Harmonic Analysis

Exercises

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Exercise 1. Let M be a smooth manifold. For each compact subset $K \subset M$ we define $s_K : C(M) \to \mathbb{R}$ by

$$s_K(f) = \sup_{x \in K} |f(x)|, \qquad (f \in C(M)).$$

(a) Show that for each compact set $K \subset M$ the function s_K is a seminorm on $C(M)$.

(b) Show that the set $\mathscr{S} = \{s_K \mid K \subset M \quad \text{compact}\}$ is a fundamental system of seminorms on $C(M)$.

(c) Show that there exists a countable fundamental system of seminorms which defines an equivalent topology on $C(M)$.

(d) Show that $C(M)$ equipped with the topology defined by $\mathscr{S}$ is complete. Accordingly, $C(M)$ is a Fréchet space.

(d) Let G be a Lie group, and $\alpha : G \times M \to M, (x,m) \to xm$ a continuous left action of G on M. For $f \in C(M)$ and $x \in G$ we define the function $\pi(x)f : M \to \mathbb{C}$ by $[\pi(x)f](m) = f(x^{-1}m)$. Show that $(\pi, C(M))$ defines a continuous representation of G in $C(M)$.

Exercise 2. Let $(\pi, \mathscr{H})$ be a unitary representation of G in a Hilbert space $\mathscr{H}$. The space of continous G-intertwining endomorphisms of $\mathscr{H}$ is denoted by $\mathrm{End}_G(\mathscr{H})$.

(a) Let $A \in \mathrm{End}_G(\mathscr{H})$ be Hermitian symmetric. Show that the spectral resolution of A consists of G-intertwining projections.

(b) Show that the equivalent of Schur's lemma is valid:

$$\pi \text{ is irreducible} \quad \Longleftrightarrow \quad \mathrm{End}_G(\mathscr{H}) = \mathbb{C}\mathrm{I}_{\mathscr{H}}.$$

Exercise 3. Let $(\pi, \mathscr{H})$ be an irreducible unitary representation of a Lie group G in a Hilbert space $\mathscr{H}$. Let $D \subset \mathscr{H}$ be a dense subspace of $\mathscr{H}$ and let T be a self-adjoint operator on $\mathscr{H}$ with domain D. We assume that D is G-invariant and that

$$\forall x \in G : \quad T \circ \pi(x) = \pi(x) \quad \text{on } D.$$

Show that $T = \lambda I_{\mathscr{H}}$ for a suitable scalar $\lambda \in \mathbb{C}$.

Exercise 4. Let V_1, V_2 be two complete locally convex spaces (Hausdorff, over $\mathbb{C}$). We define the conjugate space $\bar{V}_j$ to be the space V_j equipped with the same addition and topology, but with the conjugate scalar multiplication given by

$$\lambda \cdot v = \bar{\lambda} v \qquad (v \in V, \ \lambda \in \mathbb{C}).$$

Let G be a unimodular Lie group and dx be left Haar measure on G.

(a) Show that for every continuous linear operator $A : \bar{V}_1 \to \bar{V}_2$ and every $f \in C_c(G, V_1)$ we have
$$A(\int_G f(x)\, dx) = \int_G A(f(x))\, dx.$$

(b) Let π be a unitary representation of G in a Hilbert space $\mathscr{H}$. Show that for every $f \in C_c(G)$ we have
$$\pi(f)^* = \pi(f^*),$$
where $f^*(x) = \overline{f(x^{-1})}$.

Exercise 5. Let K be a compact Lie group. By a K-module with finite K-action, we shall mean a complex linear space V equipped with a representation π of K such that

- for each $v \in V$ the linear span of $\pi(K)v$ is finite dimensional
- for each $v \in V$ the restriction of π to span$\,(\pi(K)v)$ is a continuous representation of K.

Let now (δ, V_δ) belong to $\widehat{K}$ and let W be an arbitrary complex linear space. On $V_\delta \otimes W$ we define the representation $\delta \otimes 1$ of K by $(\delta \otimes 1)(k) = \delta(k) \otimes \mathrm{I}_W$.

(a) Show that $V_\delta \otimes W$ is a K-module with finite K-action.

(b) Show that $\mathrm{Hom}_K(V_\delta, V_\delta \otimes W) \simeq W$, naturally.

(c) Let W' be complex linear space. Show that
$$\mathrm{Hom}_K(V_\delta \otimes W, V_\delta \otimes W') \simeq \mathrm{Hom}_{\mathbb{C}}(W, W'),$$
naturally. Observe that Schur's lemma can be viewed as a special case of this assertion.

