Corrections to proofs in "An equilibrium closure result for discontinuous games"*

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G. Carmona has kindly pointed out to me that two proofs in my paper "An equilibrium closure result for discontinuous games" (Economic Theory 48 (2011), 47-65), a paper from now on referred to as ECR, contain errors. Here I correct these errors. As a consequence, all results in ECR, and in particular the main result in Theorem 1, remain valid as stated. Below notation, references, etc. are borrowed freely from ECR.

Proof of Lemma 5 in ECR. The proof must be corrected as follows. By compactness of $S$ and $Z$ the function $s \mapsto \phi_i(s) := \inf_{z \in L_s} z_i$ is not upper semicontinuous, as I claimed in ECR, but lower semicontinuous (as a direct consequence of Berge’s theorem). That still causes $(s, z) \mapsto \phi_i(s) - z_i$ to be Borel measurable on $S \times Z$, so the set $E$, consisting of all $(s, z) \in L$ such that $\phi_i(s) - z_i \geq 0$, is Borel measurable. Now observe that for every $s \in S$ the section $E_s$, which is precisely equal to $\arg\min_{z \in L_s} z_i$, is compact. This allows me to apply Theorem 1 in reference [7] of ECR, and it yields that $s \mapsto \arg\min_{z \in L_s} z_i$ has a measurable selection $\hat{q}_i : S \to Z$, just as was stated in Lemma 5.

Proof of Case 2 on p. 64 of ECR. As stated, the proof of case 2 makes sense until its last line: “So (19) now follows from ... $\hat{q}_i \leq q_i^{**}$”. This line must be replaced by an application of the following lemma, which implies that (19) of ECR is indeed valid in case 2.¹

**Lemma** For every fixed $\bar{s}_i \in S_i$  

\[
\int_{S} q_i^{**} d\tilde{\beta} \geq \int_{S_{-i}} \hat{q}_i(\bar{s}_i, \cdot) d\tilde{\beta}_{-i}.
\]  

¹I am very grateful to Guilherme Carmona (University of Cambridge, UK) for bringing these omissions to my attention.

¹In addition, in lines 15-16 of p. 64 $\hat{q}_i$ must be read as $\hat{q}_i^i$.  

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Proof. By assumption (3) in ECR there exists a subsequence \( \{S_i^{(n')}\}_{n'} \) (which is \( \tilde{s}_i \)-dependent, of course) and associated elements \( \tilde{s}_{i,n'} \in S_i^{(n')} \) such that \( \tilde{s}_i = \lim_{n' \to \infty} \tilde{s}_{i,n'} \). By the definition of the mixed Nash equilibrium profiles (\( \alpha_i \) dependent, of course) and associated elements \( \bar{\alpha}_{i-} \), to handle the right side in (2), I define a sequence \( \{\rho_i^{(n')}\}_{n'} \) in Prob(\( S_{-i} \times Z \)) by setting \( \rho_i^{(n')} (A \times B) := \bar{\alpha}_{i-}^{(n')} (A \cap (\tilde{q}_i^{(n')})^{-1}(B)) \), where \( \tilde{q}_i^{(n')} := q_i^{(n')} (\tilde{s}_{i,n'}, \cdot) : S_{-i} \to Z \) defines a continuous function (use (4) in ECR). The definition of \( \rho_i^{(n')} \) is motivated by the identity
\[
\int_{S_{-i}} q_i^{(n')} (\tilde{s}_{i,n'}, \cdot) d\bar{\alpha}_{i-}^{(n')} = \int_{S_{-i} \times Z} z_i \rho_i^{(n')} (d(s_{-i}, z)),
\]
and its definition mimicks the definition of \( \pi_i^{(n)} \in \text{Prob}(S \times Z) \), which plays a major role in ECR. Recall here the following standard convention: any probability measure on \( S_{-i} \), such as \( \bar{\alpha}_{i-}^{(n')} \), is automatically extended to a probability measure on the larger set \( S_{-i} \), simply by setting it to zero on \( S_{-i} \setminus S_{-i}^{(n')} \); see p. 49 in ECR. I can now mimick arguments given in ECR by replacing the \( \pi_i^{(n)} \) by the current \( \rho_i^{(n')} \). This gives me (i)–(iii) stated below. Namely, just as in the proof of Lemma 1 in ECR, the tightness of \( \{\rho_i^{(n')}\}_{n'} \), which results from combining the compactness of \( Z \) with the weak convergence of \( \{\bar{\alpha}_{i-}^{(n')}\}_{n'} \) to \( \bar{\beta}_{-i} \) as in ECR, implies

