Solutions Final Exam M & I, 24-6-10

Erik J. Balder

Problem 1 [16 pt]. Let \((X, \mathcal{A}, \mu)\) be a finite measure space and let \(T : X \to X\) be an \(\mathcal{A}/\mathcal{A}\)-measurable mapping. Then \(T\) is said to preserve the measure \(\mu\) if \(\mu(T^{-1}(A)) = \mu(A)\) for every \(A \in \mathcal{A}\).
a. Denote by \(T^n : X \to X\) the \(n\)-fold composition of \(T\) with itself (i.e., \(T^1 := T\), \(T^2 := T \circ T\), \(T^3 := T \circ T \circ T\), etc.). Prove by means of induction that \(b.\) For fixed \(\mu\) prove that \(\mu(T^n) = \mu(T)\).
\(c.\) For \(m \in \mathbb{N}\) define \(C_m := (T^{-1})^m(C)\). Prove that the sets \(C_m\) are mutually disjoint.
\(d.\) Prove that \(\mu(C) = 0\).
\(e.\) Provide a concrete counterexample to show that the result in \(d\) does not continue to hold if \(\mu(X) = \infty\).

Solution. a. Let \((H_n) : T^n\) is measure preserving. Then \((H_n) \Rightarrow (H_{n+1})\) by \(\mu((T^{n+1})^{-1}(A)) = \mu(T^{-1}((T^n)^{-1}(A))) = \mu(T^n)^{-1}(A) \equiv \mu(A)\).
b. Clearly, \(C = B \cap (T_n D_n)\), with \(D_n := (T^n)^{-1}(X \setminus B)\). Every \(D_n\) belongs to \(\mathcal{A}\), because \(T^n\), the composition of measurable mappings, is \(\mathcal{A}/\mathcal{A}\)-measurable. Hence, \(C \in \mathcal{A}\).
c. Consider \(k \neq m\) and suppose \(k > m\) without loss of generality. Then \(x \in C_k \cap C_m\) would imply \(T^m(x) \in C\), whence \(T^{m+n}(x) \notin B\) for all \(n \in \mathbb{N}\). Hence, \(n = k - m\) gives \(T^k(x) \notin B\), which contradicts \(x \in C_k\), because the latter implies \(T^k(x) \in C \subset B\).
d. We have \(\mu(C_m) = \mu(C)\) by part a. So \(\mu(C) > 0\) would imply \(\mu(\cup_m C_m) = \sum_m \mu(C_m) = \infty\) by part c. By \(\mu(X) < \infty\) this is impossible. Conclusion: \(\mu(C) = 0\).

Problem 2 [16 pt]. Let \((X_i, \mathcal{A}_i, \mu_i)\), \(i = 1, 2, 3\) be three finite measure spaces. By complete analogy to the case of two measure spaces in the book, one can introduce the following objects (you need not prove this):
\(i.\) \(\mathcal{A} := \sigma(\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3)\); this is called the product \(\sigma\)-algebra on \(X := X_1 \times X_2 \times X_3\).
\(ii.\) The unique extension \(\rho : \mathcal{A} \rightarrow [0, \infty]\) which extends \(\rho : \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \rightarrow [0, \infty]\), given by \(\rho(\mathcal{A} \times \mathcal{B} \times \mathcal{C}) := \mu_1(A)\mu_2(B)\mu_3(C)\), to a \(\sigma\)-finite measure on \((X, \mathcal{A})\); this is called the product measure of \(\mu_1\), \(\mu_2\) and \(\mu_3\).

Solution. a. We invoke Lemma 13.3\(^1\) with \(\mathcal{F} = \mathcal{A}_1 \times \mathcal{A}_2 \) and \(\mathcal{G} = \mathcal{A}_3\) (here the exhaustive sequences are trivial: use \(F_j \equiv X_1 \times X_2\) and \(G_i \equiv X_3\)). This yields the desired identity by \(A := \sigma(\mathcal{F} \times \mathcal{A}_3)\) \(L =_{13.3} \sigma(\mathcal{F}) \otimes \mathcal{A}_3\), with \(\sigma(\mathcal{F}) =: \mathcal{A}_1 \otimes \mathcal{A}_2\). Of course, an independent proof of the nontrivial inclusion \(\supset\) in \(i\) (which rather resembles the proof of Lemma 13.3) can also be given.

b. Let \(\pi := (\mu_1 \times \mu_2) \times \mu_3\). Then, by the definition of the product of two measures applied twice, we have \(\pi(A \times B \times C) = \mu_1(A)\mu_2(B)\mu_3(C) = \rho(A \times B \times C)\) for every \(A \times B \times C\)

