On subdifferential calculus – highlights

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Fundamentals:

Working with $+\infty$ and $-\infty$:

- $\forall_{\alpha\in(-\infty,+\infty]}\ \alpha + (+\infty) = (+\infty) + \alpha = +\infty.$
- $\forall_{\alpha\in[-\infty,+\infty)}\ \alpha - (+\infty) = \alpha + (-\infty) = -\infty.$
- neither $(+\infty) - (+\infty)$ nor $(+\infty) + (-\infty)$ etc. defined!
- careful! $2 + (+\infty) = 3 + (+\infty) \not\Rightarrow 2 = 3$
- $\forall_{\alpha\in(0,+\infty]}\ \alpha \cdot (+\infty) = +\infty$
- $\forall_{\alpha\in[-\infty,0)}\ \alpha \cdot (+\infty) = -\infty$
- By definition: $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0.$
- $\forall_{\alpha\in\mathbb{R}}\ \alpha/(+\infty) = \alpha/(-\infty) = 0.$
- neither $(+\infty)/(+\infty)$ nor $(+\infty)/(-\infty)$ etc. defined!
- $(+\infty)/(+\infty)$, etc. undefined.
- careful! $2/(+\infty) = 3/(+\infty) \not\Rightarrow 2 = 3$

Convex sets in $\mathbb{R}^n$:

Definition A.1: $S \subset \mathbb{R}^n$ is *convex* if

$$\forall_{x_1,x_2 \in S} \forall_{\lambda \in [0,1]} \lambda x_1 + (1-\lambda) x_2 \in S.$$

Convex functions:

Definition 2.1: Let $S \subset \mathbb{R}^n$ be convex. Then $f : S \to (-\infty, +\infty]$ is *convex on* S if

$$\forall_{x_1,x_2 \in S} \forall_{\lambda \in [0,1]} f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

Also, f is *strictly convex on* S if

$$\forall_{x_1,x_2 \in S, x \neq x_2} \forall_{\lambda \in (0,1)} f(\lambda x_1 + (1-\lambda) x_2) < \lambda f(x_1) + (1-\lambda) f(x_2).$$

Remark: $f \neq -\infty$, so $\lambda(+\infty) + (1-\lambda)(-\infty)$ cannot confuse us.

Associated definition: Let $f : S \to [-\infty, +\infty)$. Then: f is *(strictly) concave* on $S \Leftrightarrow -f$ is (strictly) convex on S.

Example: $f_1(x) := p^t x + \alpha$ is *affine*, i.e., both convex and concave, on $\mathbb{R}^n$ for any $p \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. It is neither strictly convex nor strictly concave.

Example: $f_2(x) := \beta |x|^2$ is strictly convex on $\mathbb{R}^n$ if $\beta > 0$. It is strictly concave on $\mathbb{R}^n$ if $\beta < 0$.

Example (Exercise 2.1c): Let $S := \mathbb{R}_+$. Define $f_3 : S \to (-\infty, +\infty]$ by $f_3(x) := 1/x$ if $x > 0$ and by $f_3(0) := \gamma$. Then f_3 can only be made convex on S by setting $\gamma = +\infty$.

Example (Exercise 2.7b): Define $f_4 : \mathbb{R} \to (-\infty, +\infty]$ by $f_4(x) := 1 - \sqrt{1 - x^2}$ if $|x| \leq 1$ and $f_4(x) = +\infty$ if $|x| > 1$. Then f_4 is convex on $\mathbb{R}$.

Definition (Exercise 2.2): Let $S \subset \mathbb{R}^n$ be convex. Then $f : S \to (-\infty, +\infty]$ is *quasiconvex on* S if

$$\forall_{\alpha \in \mathbb{R}} S_\alpha := \{x \in S : f(x) \leq \alpha\} \text{ is convex}$$

Every convex function on $\mathbb{R}^n$ is quasiconvex, but not conversely.

