Recall from Thm. 3.1(iii):

\[ h \text{ continuous at } 0 \Rightarrow \partial h(0) \neq \emptyset \Rightarrow \]
\[ \Rightarrow \text{set of dual solutions } = -\partial h(0) \neq \emptyset \]

Seek to relax such continuity for special perturbation functions \( h \), namely \textit{polyhedral} \( h \).

\textbf{Def.} \( S \subseteq \mathbb{R}^n \) is \textbf{polyhedral} if

\[ S = \cap_{j=1}^{J} \{ x \in \mathbb{R} : y_j^t x \leq \alpha_j \} \]

for some \( \{y_1, \ldots, y_J\} \subseteq \mathbb{R} \) and \( \{\alpha_1, \ldots, \alpha_J\} \subseteq \mathbb{R} \). This is \textit{special} type of convex set.

\textbf{Def.} \( f : \mathbb{R}^n \rightarrow [-\infty, +\infty] \) is \textbf{polyhedral} if \( \text{epi } f \subset \mathbb{R}^n \times \mathbb{R} \) is polyhedral. This is \textit{special} type of convex function.
Prop. 1.2. Let \( f : \mathbb{R}^n \to (-\infty, +\infty], f \neq +\infty, \) be polyhedral. Then \( f \) is of the following form: there exist a polyhedral set \( P \subset \mathbb{R}^n \) and affine functions \( a_1, \ldots, a_N : \mathbb{R}^n \to \mathbb{R} \) such that

\[
 f(x) = \chi_P(x) + \max_{1 \leq i \leq N} a_i(x).
\]

Moreover,

\[
 \partial f(x_0) \neq \emptyset \text{ for every } x_0 \in \text{dom } f.
\]

Obs: The latter does not require the usual condition

\[
 f \text{ continuous at } x_0 \iff x_0 \in \text{int dom } f \quad (\ast)
\]

Example. a. Recall: for \( n = 1 \) the convex non-polyhedral function

\[
 f(x) := \begin{cases} 
 1 - \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\
 +\infty & \text{otherwise}
\end{cases}
\]

had \( \partial f(1) = \emptyset \) ”because” \((\ast)\) is not valid for \( x_0 = 1 \).

b. However, the polyhedral function

\[
 f(x) := \begin{cases} 
 |x| & \text{if } |x| \leq 1 \\
 +\infty & \text{otherwise}
\end{cases}
\]

has \( \partial f(1) \neq \emptyset \), although \((\ast)\) is not valid for \( x_0 = 1 \).
Fenchel’s duality theorem can now be modified: if 
\((P_F)\) is polyhedral, then the sufficient condition for 
strong stability becomes much weaker than before:

Let \(A\) be \(m \times n\)-matrix. Consider 
\[
(P_F) \inf_{x \in \mathbb{R}^n} [f(x) + g(Ax)],
\]
where \(f : \mathbb{R}^n \to (-\infty, +\infty]\) and \(g : \mathbb{R}^m \to (-\infty, +\infty]\) 
are **polyhedral** functions. Suppose \(\inf(P_F) \in \mathbb{R}\).

Define associated *Fenchel dual problem*:
\[
(D_F) \sup_{q \in \mathbb{R}^m} [-f^*(A^tq) - g^*(-q)].
\]

**Fenchel’s duality thm – polyhedral version**

(i) For all \(x \in \mathbb{R}^n\) and \(q \in \mathbb{R}^m\)

\[-f^*(A^tq) - g^*(-q) \leq f(x) + g(Ax).\]

(ii) If \(0 \in \text{int} (\text{dom } g - A(\text{dom } f))\), then

\[\inf(P_F) = \max(D_F).\]

Moreover, then \(\bar{x} \in \mathbb{R}^n\) is optimal for \((P_F)\) and \(\bar{q} \in \mathbb{R}^k\) is optimal for \((D_F)\) if and only if

\[A^t\bar{q} \in \partial f(\bar{x}) \text{ and } -\bar{q} \in \partial g(A\bar{x}).\]
Application: semidefinite programming duality

$\mathcal{S}^n$: set of all symmetric $n \times n$-matrices

$\mathcal{S}^n$ is Euclidean vector space with inner product

$$\langle X, Y \rangle := \text{tr}(XY) = \sum_{i,j} X_{i,j}Y_{i,j}.$$ 

Two special convex cones in $\mathcal{S}^n$:

$\mathcal{S}^n_+$: set of all **positive semidefinite** matrices

$\mathcal{S}^n_{++}$: set of all **positive definite** matrices

Recall: $A \in \mathcal{S}^n_+ \iff \forall x x^tAx \geq 0$

Recall: $A \in \mathcal{S}^n_{++} \iff \forall x \neq 0 x^tAx > 0$

Fact: $\mathcal{S}^n_+$ is closed and $\mathcal{S}^n_{++} = \text{int} \mathcal{S}^n_+$. 
Let \( A_i \in \mathbb{S}^n \) and \( b_i \in \mathbb{R} \) for \( i = 1, \ldots, m \).

The **primal semidefinite program** is

\[
(P_{SDP}) \inf_{X \in \mathbb{S}^n_+} \{ \text{tr}(CX) : \text{tr}(A_iX) = b_i, i = 1, \ldots, m \}.
\]

Can be written as

\[
(P_{SDP}) \inf_{X \in \mathbb{S}^n_+} \{ \langle C, X \rangle : \langle A_i, X \rangle = b_i, i = 1, \ldots, m \},
\]

Compare to standard linear program

\[
\inf_{x \in \mathbb{R}^n_+} \{ \langle c, x \rangle : \langle a_i, x \rangle = b_i, i = 1, \ldots, m \},
\]

Define **dual problem** of \((P_{SDP})\) by

\[
(D_{SDP}) \sup_{q \in \mathbb{R}^m} \{ b^t q : C - \sum_{i=1}^m q_i A_i \in \mathbb{S}^n_+ \}.
\]