Handout: using SAD to unify the examples in sections 1.3.1-1.3.2 of "Game Theory" – H. Peters

Erik J. Balder

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This handout shows how the examples in sections 1.3.1-1.3.2 of "Game Theory" by H. Peters can be explained from a simple unifying point of view, based on *stability against (individual) deviations* (SAD). Because it allows greater flexibility in terminology, SAD is used here as a pedagogical precursor to the notion of a Nash equilibrium.

1 SAD for zero-sum games

Let X and Y be arbitrary sets, where you can think of X as the set of all choices available to player 1 and Y as the set of all choices available to player 2. To discuss SAD for the zero-sum situation, it is enough to consider only the payoff function $F : X \times Y \to \mathbb{R}$ of player 1, who is striving to maximize his payoff. The payoff for the minimizing player 2 follows then implicitly: if choices $x \in X$ by player 1 and $y \in Y$ by player 2 are realized, then player 1 receives $F(x, y)$ monetary units from the player 2.

Definition 1 *The function $F : X \times Y \to \mathbb{R}$ is* **stable against (individual) deviations** *(SAD) at a point $(\bar{x}, \bar{y}) \in X \times Y$ if* (1) $F(\bar{x}, \bar{y}) \geq F(x, \bar{y})$ *for all $x \in X$ and* (2) $F(\bar{x}, \bar{y}) \leq F(\bar{x}, y)$ *for all $y \in Y$.*

In this zero-sum situation SAD just describes a saddle point property. The name SAD comes from observing that (1) in Definition 1 makes it unattractive for player 1 to individually deviate from his choice $\bar{x}$ (assuming that player 2 holds on to $\bar{y}$!) and that (2) makes it unattractive for player 2 (who is the minimizing player) to deviate from her choice $\bar{y}$ (assuming that player 1 holds on to $\bar{x}$).

This zero-sum game is said to **have a value** if

$$v_F := \min_{y \in Y} \max_{x \in X} F(x, y) = \max_{x \in X} \min_{y \in Y} F(x, y) \tag{1}$$

and v_F is then called the **value** of the game. Games need not have a value. For instance, for a matrix game with X as the set of row indices and Y as the set of column indices the existence of a value is equivalent to the existence of a saddlepoint in the sense of Definition 2.3 in the book; so if the matrix does not have a saddlepoint, then the game has no value. However, for a *mixed* matrix game, where X is the set of all probability distributions over the row indices and Y is the set of all probability distributions over the column indices, von Neumann's minimax theorem states that the corresponding matrix game has a value. See (2) below and p. 22-23 in the book for details. [Ordinarily, matrices are understood to have finitely many rows and finitely many columns and this is essential for von Neumann's result to hold, as is seen by considering the zero-sum game "largest number wins", which has $X = \mathbb{N}$, $Y := \mathbb{N}$.[1]

Proposition 1 *Let $F : X \times Y \to \mathbb{R}$. The following are equivalent for any $(\bar{x}, \bar{y}) \in X \times Y$:*
i. F is SAD at $(\bar{x}, \bar{y})$.
ii. $\max_{x\in X} F(x, \bar{y}) = \min_{y\in Y} F(\bar{x}, y)(= F(\bar{x}, \bar{y}))$.
iii. There exists a real number w such that $F(\bar{x}, y) \geq w$ for all $y \in Y$ and $F(x, \bar{y}) \leq w$ for all $x \in X$; in this case the game w is the the value $v_F = F(\bar{x}, \bar{y})$ of the game.
iv. $\bar{x}$ maximizes the function $F_m(x) := \min_{y\in Y} F(x, y)$ over all $x \in X$ and $\bar{y}$ minimizes the function $F^m(y) := \max_{x\in X} F(x, y)$ over all $y \in Y$ and the game has a value.

A pair $(\bar{x}, \bar{y})$ in $X \times Y$ having the equivalent properties of the above proposition is called a **minimax equilibrium** pair. Moreover, part *ii* can be rephrased as follows. Call any maximizer over X of the function $F(\cdot, y)$ (i.e., of the function $F(x, y)$ that you obtain by keeping y fixed and letting x be its only variable) **player 1's best reply (or reaction) to** y and let $\beta_1(y)$ be the set of all such best replies. Similarly, call any minimizer over Y of the function $F(x, \cdot)$ **player 2's best reply (or reaction) to** x and let $\beta_2(x)$ be the set of all such best replies. Then *ii* in the above result can be alternatively expressed as the following mutual **best reply property**:

ii′. $\bar{x} \in \beta_1(\bar{y})$ and $\bar{y} \in \beta_2(\bar{x})$, i.e., $\bar{x}$ is player 1's best reply to $\bar{y}$ and $\bar{y}$ is player 2's best reply to $\bar{x}$.

