Alternative take home exercise, to be handed in November 18

Exercise.
We consider the differential operator $P = -\Delta + e^{-\|x\|^2}$, where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

denotes the Laplacian on $\mathbb{R}^n$.

(a) Determine a symbol $p \in S^2(\mathbb{R}^n)$ such that $P = \Psi_p$. Do not forget to show that p belongs to $S^2(\mathbb{R}^n)$.

(b) Show that the operator P is properly supported.

(c) Show that the function $q(x, \xi) := (1 + \|\xi\|^2)^{-1}$ defines an element of $S^{-2}(\mathbb{R}^n)$.

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ and let $\chi' \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi' = 1$ on an open neighborhood of $\operatorname{supp} \chi$. Put $Q = M_{\chi'} \circ \Psi_q \circ M_\chi$.

(d) Show that $Q \in \Psi^{-2}$ and that Q is properly supported.

(e) Show that

$$Q \circ P - M_\chi \in \Psi^{-1}(\mathbb{R}^n).$$

Hint: use the principal symbol.