Domain extension by adding values $+\infty$:

Exercise 2.5: Let $S \subset \mathbb{R}^n$ be convex. Let $f : S \to (-\infty, +\infty]$. Define $\hat{f} : \mathbb{R}^n \to (-\infty, +\infty]$ by

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

Exercise: $\hat{f}$ convex on $\mathbb{R}^n \Leftrightarrow f$ convex on S.

Consequence: From now on we mainly consider convex functions on $\mathbb{R}^n$. This is thanks to working with $+\infty$!

New habit: Speak of "convex functions" instead of "convex functions on $\mathbb{R}^n$".

Definition 2.2: Let $f : \mathbb{R}^n \to (-\infty, +\infty]$. The *essential domain* of f is defined by

$$\text{dom } f := \{x \in \mathbb{R}^n : f(x) < +\infty\}.$$

Exercise: f convex $\Rightarrow$ dom f is convex, but not conversely.

Connections between convex sets and convex functions:

From convex sets to convex functions:

Definition 2.3: Let $S \subset \mathbb{R}^n$. The *indicator function* χ_S of S is defined by

$$\chi_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

Exercise: S convex set $\Leftrightarrow$ χ_S convex function.

From convex functions to convex sets:

Definition 2.4: Let $f : \mathbb{R}^n \to (-\infty, +\infty]$. The *epigraph* epi $f \subset \mathbb{R}^{n+1}$ is defined by

$$\text{epi } f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq y\}.$$

Exercise: f convex function $\Leftrightarrow$ epi f convex set.

Remark: Many proofs of results for convex functions "work" on their convex epigraphs by means of separation results (see Appendix A).

Example: For $S \subset \mathbb{R}^n$ let $f := \chi_S$. Then $epi f = S \times \mathbb{R}_+$.

From convex functions to more convex functions:

Easy: Let $f_i : \mathbb{R}^n \to (-\infty, +\infty]$ be convex and let $\alpha_i \in [0, +\infty]$ for $i = 1, \ldots, m$. Then $f(x) := \sum_{i=1}^m \alpha_i f_i(x)$ defines a convex function, as does $f(x) := \max_{1 \leq i \leq m} \alpha_i f_i(x)$.

Exercise 2.6: Let $S \subset \mathbb{R}^n$ be convex. Let $f : S \to \mathbb{R}$ be convex and let $g : D \to \mathbb{R}$ be **convex and nondecreasing** on a convex interval $D \subset \mathbb{R}$, with $D \supset f(S)$. Then $h(x) := g(f(x))$ defines a convex function $h : S \to \mathbb{R}$.

Example (Exercise 2.7): a. If $f : \mathbb{R}^n \to [0, +\infty]$ is convex on $\mathbb{R}^n$, then so is f^2. However, f^2 need not be convex if f can also take negative values.
b. $f(x) := 1 - \sqrt{1 - x^2}$ is convex on $[-1, +1]$.
c. $f(x) := \exp(x^2)$ is convex on $\mathbb{R}$.

Subdifferentials and subgradients of convex functions

Definition 2.5: Let $f : \mathbb{R}^n \to (-\infty, +\infty]$, $f \not\equiv +\infty$, and let $x_0 \in \text{dom} f$ (so $f(x_0) \in \mathbb{R}$).
a. A *subgradient of* f *at* x_0 is a $\xi \in \mathbb{R}^n$ with

$$f(x) \geq f(x_0) + \xi^t(x - x_0) \text{ for all } x \in \mathbb{R}^n.$$

b. The *subdifferential* of f at x_0 is the **set**

$$\partial f(x_0) := \{\xi \in \mathbb{R}^n : \xi \text{ is subgradient of } f \text{ at } x_0\}.$$

This set may be empty!

Observation: If $x_0 \notin \text{dom} f$ (so $f(x_0) = +\infty$) then $\partial f(x_0) = \emptyset$. But $\partial f(x_0) = \emptyset$ is also possible for $x_0 \in$ dom f.

