Advanced linear programming

11.1, 11.2: Solving integer linear programming problems:

Totally Unimodular Matrices

Cutting planes

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This lecture: More on solving ILPs

- Totally Unimodular Matrices: LP-relaxation is integral ☺

If LP-relaxation is fractional (usually)

- Cutting planes
- Branch-and cut

Cutting plane algorithm

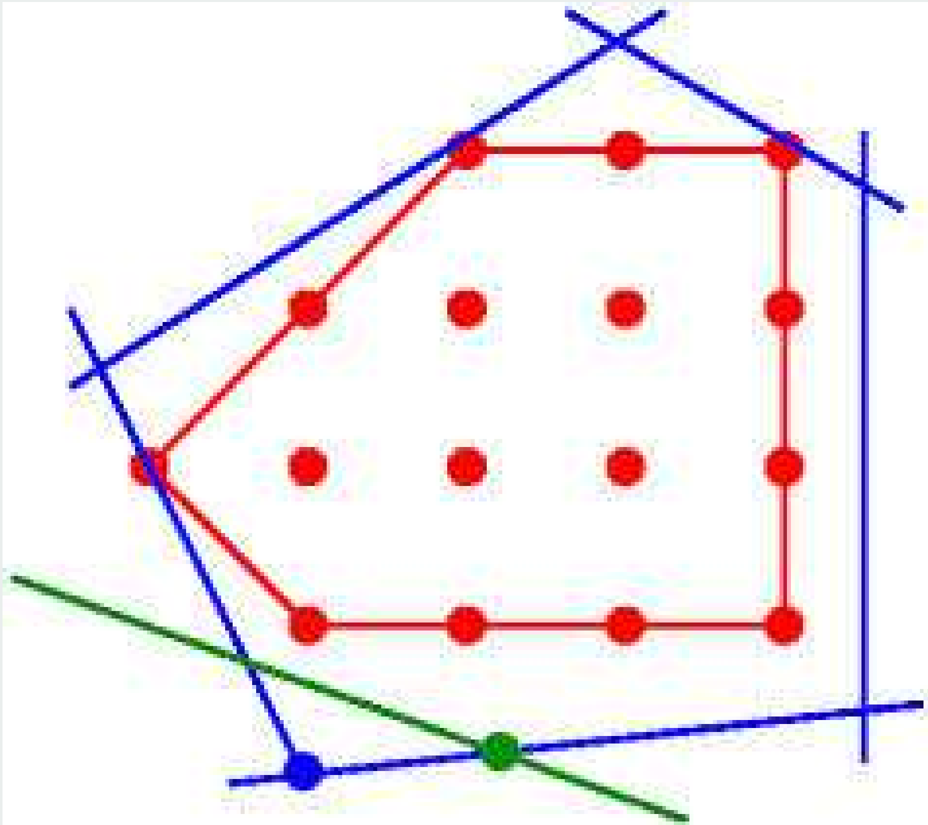
1. Solve the LP-relaxation. Let *x** be an optimal solution.
2. If *x** is integral, stop *x** optimal solution to the integer linear programming problem.
3. If not, add a *valid inequality* that is not satisfied by *x**

and go to Step 1. Solve separation problem.

Definition: *A valid inequality* is a linear constraint that is satisfied by all integral solutions.

If it cuts of fractional solutions it acts as *cutting plane*.

Cutting plane algorithm



Gomory cuts

- Take row from end tableau corresponding to fractional basic variable

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{a}_{io} \text{ with } \bar{a}_{io} \text{ fractional}$$

- Gomory cut

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{a}_{io} \rfloor$$

- You can find the optimum by adding Gomory cuts, no practical method since you may need exponentially many steps

Problem-specific valid inequalities

- Usually classes of inequalities

- Cover inequalities for knapsack problems
- Weighted independent set
- Single-machine scheduling

"Cutting plane"

Branch-and-cut

- Combination of cutting planes and ILP solving by branch-and-bound
- Applied in well-known ILP-solvers:
  - CPLEX,
  - GUROBI,
  - EXPRES-MP
  - COIN-OR

Solving ILP by branch-and-bound or branch-and-cut

Let x* be the best known feasible solution

Search strategy

1. Select an active sub problem F_i (unevaluated node)
2. If F_i is infeasible: delete node
3. Compute upper bound $Z_{LP}(F_i)$ by solving LP-relaxation and feasible solution x_f (by rounding)

   If $Z_{LP}(F_i) \leq$ value x* delete node (bounding)

   If x_f is better than x*: update x*

   If solution x_{LP} to LP-relaxation is integral,

   then If x_{LP} is better than x*: update x* and node finished,

   otherwise split node into two new subproblems (branching)
4. Go to step 1

How many cuts? Which classes?

Primal heuristic

Branching strategy

Optional

This if for maximization problem, the book uses a minimization problem.

Branch-and-cut is a framework algorithm!!

Last century, tailoring to your own problem was necessary in most cases and a huge amount of research has been undertaken in doing this:

- Pre-processing
- Classes of valid inequalities
- How many cuts in each node?
- Search strategy
- Branching strategy
- Primal heuristics
- Reduced cost fixing

Now solvers like CPLEX and Gurobi successfully (and secretly) apply a lot of the above techniques.