\(^1\)Lemma 13.3: if \(\mathcal{B} = \sigma(\mathcal{F}), \mathcal{C} = \sigma(\mathcal{G})\) then \(\sigma(\mathcal{F} \times \mathcal{G}) = \mathcal{B} \otimes \mathcal{C}\), provided that \(\mathcal{F}\) and \(\mathcal{G}\) contain exhaustive sequences \((F_i)\) and \((G_i)\).
in the class $\mathcal{H}$ of all measurable rectangles. We now apply the uniqueness Theorem 5.7.\footnote{Theorem 5.7: if two measures $\pi$ and $\rho$ coincide on a class $\mathcal{H} \subset \mathcal{A}$, closed for finite intersections and generating $\mathcal{A}$, and if a monotone sequence $(H_j)_j$ exists with $\pi(H_j) = \rho(H_j) < \infty$ for all $j$ and $H_j \uparrow X$, then $\pi$ and $\rho$ coincide on $\mathcal{A}$.}

Problem 3 [18 pt]. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $f : X \to \mathbb{R}_+$ be a nonnegative $\mu$-integrable function with the following property: there exists a constant $c \in \mathbb{R}$ such that $\int_X f^n \, d\mu = c$ for every $n \in \mathbb{N}$. Prove that there exists $A \in \mathcal{A}$ such that $f(x) = 1_A(x)$ for almost every $x$ in $X$.

Solution. Step 1: $0 \leq f \leq 1$ a.e. Let $B := \{ f > 1 \}$; then $\infty > c \geq \int_B f^n$ and on $B$ we have $f^n(x) = (f(x))^n \uparrow \infty$, so $\mu(B) = 0$ by the monotone convergence theorem.

Step 2: $f \in \{ 0, 1 \}$ a.e. For $C := X \setminus B$ step 1 implies $\int_C f^2 = c = \int_C f$, so $\int_C (f - f^2) = 0$, where $f - f^2 \geq 0$. Hence, $f = f^2$ a.e. on $C$, i.e., $f \in \{ 0, 1 \}$ a.e. on $C$.

Step 3. By steps 1-2 we have $f \in \{ 0, 1 \}$ a.e. on $X$. Let $A := \{ f = 1 \}$ and we are done.

Alternative step 2: $c = \mu(\{ f = 1 \})$. By $f^n \downarrow 0$ on $\{ f = 1 \}$ we get $c = \int_{\{ f = 1 \}} 1 + \int_{\{ 0 < f < 1 \}} f^n \to \mu(\{ f = 1 \})$ (MCT), so $c = \mu(\{ f = 1 \})$, but then $\Rightarrow$ with $n := 1$ becomes $0 = \int_{\{ 0 < f < 1 \}} f$, causing $\mu(\{ 0 < f < 1 \}) = 0$. Now go to step 3.

Problem 4 [16 pt]. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(u_j)_j$ be a sequence of $\mathcal{A}$-measurable functions $u_j : X \to \mathbb{R}$. Let $u : X \to \mathbb{R}$ also be $\mathcal{A}$-measurable. Prove the following equivalence: the sequence $(u_j)_j$ converges to $u$ in measure if and only if $\int_X \frac{|u_j - u|}{1 + |u_j - u|} \, d\mu \to 0$ for $j \to \infty$.

Solution. Write $v_j := u_j - u$ and note: $\xi \mapsto \xi/(1 + \xi)$ is strictly increasing on $\mathbb{R}_+$.\footnote{This was course Exercise 16.8.}

$\Rightarrow$: Give $\eta > 0$; then for any $\epsilon > 0$ we have $\mu(\{ |v_j| > \epsilon \}) < \eta/2$ for $j$ large enough, so $\int_{\{ |v_j| > \epsilon \}} |v_j|/(1 + |v_j|) + \int_{\{ |v_j| \leq \epsilon \}} |v_j|/(1 + |v_j|) \leq \eta/2 + \epsilon/(1 + \epsilon) \mu(X) < \eta$ for $j$ large enough; namely, choose $\epsilon < \eta/(2\mu(X))$.

$\Leftarrow$: Give $\epsilon > 0$. By Markov’s inequality $\mu(|v_j| > \epsilon)/(1 + \epsilon) \leq \int_X |v_j|/(1 + |v_j|) \to 0$. This implies $\mu(|v_j| > \epsilon) \to 0$.

Problem 5 [18 pt]. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\lambda$ be another measure on $(X, \mathcal{A})$, with $\lambda(X) < \infty$. Recall that $\lambda$ is defined to be absolutely continuous with respect to $\mu$ if $\mu(A) = 0 \Rightarrow \lambda(A) = 0$ for every $A \in \mathcal{A}$. Prove that $\lambda$ is absolutely continuous with respect to $\mu$ if and only if $\lim_n \lambda(A_n) = 0$ holds for every sequence $(A_n)_n$ in $\mathcal{A}$ with $\lim_n \mu(A_n) = 0$. Hint. Use contradiction and apply the Borel-Cantelli lemma (exercise 6.9, week 9): $\sum_n \nu(B_k) < \infty$ implies $\nu(\cap_{n=1}^\infty \cup_{k \geq n} B_k) = 0$; this holds for any measure $\nu$ on $(X, \mathcal{A})$.

