Advanced linear programming

Stochastic programming
Benders decomposition

Marjan van den Akker

Universiteit Utrecht

[Faculty of Science
Information and Computing Sciences]

Two stage stochastic linear programming example: farmer's problems

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Minimum requirement (T)	200	240	–
Total available land: 500 acres			

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Optimal solution

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–
Overall profit: $118,600			

Scenarios

- Harvest yield very sensitive to weather:
  - Yield 20% more in good year, 20% reduced in bad year
- Suppose type of year would be known beforehand:
  - Good year: 167.667
  - Average year: 118.600
  - Bad year: 59.950 (buy corn)
  - Average: 115.406
- This is not realistic!!!!

Two-stage model

- First stage: land assignment
- Second stage: sales and purchase
- Maximize expected profit

- Solution value 108.390

- Expected value of perfect information:
  - 115.406-108.390= 7.016

Two-stage stochastic linear programming problem

- Scenario's: 1,2,…,K
- α_ω :probability scenario *s* occurs
- First stage decision variables: x
- Second stage decision variables: y_ω (ω=1,2,…,K)

- Minimize expected cost:

$$cx + \alpha_1 fy_1 + \alpha_2 fy_2 + \ldots + \alpha_K fy_K$$

Two stage stochastic linear programming problem

$$\min cx + \alpha_1 fy_1 + \alpha_2 fy_2 + \ldots + \alpha_K fy_K$$

$$s.t$$

$$\begin{array}{lllll} Ax & & & & = b \\ B_1 x + & Dy_1 & & & = d_1 \\ B_2 x + & & Dy_2 & & = d_2 \\ \vdots & & & \ddots & \vdots \\ B_k x + & & & Dy_K & = d_K \\ x, y_1, \ldots, y_K \geq 0 & & & & \end{array}$$

Benders decomposition sub problem

- For fixed *x*:

$$P_\omega : z_\omega(x) = \min\ f y_\omega$$
$$\text{subject to } B_\omega x + D y_\omega = d_\omega$$
$$y_\omega \geq 0$$

- Equivalent to:

$$P_\omega :\ z_\omega(x) = \min\ f y_\omega$$
$$\text{subject to } D y_\omega = d_\omega - B_\omega x$$
$$y_\omega \geq 0$$

Benders decomposition

- Dual of P_ω:

$$\max\ p_\omega(d_\omega - B_\omega x)$$
$$s.t.\ p_\omega D \le f$$

$$P = \{p \mid pD \le f\}$$
$$p^i\ (i \in I) \text{ extreme points}$$
$$\text{w}^{\text{j}}\ (j \in J) \text{ extreme rays}$$

- We have

$$w^j(d_\omega - B_\omega x) \le 0 \quad \forall j \qquad \text{(required for primal feasibility)}$$

$$z_w = \max_{i \in I}\ p^i(d_\omega - B_\omega x)$$

$$\text{Hence}\quad p^i(d_\omega - B_\omega x) \le z_\omega \quad \forall i$$

Benders decomposition: master problem

$$\min cx + \sum_{\omega=1}^{K} \alpha_\omega z_\omega$$

$$s.t.$$

$$Ax = b$$

$$p^i (d_\omega - B_\omega x) \le z_\omega \quad \forall i, \omega$$

$$w^j (d_\omega - B_\omega x) \le 0 \quad \forall j, \omega$$

$$x \ge 0$$

Number of constraints increased
Number of variables has decreased
Cutting plane algorithm!

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Benders decomposition: initial relaxed master problem

$$\min cx + \sum_{\omega=1}^{K} \alpha_{\omega} z_{\omega}$$

$$s.t.$$

$$Ax = b$$

$$x \geq 0$$

Benders decomposition: Solution algorithm

1. Solve relaxed master problem: x^*, z_1^*,…, z_K^*
2. For all ω do
   1. Solve sub problem by dual simplex

$$\min\ fy_\omega \quad s.t.\ Dy_\omega = (d_\omega - B_\omega x^*), \quad y_\omega \geq 0.$$

   2. If optimum > z^*_ω add

$$p^i(d_\omega - B_\omega x) \leq z_\omega, \text{ where } p^i \text{ optimal extreme point in dual}$$

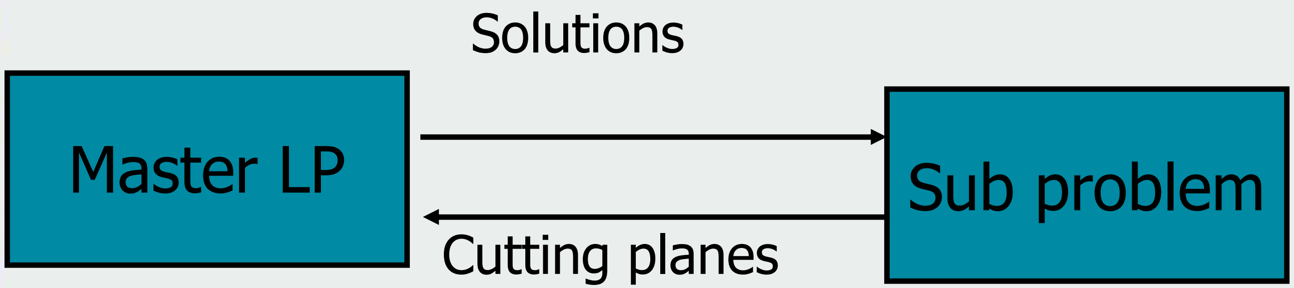
   3. If sub problem infeasible, then dual has infinite solution add

$$w^j(d_\omega - B_\omega x) \leq 0, \text{ where } w^j \text{ extreme ray in dual}$$

$$\text{which gives infinite solution value}$$

   4. If feasible and optimum ≤ z^*_ω do nothing
3. If feasible and optimum ≤ z^*_ω for all ω, problem solved to optimality, otherwise solve relaxed problem again (go to step 1)

Benders decomposition: use cutting plane algorithm



Example: facility location, just one scenario

Depot i	1	2	3
Fixed c F_i	2	3	3
customers	c_{ij}		
1	2	4	5
2	3	3	4
3	4	1	2
4	5	2	1
5	7	6	3

v_j and w_{ij} first round

Cust j	v_j	w_{1j}	w_{2j}	w_{3j}
1	2	0	0	0
2	3	0	0	0
3	4	0	3	2
4	5	0	3	4
5	7	0	1	4