- Tailoring of branch-and-cut sometimes successful, especially valid inequalities in case of a weak LP-relaxations from http://www.lnmb.nl/conferences/2015/programlnmbconference/LNMB-NGB_Bixby.pdf

1998 … A New Generation of MIP Codes

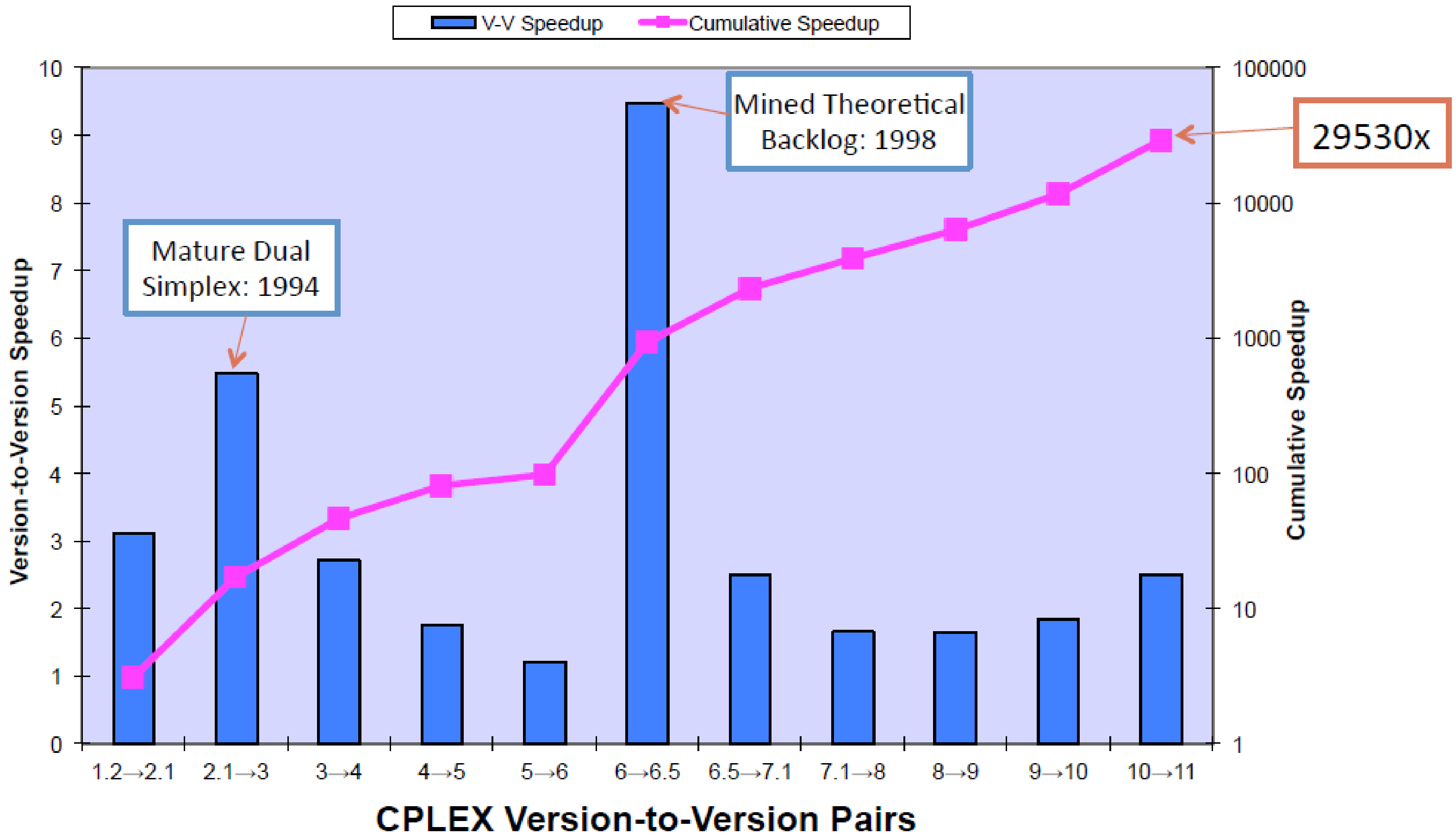
- Linear programming
  - Stable, robust dual simplex
- Variable/node selection
  - Influenced by traveling salesman problem
- Primal heuristics
  - 12 different tried at root
  - Retried based upon success
- Node presolve
  - Fast, incremental bound strengthening (very similar to Constraint Programming)
- Presolve – numerous small ideas
  - Probing in constraints:

    $\sum x_j \leq (\sum u_j)\, y, \quad y = 0/1$

    $\rightarrow x_j \leq u_j y$ (for all j)
- Cutting planes
  - Gomory, mixed-integer rounding (MIR), knapsack covers, flow covers, cliques, GUB covers, implied bounds, zero-half cuts, path cuts

Gurobi Optimization

Speedups 1991-2008



World record exact TSP solving

- Branch-and-cut

CONCORDE:

http://www.math.uwaterloo.ca/tsp/concorde/index.html

Still we are talking about combinatorial optimization

- We have many, many, many variables
- Still large computation times
- Decomposition approaches often help
- **We now go to Chapter 6: Large-scale optimization**
  - *Decompositions for large LP's*
  - *Since these principles are usually used for solving ILP''s, Ch 10 was done first.*