Proof. *i* $\Leftrightarrow$ *ii*: It is enough to observe that $\max_{x\in X} F(x, \bar{y}) = F(\bar{x}, \bar{y})$ is equivalent to $F(x, \bar{y}) \leq F(\bar{x}, \bar{y})$ for all $x \in X$ and that, vice versa, $\min_{y\in Y} F(\bar{x}, y) = F(\bar{x}, \bar{y})$ is equivalent to $F(\bar{x}, y) \geq F(\bar{x}, \bar{y})$ for all $y \in Y$.

i $\Rightarrow$ *iii*: Take $w := F(\bar{x}, \bar{y})$, then the two inequalities hold. Also, *i* gives

$$\max_x \min_y F(x, y) \geq \min_y F(\bar{x}, y) \geq F(\bar{x}, \bar{y}) \geq \max_x F(x, \bar{y}) \geq \min_y \max_x F(x, y)$$

and the reverse inequality holds always (prove it yourself). So the game has the value $w = F(\bar{x}, \bar{y})$.

iii $\Rightarrow$ *i*: For w as stated it follows that both $F(\bar{x}, \bar{y}) \geq w$ and $F(\bar{x}, \bar{y}) \leq w$ hold. Hence $w = F(\bar{x}, \bar{y})$, so *i* follows immediately.

i $\Rightarrow$ *iv*: For any $x \in X$ the inequality $F_m(x) \leq F(x, \bar{\mathbf{y}})$ follows by definition of the minimum. Hence, $F_m(x) \leq F(\bar{x}, \bar{y})$ follows from *i*. Now *ii* (which has already been shown to be equivalent to *i*) states that $F_m(\bar{x}) = F_m(\bar{y}) = F(\bar{x}, \bar{y})$. Hence, $F_m(x) \leq F_m(\bar{x})$ holds for any $x \in X$. Conclusion: $\bar{x}$ maximizes $F_m(x)$ over all $x \in X$. In the same way it follows that $\bar{y}$ minimizes $F^m(y)$ over all $y \in Y$. The properties just proven yield $\max_x \min_y F(x, y) = \max_x F_m(x) = F_m(\bar{x}$ and $\min_y \max_x F(x, y) = \min_y F^m(y) = F^m(\bar{y}$. Because *ii* comes down to $F_m(\bar{x}) = F^m(\bar{y}) = F(\bar{x}, \bar{y})$, it follows that (1 holds.

iv $\Rightarrow$ *i*: It is easy to show that v_F, the value of the game mentioned in *iv*, must be equal to $F(\bar{x}, \bar{y})$. Hence, *iv* gives $F_m(\bar{x}) = v_F = F(\bar{x}, \bar{y}) = F^m(\bar{y})$. By definition of the functions F_m and F^m this comes down to *i*. QED

At this point of the development of SAD, you can already understand the examples in Peters' section 1.3.1, which I have worked out in section 3. However, a little more about SAD is needed to understand the examples in Peters' section 1.3.2, where each player has his/her own payoff function. This I do in the next section.

[1]Exercise 26-9-12a: show that this game does not have a value.

2 SAD for nonzero-sum games

Let X and Y be arbitrary sets. Just as in the previous section, you can think of X as the set of all choices available to player 1 and Y as the set of all choices available to player 2.

Definition 2 *Two functions $F_1 : X \times Y \to \mathbb{R}$ and $F_2 : X \times Y \to \mathbb{R}$ are* **stable against (individual) deviations** *(SAD) at a point $(\bar{x}, \bar{y}) \in X \times Y$ if (1) $F_1(\bar{x}, \bar{y}) \geq F_1(x, \bar{y})$ for all $x \in X$ and (2) $F_2(\bar{x}, \bar{y}) \geq F_2(\bar{x}, y)$ for all $y \in Y$.*

In Definition 1 a pair where SAD holds turned out to be precisely a minimax equilibrium pair, but this name could only be justified by Proposition 1. SAD as defined in Definition 2 needs no additional result to justify its name: a pair where SAD holds is called a **Nash equilibrium** pair (also **Cournot-Nash equilibrium** pair) after Cournot (1838) and Nash (1951).