(i) \( \{\rho_i^{(n')}\}_{n'} \) contains a subsequence \( \{\rho_{i''}^{(n'')}\}_{n''} \) which converges weakly to some limit probability measure \( \rho^* \)

by applying both Prohorov’s theorem and its converse (both are recalled in ECR). Here I accept that both \( \{\rho_{i''}^{(n'')}\}_{n''} \) and \( \rho^* \) may depend on the initial choice of the fixed \( \tilde{s}_i \in S_i \). By Propositions 2 and 3 of ECR it follows from (i) that the support \( \text{supp} \rho^* \) is contained in \( L_{s_{i,n'}} \supset \text{supp} \rho_i^{(n')} \subset \text{supp} \rho_i^{(n')} \), whence in \( L_{s_{i,n'}} \{(s_{-i}, \tilde{q}_i^{(n')} (s_{-i})) : s_{-i} \in S_{-i}^{(n')} \} \). By \( \tilde{s}_{i,n'} \to \tilde{s}_i \) and the definition of the Kuratowski limes superior set \( L := L_{s_{i,n'}} \{(s, \tilde{q}_i^{(n)} (s)) : s \in S^{(n)} \} \) in Lemma 1 of ECR, this implies

(ii) \( \text{supp} \rho^* \) is contained in the section \( L_{\tilde{s}_i} := \{(s_{-i}, z) \in S_{-i} \times Z : (\tilde{s}_i, s_{-i}, z) \in L \} \).

Because \( \bar{\alpha}_{i-}^{(n'')} \) is the \( S_{-i} \)-marginal of \( \rho_i^{(n'')} \) for every \( n'' \), I also conclude directly from (i) that

(iii) \( \bar{\beta}_{-i} \) is the marginal of \( \rho^* \) on \( S_{-i} \);

cf. step 1 in the proof of Lemma 2 in ECR. Now for \( n' \to \infty \) the left side in (2) converges to the left side of (1) by Lemma 4.b in ECR. Also, if I replace \( n' \) on the right of (2) by \( n'' \), then the resulting expression converges to \( \int_{S_{-i} \times Z} z_i \rho^* (d(s_{-i}, z)) \) for \( n'' \to \infty \), by (i) and the identity (3). So by the above I obtain
\[
\int_S q_i^{(n'')} d\tilde{\beta}_i \geq \int_{S_{-i} \times Z} z_i \rho^* (d(s_{-i}, z)) = \int_{L_{\tilde{s}_i}} z_i \rho^* (d(s_{-i}, z)),
\]
where the equality on the right holds by (ii). Finally, I observe that for all \( (s_{-i}, z) \in L_{\tilde{s}_i} \), the definition of \( \tilde{q}_i \) gives \( z_i \geq \tilde{q}_i (\tilde{s}_i, s_{-i}) \), so the rightmost expression above is
larger than or equal to

\[ \int_{L_i} q_i^l(\bar{s}_i, s_{-i}) \rho^*(d(\bar{s}_i, z)) = \int_{S_{-i} \times Z} q_i^l(\bar{s}_i, s_{-i}) \rho^*(d(s_{-i}, z)) = \int_{S_{-i}} q_i^l(\bar{s}_i, \cdot) d\bar{\beta}_{-i}, \]

where I use (ii) for the first identity and (iii) for the second one. This proves (1).