Solution of Problem 3.7

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**Problem 3.7.** Determine, for every possible value of the parameter $a$ in $\mathbb{R}$, the Nash equilibria of the bimatrix game

$$(A, B) = \begin{pmatrix} 1, 1 & a, 0 \\ 0, 0 & 2, 1 \end{pmatrix}.$$ 

**Solution.** The usual expected payoff functions are

$$F_A(p, q) = pq + ap(1-q) + 0*(1-p)q + 2(1-p)(1-q) = (3-a)pq + (a-2)p - 2q + 2,$$

where $s_a(q) := 3 - aq + a - 2$, and

$$F_B(p, q) = pq + (1-p)(1-q) = 2pq - p - q + 1 = (2p - 1)q - p + 1.$$

To determine player 1’s best replies if player 2 chooses any given $q \in [0, 1]$ for column 1, you must maximize $s_a(q)p - 2q + 2$ over all $p \in [0, 1]$. For player 1’s best reply set this gives

$$\beta_1(q) = \begin{cases} \{1\} & \text{if } s_a(q) > 0, \\ [0, 1] & \text{if } s_a(q) = 0, \\ \{0\} & \text{if } s_a(q) < 0. \end{cases} \tag{1}$$

and this will be worked out further below. Vice versa, to determine player 2’s best replies if player 1 chooses any given $p \in [0, 1]$ for row 1, you must maximize $(2p - 1)q - p + 1$ over all $q \in [0, 1]$. For player 2’s best reply set this gives

$$\beta_2(p) = \begin{cases} \{1\} & \text{if } p > \frac{1}{2}, \\ [0, 1] & \text{if } p = \frac{1}{2}, \\ \{0\} & \text{if } p < \frac{1}{2}. \end{cases}$$

Next, you must still work out the consequences of the formula (1), which by itself is too indirect to be of use. To determine for a given value of the parameter $a$, which $q$’s lead to $s_a(q) > 0$, the easiest solution is to plot the linear function $s_a(q)$ on the interval $[0, 1]$. For $q = 0$ it takes the value $s_a(0) = a - 2$ and for $q = 1$ it is $s_a(1) = 1$. Because $a$ can be any value, this plot suggests distinguishing between the following three cases:

**Case 1:** $a > 2$. In this case the entire plotted line takes strictly positive values, i.e., $s_a(q) > 0$ for all $q \in [0, 1]$. This leads to the following rewriting of (1) in case 1:

$$\beta_1(q) = \{1\} \text{ for all } q \in [0, 1].$$

**Case 2:** $a = 2$. In this border case, the plotted line takes strictly positive values, except for its value in $q = 0$, which is $s_2(0) = 2 - 2 = 0$. So the rewriting of (1) in case 2 gives:

$$\beta_1(q) = \begin{cases} \{1\} & \text{if } q > 0, \\ [0, 1] & \text{if } q = 0. \end{cases}$$

**Case 3:** $a < 2$: In this case the plotted line intersects the horizontal axis at $q = \frac{2-a}{3-a}$ (note that in the present case $0 < \frac{2-a}{3-a} < 1$). Consequently, this shows that $s_a(q) < 0$ for all $q < \frac{2-a}{3-a}$ and
\( s_a(q) > 0 \) for all \( q > \frac{2-a}{3-a} \). So the rewriting of (1) in case 3 gives:

\[
\beta_1(q) = \begin{cases} 
\{1\} & \text{if } q > \frac{2-a}{3-a}, \\
[0,1] & \text{if } q = \frac{2-a}{3-a}, \\
\{0\} & \text{if } q < \frac{2-a}{3-a}.
\end{cases}
\]

In each of these three cases you can draw the two reaction curves in the same way as shown on pp. 36-37. This leads to the following conclusions for the mixed NE pairs, which should officially be denoted by \( ((\bar{p},1-\bar{p}),(q,1-q)) \), but which you can more conveniently denote by \((\bar{p},\bar{q})\), as is done below:

**Case 1:** \( a > 2 \). The only NE is \((\bar{p},\bar{q}) = (1,1)\); observe that this is not surprising: \( a > 2 \) leads to row 1 strictly dominating row 2.\(^1\)

**Case 2:** \( a = 2 \). There is a multitude of NE’s \((\bar{p},\bar{q})\), namely \((1,1)\) and all \((p,0)\) with \( 0 \leq p \leq \frac{1}{2} \).

**Case 3:** \( a < 2 \). There are three NE’s \((\bar{p},\bar{q})\), namely \((0,0)\), \((1,1)\) and \((\frac{1}{2},\frac{2-a}{3-a})\).

**Remark.** Without the above idea to plot the function \( s_a(q) \), another, more laborious method still works as well: it is based on keeping track of the signs of numerator \( a - 2 \) and denominator \( a - 3 \) in the aforementioned intersection point \( q = \frac{2-a}{3-a} \). In principle, this method distinguishes five cases (namely \( a > 3 \), \( a = 3 \), \( 2 < a < 3 \), \( a = 2 \) and \( a < 2 \)) instead of the above three.

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\(^1\)By making this observation initially, a small amount of work, such as plotting the two reaction curves in case 1, could have been saved.