You should carefully note that in (2) above the inequality is reversed with respect to the one used in (2) of Definition 1. Of course, that comes from the fact that in the setting of Definition 2 *both* players are understood to be striving to maximize their respective payoff. Hence, (1) in Definition 2 makes it unattractive for player 1 to individually deviate from his choice $\bar{x}$ (assuming that player 2 holds on to $\bar{y}$) and (2) in Definition 2 makes it unattractive for player 2 to deviate from her choice $\bar{y}$ (assuming that player 1 holds on to $\bar{x}$). Quite similar to the equivalence $i \Leftrightarrow ii$ in Proposition 1, the following equivalence holds.

Proposition 2 *Let $F_1 : X \times Y \to \mathbb{R}$ and $F_2 : X \times Y \to \mathbb{R}$. The following are equivalent for any $(\bar{x}, \bar{y}) \in X \times Y$:*

i. F_1 and F_2 are SAD at $(\bar{x}, \bar{y})$ (i.e., $(\bar{x}, \bar{y})$ is a Cournot-Nash equilibrium pair).
ii. $\max_{x \in X} F_1(x, \bar{y}) = F_1(\bar{x}, \bar{y})$ and $\max_{y \in Y} F_2(\bar{x}, y) = F_2(\bar{x}, \bar{y})$.

For obvious reasons the value notion for a game, introduced just before Proposition 1, has no counterpart for games where each player has his/her own payoff function, such as bimatrix games or Cournot's market game.

On the other hand, part *ii* above justifies the following terminology, used on p. 7. For the zero-sum case it coincides with what was already done in section 1. Call any maximizer over X of the function $F_1(\cdot, y)$ (i.e., of the function $F_1(x, y)$ that you obtain by keeping y fixed and letting x be its only variable) **player 1's best reply (or reaction) to** y and denote the set of all such best replies by $\beta_1(y)$. Also, call any maximizer over Y of the function $F_2(x, \cdot)$ **player 2's best reply (or reaction) to** x and denote the set of all such best replies by $\beta_2(x)$. Just as in the previous section, *ii* in the above result can be alternatively expressed as a mutual **best reply property**:

ii′. $\bar{x} \in \beta_1(\bar{y})$ and $\bar{y} \in \beta_2(\bar{x})$, i.e., $\bar{x}$ is player 1's best reply to $\bar{y}$ and $\bar{y}$ is player 2's best reply to $\bar{x}$.

The following result holds, which you should prove yourself.[2] It essentially says that the two definitions of SAD coincide for a zero-sum game.

Proposition 3 *If $F_2 = -F_1$, then the following are equivalent for any $(\bar{x}, \bar{y}) \in X \times Y$:*
i. $(\bar{x}, \bar{y})$ is a minimax equilibrium pair for the (single) payoff function F_1.
ii. $(\bar{x}, \bar{y})$ is a Nash equilibrium for the functions F_1 and $F_2 (= -F_1)$.

[2] Call this Exercise 26-9-12b.

To summarize, in section 1 it was demonstrated that in the zero-sum game situation a pair is SAD if and only if it is a minimax equilibrium pair, thanks to Proposition 1. For a non-zero-sum game, as discussed in the present section, a pair is SAD if and only if it is a Nash equilibrium pair (simply by giving it another name).

One final observation about multiplayer games: it is easy to extend Definition 2 and Proposition 2 to a game with three players (and then also, with little left to imagination, to games with any (finite) number of players). Namely, let Z be another set and let F_1, F_2 and F_3 be real-valued functions on $X \times Y \times Z$. Then F_1, F_2 and F_3 are defined to have a Nash equilibrium (or to be SAD) at the triple $(\bar{x}, \bar{y}, \bar{z})$ if (1) $F_1(\bar{x}, \bar{y}, \bar{z}) \geq F_1(x, \bar{y}, \bar{z})$ for all $x \in X$ and (2) $F_2(\bar{x}, \bar{y}, \bar{z}) \geq F_2(\bar{x}, y, \bar{z})$ for all $y \in Y$ and (3) $F_3(\bar{x}, \bar{y}, \bar{z}) \geq F_3(\bar{x}, \bar{y}, z)$ for all $z \in Z$.

3 Applications of SAD

The flexibility of SAD will allow me to use three really different substitutions for the sets X and Y as used in the previous sections, and also for the payoff functions defined on the product set $X \times Y$.

