Solution of Quiz 1, WISB372, 7-10-2011

Problem. The following well-known "mod 4" optimal control problem is also studied in Dowling's textbook: maximize $J(y) := \int_0^T f(x(t), y(t), t)dt$ over all control functions $y(\cdot) : [0, T] \to \Omega := \mathbb{R}$ such that the dynamical system $\dot{x}(t) = g(x(t), y(t), t)$ holds with initial condition $x(0) = x_0$ and terminal condition $x(T) \geq x_{min}$. Here $x_{min} \in \mathbb{R}$ is given. Let $y^*(\cdot)$ be a given control function, with associated trajectory $x^*(\cdot)$.

a. [20 pts] State *sufficient* conditions for $y^*(\cdot)$ to be optimal for the above problem. Note that, similar to certain homework problems already distributed (e.g. for "mod 1", "mod 2a" and "mod 2b"), these sufficient conditions must be patterned after a combination of the following two: (1) the sufficiency theorem for the "simplest problem in optimal control theory", as given and proved in class[1] and (2) the well-known necessary conditions for this problem, which you know from Dowling.

b. [30 pts] Give a complete proof of the fact that the conditions which you stated in part a are indeed sufficient for $y^*(\cdot)$ to be optimal. *Advice:* Pay careful attention to the role played by the different conditions for $\lambda(T)$ (same notation as used in Dowling) and make it quite clear how your reasoning goes.

c. [20 pts] Suppose now that the concavity conditions for the Hamiltonian (see footnote 1) are strengthened into *strict* concavity conditions. What more do the sufficient conditions in part a then imply? Prove this as well.

d. [15 pts] Consider the above "mod 4" optimal control problem again, *but* with the terminal condition $x(T) \geq x_{min}$ replaced by the following one: $x_{min} \leq x(T) \leq x_{max}$. Here $x_{min}, x_{max} \in \mathbb{R}$ are given with $x_{min} < x_{max}$. Similar to part a, state *sufficient* conditions for $y^*(\cdot)$ to be an optimal control function for the new optimal control problem.

e. [15 pts] Similar to part b, give a proof of the fact that the conditions which you stated in part d are indeed sufficient for optimality of $y^*(\cdot)$.

Solution. a. The method to create sufficient optimality conditions for the original OC-problem in Dowling and other mod's was to copy their respective necessary optimality conditions and augment these by additional concavity conditions. So here that simply means that we should restate the necessary conditions for "mod 4" from Dowling and adding condition of concavity for the Hamiltonian:

1. $\forall_t H_\theta(x^*(t), y^*(t), \lambda(t), t) = 0$,
2. $\forall_t \dot{\lambda}(t) = -H_\xi(x^*(t), y^*(t), \lambda(t), t)$ and $\dot{x}^*(t) = g(x^*(t), y^*(t), t)$,
3. $x^*(0) = x_0$ and either (i) $x^*(T) > x_{min}$ and $\lambda(T) = 0$ or (ii) $x^*(T) = x_{min}$ and $\lambda(T) \geq 0$,
4. $\forall_t H(\xi, \theta, \lambda(t), t)$ is concave in (ξ, θ).

Of course, we still need to *prove* that this really works, which will be done in part b!

b. Let the above four conditions hold. To prove that the given $y^*(\cdot)$ is optimal for the above optimal control problem, it is enough to prove that the inequality $J(y^*) \geq J(y)$ holds for any arbitrary control function $y(\cdot)$ whose trajectory $x(\cdot)$ (associated to y via DS and $x(0) = x_0$) meets the terminal condition $x(T) \geq x_{min}$. Just as seen in class and in the homework exercises about "mod 1", "mod 2a" and "mod 2b", we have for any fixed $t \in [0, T]$ that

$$H(\xi, \theta, \lambda(t), t) \leq H(x^*(t), y^*(t), \lambda(t), t) + R \text{ for every } (\xi, \theta) \tag{1}$$

by condition 4, where we set

$$R := \underbrace{H_\xi(x^*(t), y^*(t), \lambda(t), t)}_{=-\dot{\lambda}(t)}(\xi - x^*(t)) + \underbrace{H_\theta(x^*(t), y^*(t), \lambda(t), t)}_{=0}(\theta - \dot{x}^*(t))$$

and already indicate the future effect of conditions 1. and 2. by the underbraces. For $\xi := x(t)$ and $\theta := \dot{x}(t)$, combined with conditions 1.-2., (1) gives

$$\underbrace{H(x(t), y(t), \lambda(t), t)}_{=f(x(t),y(t),t)+\lambda(t)\dot{x}(t)} \leq \underbrace{H(x^*(t), y^*(t), \lambda(t), t)}_{=f(x^*(t),y^*(t),t)+\lambda(t)\dot{x}^*(t)} - \dot{\lambda}(t)(x(t) - x^*(t)) \tag{2}$$

[1] Recall that these involved concavity conditions for the Hamiltonian.