(d) Let $\delta' \in \widehat{K}$, $\delta' \not\sim \delta$. Show that
$$\mathrm{Hom}_K(V_\delta \otimes W, V_{\delta'} \otimes W') = \{0\}.$$

Exercise 6. We consider a two finite dimensional continuous representations (π_j, V_j), for $j = 1, 2$, of a compact group K. Let $\pi_1 \widehat{\otimes} \pi_2$ be the associated representation of $K \times K$ on $V_1 \otimes V_2$.

(a) Show that the natural isomorphism $\mathrm{End}(V_1) \otimes \mathrm{End}(V_2) \simeq \mathrm{End}(V_1 \otimes V_2)$ maps $\mathrm{End}_K(V_1) \otimes \mathrm{End}_K(V_2)$ onto $\mathrm{End}_{K \times K}(V_1 \otimes V_2)$.

(b) Show that $\pi_1 \widehat{\otimes} \pi_2$ is irreducible if and only if both π_1 and π_2 are irreducible representations of K.

(c) Let (π, V) be finite dimensional irreducible representation of K. Show that π is an irreducible representation of K if and only if $\pi \hat{\otimes} \pi^\vee$ is an irreducible representation of $K \times K$.

Exercise 7. We assume that K is a compact Lie group and that (π, V) is an irreducible representation of K. Let $\langle\cdot\,,\,\cdot\rangle_j$, for $j = 1,2$ be two positive definite Hermitian inner products on $\mathscr{H}$ for which π is unitary.

(a) Show that there exists a positive scalar $c > 0$ such that

$$\langle v\,,\,w\rangle_1 = c\langle v\,,\,w\rangle_2 \qquad (v, w \in V).$$

Hint: relate the two inner products by a Hermitian tranformation.

(b) For $j = 1,2$, let $\langle\cdot\,,\,\cdot\rangle_{\mathrm{HS},j}$ denote the Hilbert-Schmid inner product on $\mathrm{End}(V)$, induced by $\langle\cdot\,,\,\cdot\rangle_j$. Show that

$$\langle\cdot\,,\,\cdot\rangle_{\mathrm{HS},1} = \langle\cdot\,,\,\cdot\rangle_{\mathrm{HS},2}.$$

Exercise 8. Show that the space $\mathscr{R}(K)$ of representative functions, equipped with pointwise addition and pointwise multiplication of functions, is an algebra over $\mathbb{C}$ with unit.

Exercise 9. Show that the map $m_\delta : V_\delta \otimes V_\delta^* \to \mathscr{R}(x)$, given by

$$m_\delta(v \otimes \eta)(x) = \eta(\delta(x^{-1}v))$$

intertwines the representations $\delta \widehat{\otimes} \delta^*$ and $L \times K$ of $K \times K$.

Exercise 10. Let K be a compact group. Show that for $f, g \in C(K)$ we have

(a) $(f * g)^* = g^* * f^*$,

(b) $f * g^*(e) = \langle f\,,\,g\rangle_{L^2(K)}$.

Let $(\pi, \mathscr{H})$ be a unitary representation of K. Show that for $f \in C(K)$,

(c) $\pi(f)^* = \pi(f^*)$.

Exercise 11. Let (π, V) be a continuous representation of the compact group K in a Fréchet space. Let $\delta \in V[\delta]$. Let $P_1, P_2 : V \to V$ be K-equivariant continuous linear operators such that $P_j^2 = P_j$ (projections) and $\mathrm{im}(P_j) = V[\delta]$. Show that $P_1 = P_2$.
Hint: determine $P_1 - P_2$ on each of the isotypical components $V[\delta']$, for $\delta' \in \widehat{K}$.

Exercise 12. Let H be a compact group and let (π, V) be a continuous finite dimensional representation of H. Let $\iota : V^H \to V$ be the inclusion map, and let $p : V \to V^H$ be the unique H-equivariant projection. Show that the map $A \mapsto \iota \circ A \circ p$ defines a linear embedding

$$\mathrm{End}(V^H) \hookrightarrow \mathrm{End}(V)$$

of algebras.

Exercise 13. Let G be a unimodular Lie group and K a compact subgroup. Let dx be a choice of Haar measure for G. Assume that the associated convolution algebra $(C_c(K\backslash G/K), *)$ is commutative. Assume that $(\pi, \mathscr{H})$ is an irreducible unitary representation of G in a Hilbert space. space. The purpose of this exercise is to show that $\dim(\mathscr{H}^K) \leq 1$.

(a) If $v \in \mathscr{H} \setminus \{0\}$, show that $\pi(C_c(G))v$ is dense in $\mathscr{H}$.