Example 1 (battle of the Bismarck sea) *Choose $X = \{1, 2\}$ to index rows and $Y = \{1, 2\}$ to index columns, and let $F(i, j) :=$ (i, j)-th element of the 2×2-matrix on p. 3. Thus, $F(2, 1) = 1$, etc. Then F is easily seen to be SAD at the point $\bar{i}, \bar{j} = (1, 1)$ (and only there), which corresponds to the choice "North" by each player.*

Example 2 (matching pennies) *1. In a first attempt, you try $X = \{1, 2\}$ to index rows and $Y = \{1, 2\}$ to index columns, and let $F(i, j) :=$ (i, j)-th element of the 2×2-matrix at the top of p. 5. However, that does not work, because to find a point where F is SAD comes down to finding a saddle point (see the comment following Definition 1), but the matrix in question does not have a saddle point.*

2. Next you try the idea to randomize, due to gambler de Waldegrave (1713), as reinvented by von Neumann: each player randomizes independently *from the other in his/her choice of rows/columns and the payoff of the game is measured* in expectation*. For this, you choose $X = [0, 1]$ and $Y = [0, 1]$; then player 1 choosing $p \in X$ is interpreted as having him randomize between row 1 and row 2 with probabilities p and $1 - p$ respectively. A similar interpretation holds for the choice $q \in Y$: it means that player 2 randomizes between columns 1 and 2 with probabilities q and $1 - q$ respectively. Note already that the special choice $p = 1$ means that player 1 plays row 1; likewise, the special choice $p = 0$ means that he only plays row 2, and a similar interpretation holds for player 2 choosing $q = 1$ (i.e., column 1) and $q = 0$ (i.e., column 2). For $p \in X$ and $q \in Y$ the expected payoff $F(p, q)$ is*

$$F(p, q) = pq * 1 + p(1 - q) * -1 + (1 - p)q * -1 + (1 - p)(1 - q) * 1 = 4pq - 2p - 2q + 1$$

by the definition of expectation and the product rule for independent randomization. More generally, if A is any 2×2 payoff matrix of the game, then the expected payoff[3] is

$$F_A(p, q) := pqa_{1,1} + p(1 - q)a_{1,2} + (1 - p)qa_{2,1} + (1 - p)(1 - q)a_{2,2}. \tag{2}$$

[3] Check that this is a special case for $m = n = 2$ of $F_A(\mathbf{p}, \mathbf{q}) := \mathbf{p}A\mathbf{q}$ as used on $X \times Y$ on in chapter 2 of the book, where $X := \Delta^m$ and $Y := \Delta^n$.

Points $(\bar{p}, \bar{q})$ (if any) must be determined at which F is SAD. Such a minimax equilibrium pair is often called a **mixed** *minimax equilibrium pair for the game, because in game theory randomization goes by the name of* mixing. *The corresponding probability distribution $(\bar{p}, 1-\bar{p})$ on $\{1,2\}$ is called a* **mixed equilibrium strategy** *for player 1 and the probability distribution $(\bar{q}, 1-\bar{q})$ on $\{1,2\}$ is called a mixed equilibrium strategy for player 2.*

Method 1: use the equalizing property. *The so-called* **equalizing property** *appears as just a trick on p. 5. It also figures in Problems 3.8 and 12.6 of Peters' book. The use of this method requires some care and precision. For instance, in the Bismarck sea Example 1 its step 2 gives $\bar{q} = \frac{1}{2}$, which is very misleading, as you can see by determining all mixed equilibria for that example! The explanation for this is that its step 1 gives $\bar{p} = 1$. Hence player 1's strategy is not completely mixed, as required in step 3 below. In the case of a 2×2 payoff matrix A the equalizing property works by the following scheme:*

Step 1. *Determine $\bar{q}$ as a solution/solutions of $F_A(1,q) = F_A(0,q)$, i.e., randomized play with distribution $(\bar{q}, 1-\bar{q})$ over the two columns by player 2 yields the same expected payoff against player 1's choosing row 1 as against his choosing row 2.*[4]

Step 2. *Determine $\bar{p}$ as a solution/solutions of $F_A(p,1) = F_A(p,0)$, i.e., randomized play with distribution $(\bar{p}, 1-\bar{p})$ over the two rows by player 1 yields the same expected payoff against player 2's choosing column 1 as against her choosing column 2.*[5]

Step 3. *Check that $0 < \bar{p} < 1$ and $0 < \bar{q} < 1$, i.e., check that the probability distributions $(\bar{p}, 1-\bar{p})$ and $(\bar{q}, 1-\bar{q})$ are completely mixed. If so, then any pair(s) $(\bar{p}, \bar{q})$ that have been produced by steps 1 to 3 can now be proclaimed to be mixed minimax equilibrium pair(s).*

It is demonstrated in Proposition 4 that suitable adaptations of step 3 are available for situations where either $(\bar{p}, 1-\bar{p})$ or $(\bar{q}, 1-\bar{q})$ is/are not completely mixed.

