Solution of Problem 3.7

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Problem 3.7. Determine, for every possible value of the parameter a in $\mathbb{R}$, the Nash equilibria of the bimatrix game

$$(A,B) = \begin{pmatrix} 1,1 & a,0 \\ 0,0 & 2,1 \end{pmatrix}.$$

Solution. The ususal expected payoff functions are

$$F_A(p,q) = pq*1 + ap(1-q) + 0*(1-p)q + 2(1-p)(1-q) = (3-a)pq + (a-2)p - 2q + 2 = s_a(q)*p - 2q + 2,$$

where $s_a(q) := (3-a)q + a - 2$, and

$$F_B(p,q) = pq + (1-p)(1-q) = 2pq - p - q + 1 = (2p-1)q - p + 1.$$

To determine player 1's best replies if player 2 chooses any given $q \in [0,1]$ for column 1, you must maximize $s_a(q)p - 2q + 2$ over all $p \in [0,1]$. For player 1's best reply set this gives

$$\beta_1(q) = \begin{cases} \{1\} & \text{if } s_a(q) > 0, \\ [0,1] & \text{if } s_a(q) = 0, \\ \{0\} & \text{if } s_a(q) < 0. \end{cases} \tag{1}$$

and this will be worked out further below. Vice versa, to determine player 2's best replies if player 1 chooses any given $p \in [0,1]$ for row 1, you must maximize $(2p-1)q - p + 1$ over all $q \in [0,1]$. For player 2's best reply set this gives

$$\beta_2(p) = \begin{cases} \{1\} & \text{if } p > \frac{1}{2}, \\ [0,1] & \text{if } p = \frac{1}{2}, \\ \{0\} & \text{if } p < \frac{1}{2}. \end{cases}$$

Next, you must still work out the consequences of the formula (1), which by itself is too indirect to be of use. To determine for a given value of the parameter a, which q's lead to $s_a(q) > 0$, the easiest solution is to plot the linear function $s_a(q)$ on the interval $[0,1]$. For $q = 0$ it takes the value $s_a(0) = a - 2$ and for $q = 1$ it is $s_a(1) = 1$. Because a can be any value, this plot suggests distinguishing between the following three cases:

Case 1: $a > 2$. In this case the entire plotted line takes strictly positive values, i.e., $s_a(q) > 0$ for all $q \in [0,1]$. This leads to the following rewriting of (1) in case 1:

$$\beta_1(q) = \{1\} \text{ for all } q \in [0,1].$$

Case 2: $a = 2$. In this border case, the plotted line takes strictly positive values, exept for its value in $q = 0$, which is $s_2(0) = 2 - 2 = 0$. So the rewriting of (1) in case 2 gives:

$$\beta_1(q) = \begin{cases} \{1\} & \text{if } q > 0, \\ [0,1] & \text{if } q = 0. \end{cases}$$

Case 3: $a < 2$: In this case the plotted line intersects the horizontal axis at $q = \frac{2-a}{3-a}$ (note that in the present case $0 < \frac{2-a}{3-a} < 1$!). Consequently, this shows that $s_a(q) < 0$ for all $q < \frac{2-a}{3-a}$ and

$s_a(q) > 0$ for all $q > \frac{2-a}{3-a}$. So the rewriting of (1) in case 3 gives:

$$\beta_1(q) = \begin{cases} \{1\} & \text{if } q > \frac{2-a}{3-a}, \\ [0,1] & \text{if } q = \frac{2-a}{3-a}, \\ \{0\} & \text{if } q < \frac{2-a}{3-a}. \end{cases}$$

In each of these three cases you can draw the two reaction curves in the same way as shown on pp. 36-37. This leads to the following conclusions for the mixed NE pairs, which should officially be denoted by $((\bar{p}, 1-\bar{p}), (\bar{q}, 1-\bar{q}))$, but which you can more conveniently denote by $(\bar{p}, \bar{q})$, as is done below:

Case 1: $a > 2$. The only NE is $(\bar{p}, \bar{q}) = (1, 1)$; observe that this is not surprising: $a > 2$ leads to row 1 strictly dominating row 2.[1]

Case 2: $a = 2$. There is a multitude of NE's $(\bar{p}, \bar{q})$, namely $(1, 1)$ and all $(p, 0)$ with $0 \leq p \leq \frac{1}{2}$.

Case 3: $a < 2$. There are three NE's $(\bar{p}, \bar{q})$, namely $(0, 0)$, $(1, 1)$ and $(\frac{1}{2}, \frac{2-a}{3-a})$.

Remark. Without the above idea to plot the function $s_a(q)$, another, more laborious method still works as well: it is based on keeping track of the signs of numerator $a - 2$ and denominator $a - 3$ in the aforementioned intersection point $q = \frac{2-a}{3-a}$. In principle, this method distinguishes five cases (namely $a > 3$, $a = 3$, $2 < a < 3$, $a = 2$ and $a < 2$) instead of the above three.

[1] By making this observation initially, a small amount of work, such as plotting the two reaction curves in case 1, could have been saved.