Solution. $\Leftarrow$: Let $\mu(A) = 0$ and take $A_n \equiv A$. Then $\lambda(A) = 0$ follows.

$\Rightarrow$: If there is $(A_n)_n$ with $\lim_n \mu(A_n) = 0$ but $\lambda(A_n) \neq 0$, then there is a subsequence $(n_j)$ and $\epsilon > 0$ such that $\lambda(A_{n_j}) \geq \epsilon$ for all $j$. Now pick from $(n_j)$ a further subsequence $(m_k)$ as follows: let $m_1$ be the first index $n_j$ with $\lambda(A_{n_j}) < 2^{-1}$, then $m_2$ be the first index $n_j > m_1$ with $\lambda(A_{n_j}) < 2^{-2}$, etc., etc. Then still $\lambda(A_{m_k}) \geq \epsilon$ for all $k$ and now also $\sum_k \lambda(A_{m_k}) < \infty$, causing $\lambda(A_\ast) = 0$ for $A_\ast := \cap_{n=1}^\infty C_p$ with $C_p := \cup_{k \geq p} A_{m_k}$ (by Borel-Cantelli, as suggested). However, now $C_p \downarrow A_\ast$ implies $\lambda(C_p) \downarrow \lambda(A_\ast)$, for $\lambda$ is a finite measure. Also, $\lambda(C_p) \geq \epsilon$ is evident, so $\lambda(A_\ast) \geq \epsilon > 0$, which contradicts $\mu(A_\ast) = 0$ above.
**Hint 1:** Time can be saved by employing Vitali’s theorem. If you use it, then make sure that what you want to use from it is written out completely in your solution. **Hint 2:** In general, if \( \alpha := \liminf_{j} \alpha_j \) in \([-\infty, +\infty]\), then a subsequence of \((\alpha_j)\) converges to \(\alpha\).

**Solution method 1: use Vitali.** By Vitali’s theorem, applied to the sequence \((u_j^-)\), we have \(\int u_j^- \to \int u^- \in \mathbb{R}_+\) (here \((u_j^-)\) converges a.e., whence also in measure, to \(u^-\)). So \(\liminf_j \int u_j = \liminf_j \int u_j^+ - \int u^-\) and now Fatou’s lemma (p. 73) gives \(\liminf_j \int_{\mathcal{X}} u_j^+ \geq \int u^+\), because \(u_j^+ \geq 0\) and \(u_j^+ \to u^+\) a.e. Together, this gives \(\liminf_j \int u_j \geq \int u^+ - \int u^-\).

**Note 1:** hint 2 is not really needed (but handy for those not aware of identities like \(\ast\)).

**Note 2:** although \(\int u^- < \infty\), it could happen that \(\int u^+ = \infty\), but then \(\int u := \int u^+ - \int u^-\) is still a meaningful value in \([-\infty, +\infty]\), just as in the Fatou lemma on p. 73.

**Solution method 2: use Fatou and UI.** By the UI hypothesis, for every fixed \(\epsilon > 0\) there exists an integrable \(w_\epsilon : \mathcal{X} \to \mathbb{R}_+\) such that \(\sup_j \int_{\{u_j < -w_\epsilon\}} u_j^- < \epsilon\). In succession the above, \(u_j \geq -u_j^-\) and the definition of \(w_j := \max(u_j, -w_\epsilon)\) give

\[
\int u_j = \int_{\{u_j < -w_\epsilon\}} u_j + \int_{\{u_j \geq -w_\epsilon\}} u_j \geq -\epsilon + \int_{\{u_j \geq -w_\epsilon\}} w_j = -\epsilon + \int_{\mathcal{X}} w_j + \int_{\{u_j \geq -w_\epsilon\}} w_\epsilon.
\]

By \(w_\epsilon \geq 0\) this gives \(\int u_j \geq -\epsilon + \int w_j\). Now Fatou’s lemma can be applied to \((w_j)\), because of \(w_j \geq -w_\epsilon\) with \(w_\epsilon \in L^1(\mu)\) (this uses exactly the same elementary reasoning as the reverse Fatou lemma in course Exercise 9.8), so \(\liminf_j \int w_j \geq \int \max(u, -w_\epsilon)\) because \(w_j \to \max(u, -w_\epsilon)\) a.e. Combined with the above, this yields \(\liminf_j \int u_j \geq -\epsilon + \int \max(u, -w_\epsilon) \geq -\epsilon + \int u\), using \(\max(u, -w_\epsilon) \geq u\). The proof is finished by letting \(\epsilon \downarrow 0\).

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\(^4\)From Vitali’s Theorem 16.6 (for \(p = 1\)): if \((v_j)\) converges in measure to \(v\) and if \(|v_j|\) is uniformly integrable, then \(\int |v_j - v| \to 0\) and a fortiori \(\int v_j \to \int v\).