For the present example $\bar{q}$ in step 1 follows by solving $2q-1 = 1-2q$, which gives $\bar{q} = \frac{1}{2}$ and in step 2 $\bar{p}$ follows by solving $2p-1 = 1-2p$, which gives $\bar{p} = \frac{1}{2}$. Obviously, these two outcomes meet the conditions of step 3, so F is SAD at a unique point in $[0,1] \times [0,1]$, namely $(\bar{p}, \bar{q}) := (\frac{1}{2}, \frac{1}{2})$, which is therefore the unique minimax equilibrium pair. The corresponding probability distributions over $\{1,2\}$ are $(\bar{p}, 1-\bar{p}) = (\frac{1}{2}, \frac{1}{2})$ for player 1 (who thus randomizes with equal probabilities over the two rows) and $(\bar{q}, 1-\bar{q}) = (\frac{1}{2}, \frac{1}{2})$ for player 2 (who thus randomizes with equal probabilities over the two columns). The expected payoff of the game is $F(\bar{p}, \bar{q}) = F(\frac{1}{2}, \frac{1}{2}) = 0$, i.e., zero monetary units are transferred by player 2 to player 1 in expectation *(see p. 5 about such use of expectations).*

Method 2: use *ii* in Proposition 1. *By Proposition 1, part ii, all pairs $(\bar{p}, \bar{q}) \in [0,1] \times [0,1]$ for which $F_m(\bar{p}) = F^m(\bar{q})$ must be determined, where*

$$F_m(p) := \min_{q \in [0,1]} F(p,q) \textit{ and } F^m(q) := \max_{p \in [0,1]} F(p,q).^6$$

Now

$$F_m(p) := \min_{q \in [0,1]} \underbrace{q(4p-2) - 2p + 1}_{F(p,q)} = \left\{ \begin{array}{ll} 1-2p & \textit{if } p \geq \frac{1}{2} \\ 2p-1 & \textit{if } p < \frac{1}{2} \end{array} \right\} = -|2p-1|$$

[4] Exercise 26-9-12c: prove in (2) that $F_A(1,\bar{q}) = F_A(0,\bar{q})$ is equivalent to $F_A(p,\bar{q})$ being independent of p; this explains the term *equalizing* property.

[5] Here footnote 4 applies mutatis mutandis: in (2) the identity $F_A(\bar{p},1) = F_A(\bar{p},0)$ is equivalent to $F_A(\bar{p},q)$ being independent of q.

[6] For F_A as in chapter 2 (see footnote 3) this gives $F_m(\mathbf{p}) := \min_{\mathbf{q} \in \Delta^n} \mathbf{p}A\mathbf{q} (= \min_{1 \leq j \leq n} \mathbf{p}A\mathbf{e}^j)$ and $F^m(\mathbf{q}) := \max_{\mathbf{p} \in \Delta^m} \mathbf{p}A\mathbf{q} (= \max_{1 \leq i \leq m} \mathbf{e}^i A\mathbf{q})$.

and

$$F^m(q) = \left\{ \begin{array}{ll} 2q-1 & \text{if } q \geq \frac{1}{2} \\ 1-2q & \text{if } q < \frac{1}{2} \end{array} \right\} = |2q-1|$$

by elementary considerations about the sign of the slope of the linear functions that are being optimized.[7] *Because* $F_m \leq 0$ *and* $F^m \geq 0$, *the desired identity* $F_m(\bar{p}) = F^m(\bar{q})$ *can only be achieved by choosing* $\bar{p} = \frac{1}{2}$ *and* $\bar{q} = \frac{1}{2}$. *So again one finds that the original* F *is SAD at a unique point in* $[0,1] \times [0,1]$, *namely* $(\bar{p}, \bar{q}) := (\frac{1}{2}, \frac{1}{2})$, *which is therefore the unique minimax equilibrium pair.*

Method 3: use the best reply property. *According to the alternative* ii' *for* ii *used in method 2,* $\bar{p}$ *should* (1) *maximize* $F(p, \bar{q})$ *over all* $p \in [0,1]$ *and* (2) $\bar{q}$ *should minimize* $F(\bar{p}, q)$ *over all* $q \in [0,1]$. *The corresponding optimizers (i.e., best replies) have already been spelled out in footnotes 7 and 7, so you must now combine the* 3×3 *different possibilities:*

Cases 1-2: $\bar{p} > \frac{1}{2}$ and $\bar{q} \geq \frac{1}{2}$. *Impossible: by footnote 7 player 2's best reply to* $\bar{p} > \frac{1}{2}$ *is* $q = 0 \not\geq \frac{1}{2}$.

