Take home exercises, to be handed in November 11

Exercise 6.1.9 (Revised formulation).

Let $U \subset \mathbb{R}^n$ be an open subset, and let $P : C_c^\infty(U) \to C^\infty(U)$ be a linear operator such that for all $\chi, \psi \in C_c^\infty(U)$ the operator $M_\chi \circ P \circ M_\psi$ belongs to $\Psi^d(U)$.

(a) Show that for all $\chi, \psi \in C_c^\infty(U)$ there exist $p \in S^d(U)$ and $K \in C^\infty(U \times U)$ with $\operatorname{supp} p \subset \operatorname{supp} \chi \times \mathbb{R}^n$ and with $\operatorname{supp} K \subset \operatorname{supp} \chi \times \operatorname{supp} \psi$, such that

$$M_\chi \circ P \circ M_\psi = \Psi_p + T_K.$$

(b) Show that $P \in \Psi^d(U)$.

Exercise 3.8.7. (Extension of the original exercise) Let $P_r : H_r(M, E) \to H_{r-k}(M, F)$ be an operator as in Remark 3.8.6 of the lecture notes, and let $Q : H_{r-k}(M, F) \to H_r(M, E)$ be such that both $PQ - \mathrm{Id}$ and $QP - \mathrm{Id}$ are smoothing.

(a) Show that the kernel of the operator $P : \Gamma(E) \to \Gamma(F)$ is finite dimensional.

(b) What can you say about the cokernel of the operator $P : \Gamma(E) \to \Gamma(F)$.

(e) Show that the index of the operator $P_r : H_r(M, F) \to H_{r-k}(M, F)$ only depends on the principal symbol of P.