Take home exercise, to be handed in November 18

Exercise 7.3.9 (Revised formulation). Let Ω be smooth manifold and E a vector bundle on Ω. Let $\{\Omega_j\}_{j\in J}$ be an open cover of Ω. Assume that for each pair of indices (i,j) with $\Omega_{ij} := \Omega_i \cap \Omega_j \neq \emptyset$ a smooth section $g_{ij} \in \Gamma(\Omega_{ij}, E)$ is given and that

$$g_{ij} + g_{jk} + g_{ki} = 0 \qquad \text{on} \quad \Omega_{ijk} := \Omega_i \cap \Omega_j \cap \Omega_k$$

for all $i, j, k \in J$ with $\Omega_{ijk} \neq \emptyset$.

There exists a partition of unity $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ on Ω which is subordinate to the covering $\{\Omega_j\}$. The latter requirement means that there exists a map $j : \mathcal{A} \to J$ such that $\operatorname{supp} \psi_\alpha \subset \Omega_{j(\alpha)}$ for all $\alpha \in \mathcal{A}$.

(a) Show that $g_j := \sum_\alpha \psi_\alpha g_{jj(\alpha)}$ defines a smooth section in $\Gamma(\Omega_j, E)$.

(b) Show that $g_i - g_j = g_{ij}$ on Ω_{ij}, for all $i, j \in J$.

Exercise 7.3.10 (Revised formulation). Let M be a smooth manifold and let $d \in \mathbb{R}$. For $P \in \Psi^d(M)$ and every open subset $U \subset M$ the operator $P_U : f \mapsto (Pf)|_U, C_c^\infty(U) \to C^\infty(U)$ belongs to $\Psi^d(U)$. Let $U \subset V$ be open subsets of M. Then $P \mapsto P_U$ defines a map $\Psi^d(V) \to \Psi^d(U)$.

(a) Show that the map $P \mapsto P_U$ maps $\Psi^{-\infty}(V)$ to $\Psi^{-\infty}(U)$.

Thus the map $P \mapsto P_U$ induces a restriction map

$$\rho_U^V : \Psi^d(V)/\Psi^{-\infty}(V) \to \Psi^d(U)/\Psi^{-\infty}(U)$$

which is a homomorphism of vector spaces. It is obvious that the restriction maps satisfy the conditions

$$\rho_U^U = \mathrm{I}, \qquad \rho_U^V \circ \rho_V^W = \rho_U^W,$$

for all open subsets $U, V, W \subset M$ with $U \subset V \subset W$. Because of these properties, the assignment $U \mapsto \Psi^d(U)/\Psi^{-\infty}(U)$ together with the system of restriction maps ρ_U^V is called a *presheaf* of vector spaces. The purpose of this exercise is to show that the presheaf $\Psi^d/\Psi^{-\infty}$ is in fact a *sheaf*. This means that for every open covering $\{U_j\}_{j\in J}$ of M the following conditions should be fulfilled.

(1) **Restriction property.** Let $P, Q \in \Psi^d(M)$ and assume that for all $j \in J$ the operator $P_{U_j} - Q_{U_j}$ belongs to $\Psi^{-\infty}(U_j)$. Then $P - Q \in \Psi^{-\infty}(M)$. (This and the next condition can be formulated more naturally in terms of the restriction maps, see the text following this exercise).

(2) **Gluing property.** Let for each $j \in J$ an operator $P_j \in \Psi^d(U_j)$ be given and assume that $(P_i)_{U_{ij}} - (P_j)_{U_{ij}} \in \Psi^{-\infty}(U_{ij})$ for all $i, j \in J$ with $U_{ij} := U_i \cap U_j \neq \emptyset$. Then there exists an operator $P \in \Psi^d(M)$ such that $P_{U_j} - P_j \in \Psi^{-\infty}(U_j)$ for all $j \in J$.

(a) Show that condition (1) is fulfilled. Hint: the proof is an adaptation of the proof of Lemma 7.3.8 (a). As in that proof, let $\Omega \subset M \times M$ be the union of the open subsets $U_j \times U_j \subset M \times M$, for $j \in J$. Let $K_P, K_Q \in \mathcal{D}'(M \times M, \mathbb{C}_M \boxtimes D_M)$ be the distribution kernels of P and Q. Show that $K_P - K_Q$ is smooth on Ω.

(b) With P_j as in condition (2), show that there exist $T_j \in \Psi^{-\infty}(U_j)$ such that $(P_i + T_i)_{U_{ij}} = (P_j + T_j)_{U_{ij}}$ for all $i, j \in J$.

Hint: put $\Omega_j = U_j \times U_j$. For all $i, j \in J$ with $U_i \cap U_j \neq \emptyset$, let $g_{ij} \in \mathcal{D}'(\Omega_{ij}, \mathbb{C}_M \boxtimes D_M)$ be the distribution kernel of the operator $(P_i)_{U_{ij}} - (P_j)_{U_{ij}}$. Show that the g_{ij} are smooth and apply Exercise 7.3.9 to find g_j. Define T_j in terms of g_j.

(c) Use (b) combined with Lemma 7.3.9 (b) to prove that condition (2) is fulfilled.

The above conditions (1) and (2) are readily seen to be equivalent to the following conditions, formulated in terms of the restriction maps ρ^V_U.

(1)' Let $P, Q \in \Psi^d(M)$ (their images in $\Psi^d(M)/\Psi^{-\infty}(M)$ are denoted by $[P], [Q]$). Assume that $\rho^M_{U_j}([P]) = \rho^M_{U_j}([Q])$ for all $j \in J$. Then $[P] = [Q]$.

(2)' Let for each j an operator $P_j \in \Psi^d(U_j)$ be given and assume that

$$\rho^{U_i}_{U_{ij}}([P_i]) = \rho^{U_j}_{U_{ij}}([P_j])$$

for all $i, j \in J$ with $U_{ij} \neq \emptyset$. Then there exists a $P \in \Psi^d(M)$ such that $\rho^M_{U_j}([P]) = [P_j]$ for all j.