conjecture

Oort's conjecture

Every generic g-dimensional principally polarised supersingular abelian variety (X, λ) over k of characteristic p has automorphism group $C_2 \simeq \{\pm 1\}$.

This fails in general: counterexamples for $(g, p) = (2, 2)$ and $(3, 2)$.

Theorem (K.-Yobuko-Yu)

When $g = 3$, Oort's conjecture holds precisely when $p \neq 2$.

- A *generic* threefold X has $a(X) = 1$.
  Its PFTQ is characterised by $t \in C^0(k)$ and $u \notin D_t$.
- Our computations show for such (X, λ) that

$$\mathrm{Aut}((X, \lambda)) \simeq \begin{cases} C_2^3 & \text{for } p = 2; \\ C_2 & \text{for } p \neq 2. \end{cases}$$