Knapsack

The Knapsack problem is defined as follows. We are given a set of n items and a knapsack with a given volume B. Item j ($j = 1, \ldots, n$) has a given weight a_j and value c_j. We are asked to find a subset of the items with maximum value such that the total weight of the selected items amounts to no more than B. Without loss of generality, we assume that all items have a weight that is no more than B and that all values are positive.

We can formulate the Knapsack problem as an ILP as follows. For each item j ($j = 1, \ldots, n$) we introduce a binary variable x_j, where $x_j = 1$ indicates that we select item j, and $x_j = 0$ indicates that we do not put item j in the Knapsack. This leads to the following ILP

$$\max \sum_{j=1}^{n} c_j x_j \quad \text{subject to}$$

$$\sum_{j=1}^{n} a_j x_j \leq B,$$

$$x_j \in \{0,1\} \text{ for } j = 1, \ldots, n.$$

We obtain the LP-relaxation by replacing the integrality constraints $x_j \in \{0,1\}$ ($j = 1, \ldots, n$) with $0 \leq x_j \leq 1$ ($j = 1, \ldots, n$).

The Lagrange relaxation is a different form of relaxing the problem. Here we determine a set of 'nasty' constraints, the presence of which complicates solving the problem. For each one of these constraints we define a *Lagrangean multiplier*. We then remove these constraints one by one, and add the difference between the right- and lefthand-side to the objective function, weighed by the Lagrangean multiplier. For the Knapsack problem, we select the 'weight' constraint and remove it, where we use λ as the Lagrangean multiplier. This yields the following problem:

$$\max \sum_{j=1}^{n} c_j x_j + \lambda(B - \sum_{j=1}^{n} a_j x_j) \quad \text{subject to}$$

$$x_j \in \{0,1\} \text{ for } j = 1, \ldots, n.$$

It is readily verified that the solution of this problem, which is called the *Lagrangean relaxation*, yields an upper bound, if we assume that $\lambda \geq 0$; therefore, we request that $\lambda \geq 0$. The resulting upper bound is defined as $L(\lambda)$. We can interpret this relaxation as follows: we can buy additional space at a cost of λ per unit, and we can sell unused space at a price of λ per unit. Hence, we can interpret that space has a value of λ per unit.

For a given value of λ, the Lagrangean relaxation is solved as follows. Since the term λB is constant, we ignore it. Rewriting the remainder of the objective function yields

$$\max \sum_{j=1}^{n} (c_j - \lambda a_j) x_j,$$

which has to be maximized subject to the constraint that each item should be selected or left out of the knapsack. Obviously, it is advantageous to select item j if $c_j - \lambda a_j > 0$ and ignore it if $c_j - \lambda a_j < 0$; if $c_j - \lambda a_j = 0$ we can go either way. This rule fully complies with the observation that one unit of space has a value of λ.

Since we find an upper bound for each value of λ, as long it is nonnegative, we want to find the value of λ that leads to the tightest upper bound. This problem is called the *Lagrangean dual problem*, which is defined as

$$\min_{\lambda \geq 0} L(\lambda).$$

Theorem The function $L(\lambda)$ of λ is piecewise-linear, continuous, and convex.
Proof. Let Ω denote all possible subsets of the items. Let S denote any subset of the items, and let $a(S)$ and $c(S)$ denote the total weight and the total value of the items in S. Then we have that

$$L(\lambda) = \max_{S \in \Omega} c(S) + \lambda(B - a(S)).$$

Since $c(S) + \lambda(B - a(S))$ is a linear function in λ, we find that $L(\lambda)$ is the maximum of a finite set of linear functions, which implies that it is piecewise-linear, continuous, and convex.
■

Since $L(\lambda)$ is convex, we know that any local minimum will be a global minimum. Renumber the items such that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}.$$

We start with $\lambda = 0$ and we increase λ as long as the set of items selected to maximize $L(\lambda)$ does not change; from the discussion above, we know that this set will remain the same for any $\lambda \in [\frac{c_{k+1}}{a_{k+1}}, \frac{c_k}{a_k}]$. Since in that interval it is optimal to select the items $1, \ldots, k$, we find that then

$$L(\lambda) = \sum_{j=1}^{k} c_j + \lambda(B - \sum_{j=1}^{k} a_j).$$

Hence, $L(\lambda)$ increases if $B - \sum_{j=1}^{k} a_j > 0$ and decreases if $B - \sum_{j=1}^{k} a_j < 0$. Therefore, we find that the value of λ that solves the Lagrangean dual problem is equal to $\frac{c_k}{a_k}$, where k is such that

$$\sum_{j=1}^{k-1} a_j < B \leq \sum_{j=1}^{k} a_j.$$

If $\lambda < \frac{c_k}{a_k}$, then it is optimal to select at least the items $1, \ldots, k$, and hence increasing λ decreases the value of $L(\lambda)$; if $\lambda > \frac{c_k}{a_k}$, then it is optimal to select at most the items $1, \ldots, k-1$, and hence we can decrease the value of $L(\lambda)$ by decreasing λ. If we find that $\sum_{j=1}^{k} a_j = B$, then the value of the Lagrangean dual problem is equal to the value of the ILP formulation of the Knapsack problem, which implies that we have found an optimal solution then.

Theorem Geoffrion
The outcome value of the Lagrangean dual problem is less than or equal to the outcome value of the LP-relaxation. When in the Lagrangean relaxation (that is, for a given λ) the integrality constraints are redundant, then the outcome values are equal. ■

Clearly the integrality constraints are redundant, and hence we find that according to the theorem by Geoffrion the solution values of the Lagrangean dual problem and the LP-relaxation are equal. This can be shown by simply calculating both values. Let $\lambda^* = \frac{c_k}{a_k}$ denote the optimum Lagrangean multiplier, where k is defined as the smallest number such that taking the first k items (after renumbering) fills the knapsack completely. Hence, we find that the value of the Lagrangean dual problem is equal to

$$L(\lambda^*) = \sum_{j=1}^{k-1} c_j + \lambda^*(B - \sum_{j=1}^{k-1} a_j).$$

If we solve the LP-relaxation, then we find that we can select the items $1, \ldots, k-1$ entirely, and we can take a fraction of item k with a total weight of $B - \sum_{j=1}^{k-1} a_j$. Hence, we find that the optimal solution of the LP-relaxation is then equal to

$$\sum_{j=1}^{k-1} c_j + c_k \frac{(B - \sum_{j=1}^{k-1} a_j}{a_k}) \; ,$$

which is clearly equal to $L(\lambda^*)$.