Case 3: $\bar{p} > \frac{1}{2}$ and $\bar{q} < \frac{1}{2}$. *Impossible: by footnote 7 player 1's best reply to* $\bar{q} < \frac{1}{2}$ *is* $p = 0 \not> \frac{1}{2}$.

Case 4: $\bar{p} = \frac{1}{2}$ and $\bar{q} > \frac{1}{2}$. *Impossible: by footnote 7 player 1's best reply to* $\bar{q} > \frac{1}{2}$ *is* $p = 1 \neq \frac{1}{2}$.

Case 5: $\bar{p} = \frac{1}{2} = \bar{q}$. *This situation can indeed occur: by footnote 7 player 1's best reply to* $\bar{q} = \frac{1}{2}$ *can be any point in* $[0,1]$, *hence also* $\frac{1}{2}$, *and a similar statement holds for player 2 by footnote 7.*

Case 6: $\bar{p} = \frac{1}{2}$ and $\bar{q} < \frac{1}{2}$. *Impossible: by footnote 7 player 1's best reply to* $\bar{q} < \frac{1}{2}$ *is* $p = 1 \neq \frac{1}{2}$.

Cases 7-8: $\bar{p} \leq \frac{1}{2}$ and $\bar{q} > \frac{1}{2}$. *Impossible: by footnote 7 player 1's best reply to* $\bar{q} > \frac{1}{2}$ *is* $p = 1 \not\leq \frac{1}{2}$.

Case 9: $\bar{p} < \frac{1}{2}$ and $\bar{q} < \frac{1}{2}$. *Impossible: by footnote 7 player 2's best reply to* $\bar{p} < \frac{1}{2}$ *is* $q = 1 \not< \frac{1}{2}$.

Conclusion: only case 5 survives, so by property ii' *the pair* $(\bar{p}, \bar{q}) = (\frac{1}{2}, \frac{1}{2})$ *is the unique minimax equilibrium pair.*

Method 4: use *iv* in Proposition 1. *This combines successive maximization and minimization (or vice versa). In view of the outcomes in method 2,* $\bar{p}$ *follows from maximizing* $F_m(p) = -|2p-1|$ *over* $[0,1]$, *which gives* $\bar{p} = \frac{1}{2}$, *and* $\bar{q}$ *follows from minimizing* $F^m(q) = |2q-1|$ *over* $[0,1]$, *which gives* $\bar{q} = \frac{1}{2}$. *Because of von Neumann's minimax theorem, the condition that this mixed game has a value is fulfilled (so it does not have to be checked any further via calculations).*

Example 3 (prisoners' dilemma) *In a first attempt, you try* $X = \{1,2\}$ *to index rows and* $Y = \{1,2\}$ *to index columns. Also, you let* $F_1(i,j)$ *be the* (i,j)*-th element* $a_{i,j}$ *of the* 2×2*-matrix* A *for player 1 that is contained in the payoff matrix at the bottom of p. 5. Similarly, you let* $F_2(i,j)$ *be the* (i,j)*-th element* $b_{i,j}$ *of the* 2×2*-matrix* B *for player 2 that is contained in that same payoff matrix. Then you have already a success, because (only) at* $(\bar{i}, \bar{j}) = (2,2)$ *the above* F_1 *and* F_2 *are easily seen to be SAD (i.e.,* $(2,2)$ *forms a Nash equilibrium pair). In principle, this does not exclude the possibility of other, namely* mixed *Nash equilibrium pairs (think of the game with the zero matrix as payoff matrix), but for*

[7] If $q > \frac{1}{2}$, then player 1's best reply is $\beta_1(q) = \{1\}$, if $q < \frac{1}{2}$ then $\beta_1(q) = \{0\}$ and if $q = \frac{1}{2}$ then $\beta_1(q) = [0,1]$. Also, if $p > \frac{1}{2}$ then $\beta_2(p) = \{0\}$, if $p < \frac{1}{2}$ then $\beta_2(p) = \{1\}$ and if $p = \frac{1}{2}$ then $\beta_2(p) = [0,1]$.

