Steropgaven Maat en Integratie, 3-6-10

Turn in on 10-6

Motivation: the hint for problem 16.10 would seem to be very vague; this explains the formulation of the following problem. After solving it, you can easily handle problem 16.10.

Problem 1* Suppose that $\mu(X) < \infty$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of measurable functions which converges to a measurable function u (so for every $\epsilon > 0$ one has $\lim_{j\to\infty} \mu(\{|u_j - u| > \epsilon\}) = 0$).

a. Using problem 16.3, prove that for every $n \in \mathbb{N}$ there exists j_n such that $\mu(\{|u_{j_n} - u_{j_{n+1}}| \geq 2^{-n}\}) \leq 2^{-n}$.

b. Let $B := \limsup_n \{|u_{j_n} - u_{j_{n+1}}| \geq 2^{-n}\}$. Prove that $\mu(B) = 0$ by using problem 6.9 (Borel-Cantelli).

c. Prove that $\lim_{n\to\infty} u_{j_n}(x) = u(x)$ for almost every x in X.

Conclusion (at least on a finite measure space): every sequence of functions converging in measure contains a subsequence that converges a.e. to the same limit function.

Problem 2* The hint for problem 16.6 is phrased very carelessly and remains rather useless even when corrected.[1] However, the spirit of the hint can be used to produce a concrete example of two different[2] limits-in-measure in a simple situation with $X = [0, 1]$, equipped with its Borel σ-algebra. Find an appropriate measure and produce such an example.

[1] (Only) the following is correct for an infinite measure space $(X, \mathcal{A}, \mu)$: there exists a set $\hat{A} \in \mathcal{A}$ which a countable union of sets of finite measure such that $\mu(A \cup \hat{A}) = \mu(\hat{A})$ for every $A \in \mathcal{A}$ with finite measure. For instance, if $X := [0, 1]$ is equipped with the counting measure μ, then any countable subset of $[0, 1]$ can play the role of such $\hat{A}$.

[2] Here the book is also a bit vague: the intention is that the two limits differ on a set of *positive* measure.