Example: a. Let $f(x) := 1 - \sqrt{1 - x^2}$ on $[-1, +1]$ and define $f(x) := +\infty$ if $x < -1$ or $x > 1$. Then f is convex and $1 \in$ dom f. However, $\partial f(1) = \emptyset$.
b. Let $f(x) := |x|$ on $\mathbb{R}$. Then $\partial f(2) = \{1\}$, $\partial f(-3) = \{-1\}$ and $\partial f(0) = [-1, +1]$.

For differentiable convex functions: "subgradient = gradient"':

Proposition 2.6: Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be convex. If f is differentiable at $x_0 \in$ int dom f, then $\partial f(x_0) = \{\nabla f(x_0)\}$.

Here: "int" means "interior".

Example (Exercise 2.9b): In previous example with $f(x) = 1 - \sqrt{1-x^2}$ on $[-1,+1]$ and $f(x) = +\infty$ if $x < -1$ or $x > 1$, one has $\partial f(x) = \{x/\sqrt{1-x^2}\}$ for every $x \in (-1, 1)$.

How to determine convexity of functions:

Proposition 2.7: Let $S \subset \mathbb{R}^n$ be open and convex. Let $f : S \to \mathbb{R}$.

(i) If f is differentiable, then f is convex on $S \Leftrightarrow$

$$\forall_{x_1,x_2 \in S} (\nabla f(x_1) - \nabla f(x_2))^t (x_1 - x_2) \geq 0.$$

(i') If f is differentiable, then f is strictly convex on $S \Leftrightarrow$

$$\forall_{x_1,x_2 \in S, x_1 \neq x_2} (\nabla f(x_1) - \nabla f(x_2))^t (x_1 - x_2) > 0.$$

(ii) If f is twice continuously differentiable, then f is convex on $S \Leftrightarrow$ the Hessian matrix

$$H_f(x) := \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{i,j}$$

is positive semidefinite at every point x of S.

(ii') If f is twice continuously differentiable, then $H_f(x)$ is positive definite at every point x of $S \Rightarrow f$ is strictly convex on S

Definition: An $n \times n$ matrix M is *positive semidefinite* if $d^t M d \geq 0$ for all $d \in \mathbb{R}^n$. And M is *positive definite* if $d^t M d > 0$ for all $d \in \mathbb{R}^n$, $d \neq 0$.

Corollary 2.8: Let $S \subset \mathbb{R}$ be open and convex. Let $f : S \to \mathbb{R}$.

(i) If f is differentiable, then f is convex [strictly convex] on $S \Leftrightarrow f'$ is nondecreasing [increasing] on S.

(ii) If f is twice continuously differentiable, then f is convex [strictly convex] on $S \Leftrightarrow f''(x) \geq 0$ $[f''(x) > 0]$ for all $x \in S$.

MR-theorem and "small" KKT-theorem

Theorem 2.9 (Moreau-Rockafellar) Let $f, g : \mathbb{R}^n \to (-\infty, +\infty]$ be convex. Then

$$\forall_{x_0 \in \mathbb{R}^n} \partial f(x_0) + \partial g(x_0) \subset \partial(f+g)(x_0).$$

Moreover, if int dom $f \cap$ dom $g \neq \emptyset$. Then also

$$\forall_{x_0 \in \mathbb{R}^n} \partial(f+g)(x_0) \subset \partial f(x_0) + \partial g(x_0).$$

Comment: First part is trivial. Proof of second part goes by separating hyperplane Theorem A.4, applied to disjoint convex sets Λ_f and Λ_g that are "epigraph-like" – see syllabus.

Theorem 2.10 ("small KKT"): Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and let $S \subset \mathbb{R}^n$ be nonempty convex. Consider the optimization problem

$$(P) \quad \inf_{x \in S} f(x).$$

Then

$$\bar{x} \in S \text{ optimal for } (P) \Leftrightarrow \exists_{\bar{\xi} \in \partial f(\bar{x})} \forall_{x \in S} \bar{\xi}^t (x - \bar{x}) \geq 0.$$

Sketch of proof. Observe

$$\bar{x} \in S \text{ optimal for } (P) \Leftrightarrow 0 \in \partial(f + \chi_S)(\bar{x}).$$

Then apply MR-theorem to right side.