the prisoners' dilemma it can be shown[8] *that mixing, based on using the two expected payoff functions* $F_A(p,q) = 9q - p - 9$ *and* $F_B(p,q) = 9p - q - 9$ *in the sense of (2), does not yield any extra mixed Nash equilibrium pairs.*

Example 4 (battle of the sexes) *Although no mixed equilibrium for this situation is considered on p. 6-7, it is not hard to compute* all *the mixed Nash equilibria for this game, which has* $F_A(p,q) = 3pq - p - q + 1$ *for player 1 and* $F_B(p,q) = 3pq - 2p - 2q + 2$ *for player 2. Namely, the same reasoning for slopes of linear functions as used in methods 2-3 of Example 2 (note that here there can be no question of using its method 1) gives as the best replies:*

$$\text{best reply of player 1 to } q = \begin{cases} p = 1 & \text{if } q > \frac{2}{3} \\ \text{any } p \text{ in } [0,1] & \text{if } q = \frac{2}{3} \\ p = 0 & \text{if } q < \frac{2}{3} \end{cases}$$

$$\text{best reply of player 2 to } p = \begin{cases} q = 1 & \text{if } p > \frac{1}{3} \\ \text{any } q \text{ in } [0,1] & \text{if } p = \frac{1}{3} \\ q = 0 & \text{if } p < \frac{1}{3} \end{cases}$$

Then imitation of the method 3 of Example 2, based on the best reply property, gives[9] *the following three Nash equilibrium pairs:* $(1,1)$, $(0,0)$ *and* $(\frac{1}{3}, \frac{2}{3})$. *The first two pairs are called* pure, *because they do not really rely on randomization: the first one consists of both players choosing for the soccer match and the second one of both players choosing for the ballet performance. These two Nash equilibria are mentioned on p. 6. In contrast, the third Nash equilibrium, which is not mentioned in the book, is completely mixed: it prescribes player 1 to randomize over the rows of the joint matrix on p. 6 by means of the probability distribution* $(\frac{1}{3}, \frac{2}{3})$ *and player 2 to randomize over its columns by means of the probability distribution* $(\frac{2}{3}, \frac{1}{3})$. *See the comments on p. 6 about the notion of focal points as a way to help the two players to choose among these Nash equilibrium pairs.*

Example 5 (Cournot game) *Here one can choose* $X = [0,1]$ *and* $Y = [0,1]$ *and work with the payoff functions*

$$F_1(q_1,q_2) := q_1 \max(1 - q_1 - q_2, 0) \text{ and } F_2(q_1,q_2) := q_2 \max(1 - q_1 - q_2, 0).$$

Again it is possible to use the best reply method. To determine player 1's best reply to any fixed $q_2 \in [0,1]$, *note first that* $F_1(q_1,q_2)$ *must only be maximized over all* $q_1 \in [0, 1-q_2]$, *because* $F_1(q_1,q_2) = 0$ *for* $q_1 > 1 - q_2$ *and because the maximum over* $[0, 1-q_2]$ *is obviously strictly positive. Now* $F(q_1,q_2) = q_1(1 - q_1 - q_2)$ *on* $[0, 1-q_2]$, *so a unique maximum is obviously achieved by setting the derivative for* q_1 *equal to zero; this gives* $q_1 = \frac{1-q_2}{2}$ *as player 1's best reply to* q_2. *By symmetry, player 2's best reply to* $q_1 \in [0,1]$ *is* $q_2 = \frac{1-q_1}{2}$. *According to the best reply property, one should now solve the two equations* $q_1 = \frac{1-q_1}{2}$ *and* $q_2 = \frac{1-q_2}{2}$. *Therefore* $\bar{q}_1, \bar{q}_2) = (\frac{1}{3}, \frac{1}{3})$ *is the unique (and pure!) Nash equilibrium: each player should produce one third of the total capacity (which has been normalized to 1) of the market.*

I finish by deriving the equalizing property for zero-sum games with a 2×2 payoff matrix A. Recall that the expected payoff function $F_A : [0,1] \times [0,1] \to \mathbb{R}$ is defined in (2); clearly,

[8] Call this Exercise 26-9-12d.