for every $t \in [0,T]$. After integration of both sides in (2) over $[0,T]$ this gives

$$J(y) + \int_0^T \lambda \dot{x} \leq J(y^*) + \int_0^T \lambda \dot{x}^* - \int_0^T \dot{\lambda}(x - x^*). \tag{3}$$

Now

$$\int_0^T \dot{\lambda}(x - x^*) = \lambda(T)(x(T) - x^*(T)) - \lambda(0) \underbrace{(x(0) - x^*(0))}_{= x_0 - x_0 = 0} - \int_0^T \lambda(\dot{x} - \dot{x}^*)$$

holds by partial integration. We substitute this in (3) and get

$$J(y) + \int_0^T \lambda \dot{x} \leq J(y^*) - \lambda(T)(x(T) - x^*(T)) + \int_0^T \lambda \dot{x}.$$

This will imply the desired $J(y^*) \geq J(y)$ if we can prove that

$$\text{both cases } (i) \text{ and } (ii) \text{ in condition 3. imply } \lambda(T)(x(T) - x^*(T)) \geq 0. \tag{4}$$

Now in case (i) this is trivial because

$$\lambda(T)(x(T) - x^*(T)) = \underbrace{\lambda(T)}_{=0}(x(T) - x^*(T)) = 0$$

and in case (ii) it follows by

$$\lambda(T)(x(T) - x^*(T)) = \underbrace{\lambda(T)}_{\geq 0} \underbrace{(x(T) - x_{min})}_{\geq 0} \geq 0.$$

Conclusion: $J(y^*) \geq J(y)$. Because $y(\cdot)$ was chosen arbitrarily, this proves that $y^*(\cdot)$ is optimal.

c. If strict concavity is supposed in 4., then for every $t \in [0,T]$ the inequality in (2) becomes strict, *at least if* $(x(t), y(t)) \neq (x^*(t), \dot{x}^*(t))$ (!) We claim that $y \neq y^*$ implies $J(y) < J(y^*)$. Indeed, if $y \neq y^*$, then there exists $\tau \in [0,T]$ with $y(\tau) \neq y^*(\tau)$. Because both y and y^* are continuous[2], it follows that for some sufficiently small $\delta > 0$ the inequality $y(t) \neq y^*(t)$ holds for all $t \in I := (\tau - \delta, \tau + \delta)$ (so that the inequality in (2) becomes strict for all $t \in I$). This implies that instead of (3) we now obtain

$$J(y) + \int_0^T \lambda \dot{x} < J(y^*) + \int_0^T \lambda \dot{x}^* - \int_0^T \dot{\lambda}(x - x^*).$$

because

$$\int_0^T H(x(t), y(t), \lambda(t), t)dt = \int_I \underbrace{H(x(t), y(t), \lambda(t), t)dt}_{< H(x^*(t), y^*(t), \lambda(t), t) - \dot{\lambda}(t)(x(t) - x^*(t))} + \int_{[0,T]\setminus I} \underbrace{H(x(t), y(t), \lambda(t), t)dt}_{\leq H(x^*(t), y^*(t), \lambda(t), t) - \dot{\lambda}(t)(x(t) - x^*(t))}.$$

The reasoning following (3) can just be repeated, so it follows that $J(y^*) > J(y)$ for every $y \neq y^*$. Conclusion: under the additional strict concavity assumption, the control function y^* is the *unique* optimal solution, but *only* if all control functions are supposed to be continuous. **Observe that the latter clarification of what "unique" should mean comes only from the proof (consider footnote 2 once more for the total confusion that could result for someone who was unable to provide such a proof ...).**

d-e. We must now extend Dowling's 3. above by distinguishing three different cases:

[2] If they are both piecewise continuous, then the uniqueness statement must become more delicate. For instance, if in problem 21.9 of Dowling the stated optimal control function is given the value 10^{10} in the single point $\frac{1}{2} \in [0,1]$, then the resulting control function is different, but still optimal!

3'. $x^*(0) = x_0$ and either (i) $x_{min} < x^*(T) < x_{max}$ and $\lambda(T) = 0$ or (ii) $x^*(T) = x_{min}$ and $\lambda(T) \geq 0$ or (iii) $x^*(T) = x_{max}$ and $\lambda(T) \leq 0$.

From the role played by (4) in the previous proof, we see that it is now enough to prove the following:

the three cases (i), (ii) and (iii) in condition 3'. imply $\lambda(T)(x(T) - x^*(T)) \geq 0$,

where both $x^*(\cdot)$ and $x(\cdot)$ satisfy the new terminal condition, i.e., $x_{min} \leq x^*(T) \leq x_{max}$ and $x_{min} \leq x(T) \leq x_{max}$ both hold. For cases (i) and (ii) this goes exactly as was demonstrated above and for case (iii) we now have

$$\lambda(T)(x(T) - x^*(T)) = \underbrace{\lambda(T)}_{\leq 0} \underbrace{(x(T) - x_{max})}_{\leq 0} \geq 0,$$

as desired. **Observe here again, next to the comment about uniqueness in part c, that the applied mathematician who is unable to see through the structure of the above analysis would have to rely on pure guesswork when confronted with the new, very practical situation introduced in part d.**