Extra exercises about domination

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Recall the following.

Definition 1 a. A mixed strategy $\bar{\sigma}_i \in \Delta(S_i)$ is *strictly dominated* if there exists a strategy $\alpha_i \in \Delta(S_i)$ such that

$$u_i(\alpha_i, s_{-i})) > u_i(\bar{\sigma}_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}. \quad (1)$$

Consequently, a strategy combination $\sigma := (\sigma_1, \dots, \sigma_n)$ in $\Pi_{i=1}^n \Delta(S_i)$ is *strictly dominated* if there exists a player i whose strategy σ_i is strictly dominated in the above sense.

b. A mixed strategy $\bar{\sigma}_i \in \Delta(S_i)$ is *weakly dominated* if there exists a strategy $\alpha_i \in \Delta(S_i)$ such that

$$u_i(\alpha_i, s_{-i})) \geq u_i(\bar{\sigma}_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i} \text{ with at least one inequality being strict.} \quad (2)$$

Consequently, a strategy combination $\bar{\sigma} := (\bar{\sigma}_1, \dots, \bar{\sigma}_n)$ in $\Pi_i \Delta(S_i)$ is *weakly dominated* if there exists a player i whose strategy $\bar{\sigma}_i$ is weakly dominated in the above sense.

Exercise 1 a. Prove that (1) is equivalent to

$$u_i(\alpha_i, \sigma_{-i})) > u_i(\bar{\sigma}_i, \sigma_{-i}) \text{ for all } \sigma_{-i} \in \Pi_{j, j \neq i} \Delta(S_j).$$

b. Prove that (2) is equivalent to

$$u_i(\alpha_i, \sigma_{-i})) \geq u_i(\bar{\sigma}_i, \sigma_{-i}) \text{ for all } \sigma_{-i} \in \Pi_{j, j \neq i} \Delta(S_j) \text{ with at least one inequality being strict.}$$

Exercise 2 Prove that a NE cannot be strictly dominated.

Exercise 2 is not hard, but the next exercise is more complicated. By making it you obtain a separate and more direct proof of Theorem 13.20.

Exercise 3 Prove that a trembling hand perfect NE $\bar{\sigma} := (\bar{\sigma}_1, \dots, \bar{\sigma}_n)$ cannot be weakly dominated. Do this by reasoning via the following steps:

Step 1. By the Definition 13.15 of trembling hand perfectness there exists a sequence $\{\sigma^t\}_{t=1}^\infty$ of strategy combinations and an associated sequence $\{\mu^t\}_{t=1}^\infty$ of strictly positive error functions, converging pointwise to zero, such that $\sigma^t \in NE(G(\mu^t))$ for every t and such that $\{\sigma^t\}_{t=1}^\infty$ converges to $\bar{\sigma}$ in $\Pi_{i=1}^n \Delta(S_i)$.

Step 2. Fix any t and any index i. Then $\sigma^t \in NE(G(\mu^t))$ implies $\sigma_i^t(h) \geq \mu_{ih}^t$ for all $h \in S_i$, by definition of $NE(G(\mu^t))$. Now prove that for every $h \in S_i$

$$\sigma_i^t(h) > \mu_{ih}^t \text{ implies } u_i(h, \sigma_{-i}^t) = \max_{h' \in S_i} u_i(h', \sigma_{-i}^t) = u_i(\sigma^t).$$

Hint: Prove and use

$$\underbrace{(1-\sum_h \mu_{ih}^t)}_{>0} u_i(\sigma^t) = \sum_{h,\sigma_i^t(h)>\mu_{ih}^t} \underbrace{(\sigma_i^t(h)-\mu_{ih}^t)}_{>0} \underbrace{u_i(h,\sigma_{-i}^t)}_{\leq u_i(\sigma^t)}.$$

Alternatively, you can also reconstruct a complete proof of Lemma 13.18(1).

Step 3. Prove that there exists a sufficiently large t (say $t=\tau$) such that $\sigma_i^\tau(h) > \mu_{ih}^\tau$ holds for every $i \in \{1,\ldots,l\}$ and every $h \in S_i$ with $\bar{\sigma}_i(h) > 0$.

Step 4. Use steps 2-3 to prove that $u_i(\bar{\sigma}_i, \sigma_{-i}^\tau) = \max_{h' \in S_i} u_i(h', \sigma_{-i}^\tau)$ holds for every $i \in \{1,\ldots,l\}$.

Step 5. Finish the proof by supposing, by way of contradiction, that $\bar{\sigma} := (\bar{\sigma}_1,\ldots,\bar{\sigma}_n)$ would be strictly dominated. Then, by Definition 1, there would exist an index i and $\alpha_i \in \Delta(S_i)$ such that (2) would hold. Prove that this would give $u_i(\alpha_i, \sigma_{-i}^\tau) > u_i(\bar{\sigma}_i, \sigma_{-i}^\tau)$. Use the result in step 4 to conclude that this is impossible.