Advanced linear programming: exercises

(1) We consider the following packing problem. We are given n items. Item j has size p_j ($j = 1, \ldots, n$). These items have to divided over m bins. The goal is to assign the items to the bins in such a way that $P_{\max}$ is minimized, where $P_{\max}$ equals the maximum bin filling. The bin filling is defined as the sum of the sizes of the items in one bin .

(a) Give an integer linear programming formulation for this problem with a **polynomial** number of variables (do not use the formulation from part (c)).

(b) Now we are allowed to buy two additional bins. The first additional bin costs Q_1 and the second costs Q_2, where $Q_2 < Q_1$. The second additional bin can only be used in combination with the first. Extend the model of part (a) with the possibility of extra bins. The goal is to minimize the sum of $P_{\max}$ and the cost of the extra bins.

(c) Now we consider a variant of the problem of part **(a)**. Suppose our bins have a maximum capacity of C and we want to minimize the number of bins needed to pack all the items. We can formulate this problem by using sets of items assigned to one bin. Give an integer linear programming formulation for the problem based on sets of items.

(d) Describe how the LP-relaxation of this formulation can be solved by column generation. Your description should include a formulation of the pricing problem, and an algorithm to solve it.

(2) We consider the game 'Monsters and caves'. As player you have to cross a network of caves. The network is modelled by a directed graph (V, A). There is a given starting point s and you want to reach the finish t as quick as possible. For arc $(i, j) \in A$ the crossing time is t_{ij}. On some edges there are monsters. If you want to cross such an edge, you have to beat the monster in a fight. You can beat the monster but then you lose l_{ij} lives. (for an edge without monsters $l_{ij} = 0$). The total number of lives you have at the start is L. **If you have lost all your lives the game is over.**

(a) Formulate the problem of finding the best way to play the game as an Integer Linear Programming problem with a **polynomial** number of variables.

(b) Show that if the number of lives is infinite (forms no restriction), the constraint matrix of the problem is Totally Unimodular.

(c) Apply Dantzig-Wolfe decomposition to the formulation of part (a), where the constraint that you stay alive is put in the master problem

(d) Describe how the LP-relaxation of this formulation can be solved by column generation. Your description should include a formulation of the pricing problem.

(3) We consider the energy constrained max-flow problem. We are given a directed graph (V, A), where V is the set of nodes and A is the set of arcs. There is a source node $s \in V$ and a sink $t \in V$. Each node i has a battery with capacity E_i. Sending flow on edge (i, j) requires energy from the battery at node i, which amounts e_{ij} per unit flow. The objective is to find the maximal flow s to t, where the flow is required to be integral.

(a) Give an integer linear programming formulation for this problem with a **polynomial** number of variables (do not use the formulation from part (c)).

(b) Show that if we omit the energy contraints, the constraint matrix is totally unimodular.

(c) It is known that a network flow can be decomposed into a number of $s - t$ paths. An alternative way to formulate the problem is by using these paths. Give an integer linear programming formulation for the problem of **part (a)** based on paths.

(d) Describe how the LP-relaxation of this formulation can be solved by column generation. Your description should include a formulation of the pricing problem. Describe how to solve the pricing problem.