[9] Call this Exercise 26-9-12e.

this gives the following two alternative expressions for $F_A(p,q)$:

$$F_A(p,q) = p(F_A(1,q) - F_A(0,q)) + F_A(0,q) \text{ and } F_A(p,q) = q(F_A(p,1) - F_A(p,0)) + F_A(p,0). \tag{3}$$

Thus, for $(\bar{p}, \bar{q})$ to be a mixed minimax equilibrium pair is equivalent to the *linear* function $p(F_A(1,\bar{q}) - F_A(0,\bar{q})) + F_A(0,\bar{q})$ attaining its maximum over $p \in [0,1]$ at $p = \bar{p}$ and, simultaneously, the *linear* function $q(F_A(\bar{p},1) - F_A(\bar{p},0)) + F_A(\bar{p},0)$ attaining its minimum over $q \in [0,1]$ at $q = \bar{q}$.

Proposition 4 *The following are equivalent for any $(\bar{p}, \bar{q}) \in [0,1] \times [0,1]$:*
i. $(\bar{p}, \bar{q})$ is a mixed minimax equilibrium pair.
ii. The following conditions (4)-(5) hold:

$$F_A(1,\bar{q}) - F_A(0,\bar{q}) \begin{cases} = 0 & \text{if } 0 < \bar{p} < 1, \\ \leq 0 & \text{if } \bar{p} = 0, \\ \geq 0 & \text{if } \bar{p} = 1. \end{cases} \tag{4}$$

$$F_A(\bar{p},1) - F_A(\bar{p},0) \begin{cases} = 0 & \text{if } 0 < \bar{q} < 1, \\ \geq 0 & \text{if } \bar{q} = 0, \\ \leq 0 & \text{if } \bar{q} = 1. \end{cases} \tag{5}$$

Hence, in particular $F_A(\bar{p},0) = F_A(\bar{p},1)$ (i.e., the equalizing property in q) and $F_A(0,\bar{q}) = F_A(1,\bar{q})$ (i.e., the equalizing property in q) must hold if $0 < \bar{p} < 1$ and $0 < \bar{q} < 1$.

Conditions (4)-(5) can easily be recalled by drawing a picture of a linear function on $[0,1]$ that is either to be maximized or to be minimized. Before giving the proof, it is instructive to observe that this new result is in agreement with what was found in Example 1, namely that both players choose the northern route. Indeed, in that example $F_A(p,q) = 2pq - p - 2q + 3$, so for $(\bar{p}, \bar{q}) = (1,1)$ the conditions (4)-(5) only require $F_A(1,1) \geq F_A(0,1)$ and $F_A(1,1) \leq F_A(1,0)$, which is true.

Proof of Proposition 4. $i \Rightarrow ii$: Only (4) needs to be proven, because the proof of (5) goes in the same way. For notational ease, let $s := F_A(1,\bar{q}) - F_A(0,\bar{q})$. By i and what was observed following (3), the linear function $sp + F_A(0,\bar{q})$ has its maximum over $p \in [0,1]$ in $p = \bar{p}$. If $0 < \bar{p} < 1$ the maximum property implies that s, the slope of that function, is zero. On the other hand, if $\bar{p} = 0$ then the maximum property comes down to $0 = s\bar{p} \geq sp$ for all $p \in [0,1]$, i.e., to $s \leq 0$. And if $\bar{p} = 1$ that same maximum property is equivalent to $s = s\bar{p} \geq sp$ for all $p \in [0,1]$, i.e., to $s \geq 0$.

$ii \Rightarrow i$: Write $s := F_A(1,q) - F_A(0,q)$ again. It is enough to prove that in each of the three cases in (4) the linear function $F_A(p,\bar{q}) = ps + F_A(0,\bar{q})$ takes its maximum over $p \in [0,1]$ for $p = \bar{p}$, for the proof that $q(F_A(\bar{p},1) - F_A(\bar{p},0)) + F_A(\bar{p},0)$ attains its minimum over $q \in [0,1]$ at $q = \bar{q}$ goes in the same fashion. If $0 < \bar{p} < 1$, then $F_A(p,\bar{q})$ for all p, so $\bar{p}$ is trivially a maximizer. Also, if $\bar{p} = 0$, then $ps \leq \bar{p}s = 0$ for all $p \in [0,1]$ follows by $s \leq 0$. Finally, if $\bar{p} = 1$, then $ps \leq \bar{p}s = s$ for all $p \in [0,1]$ follows by $s \geq 0$.