(b) Let $P : \mathscr{H} \to \mathscr{H}$ be the unique K-equivariant continuous projection operator with image $\mathscr{H}^K$. Show that for $v \in \mathscr{H}$ the space $P(\pi(C_c(G))v)$ is dense in $\mathscr{H}^K$.

(c) Show that for $v \in \mathscr{H}^K \setminus \{0\}$ the space $\pi(C_c(K\backslash G/K)v)$ is a dense subspace of $\mathscr{H}^K$.

To complete the proof we assume that $\mathscr{H}^K$ is at least two dimensional and aim at deriving a contradiction.

(d) Show that there exists $f \in C_c(K\backslash G/K)$ such that $\pi(f)|_{\mathscr{H}^K}$ is a normal operator different from a scalar multiplication.

(e) Show that there exists an orthogonal projection $Q : \mathscr{H}^K \to \mathscr{H}^K$ such that $Q \neq 0, I$ and such that Q commutes with all operators from $\pi(C_c(K\backslash G/K))|_{\mathscr{H}^K}$. Hint: show that the spectrum of $\pi(f)|_{\mathscr{H}^K}$ consists of more than one point and use the spectral resolution of this operator.

(f) Derive a contradiction.

Exercise 14. Let G be a Lie group, and σ an involution. Let H be an open subgroup of the group G^σ of fixed points. The derivative σ_* of σ at e is denoted by σ as well.

(a) Show that the Lie algebra of G^σ equals $\mathfrak{g}^\sigma$; here, $\mathfrak{g}$ denotes the Lie algebra of G.

(b) Show that the Lie algebra $\mathfrak{h}$ of H equals $\mathfrak{g}^\sigma$.

We define $\mathfrak{q} := \ker(\sigma + \mathrm{I}_\mathfrak{g})$, the minus one eigenspace of $\sigma : \mathfrak{g} \to \mathfrak{g}$.

(c) Show that $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$.

(d) Show that $\mathfrak{q}$ is invariant under $\mathrm{Ad}_G(H)$.

(e) Show that $\mathfrak{h}$ and $\mathfrak{q}$ are perpendicular for the Killing form (which may be degenerate).

Exercise 15. We consider the Lie group $\mathrm{SO}(3)$.

(a) Show that its Lie algebra $\mathfrak{so}(3)$ is generated by the infinitesimal rotations R_1, R_2, R_3 given by $R_j(x) = e_j \times x$ (exterior product).

(b) Let J be the diagonal matrix with entries $+1, -1, -1$. Show that the map $x \mapsto JxJ$ defines an involution σ of $\mathrm{SO}(3)$.

(c) Determine the infinitesimal involution $\sigma : \mathfrak{so}(3) \to \mathfrak{so}(3)$ in terms of the generators.

(d) Show that the connected component H of $\mathrm{SO}(3)^\sigma$ is a copy of $\mathrm{SO}(2)$.

(e) Determine the -1 eigenspace $\mathfrak{q}$ of σ in $\mathfrak{so}(3)$.

(f) Show that the $+1$ eigenspace $\mathfrak{h}$ of σ is a maximal torus in $\mathfrak{so}(3)$ (this is a special feature of the present case).

(g) Argue that the irreducible representations of $\mathrm{SO}(3)$ may be parametrized as δ_n, with $n \in \mathbb{N}$, so that $\dim(\delta_n) = 2n+1$.

(h) Determine the $\mathfrak{h}$-weights of the associated nfinitesimal representation δ_n of $\mathfrak{so}(3)$.

(i) Show that $(V_{\delta_n})^{\mathrm{SO}(2)}$ has dimension 1 for every $n \geq 0$.

(j) Argue that the natural representation of $\mathrm{SO}(3)$ decomposes as

$$(L, L^2(S^2)) \simeq \widehat{\bigoplus}_{n\geq 0} \delta_n.$$

Exercise 16. Let $\mathfrak{g}, \mathfrak{h}$ be finite dimensional complex Lie algebras, and let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be be a Lie algebra homomorphism.

(a) Show that φ has a unique extension to an algebra homomorphism $U(\varphi) : U(\mathfrak{g}) \to U(\mathfrak{h})$.

(b) Let $\psi : \mathfrak{h} \to \mathfrak{l}$ be a second homorphism of Lie algebras. Show that $U(\psi \circ \varphi) = U(\psi) \circ U(\varphi)$.

(c) Show: if φ is an isomorphism then $U(\varphi)$ is an isomorphism as well.

Remark: it follows that $\mathfrak{g} \to U(\mathfrak{g})$ defines a functor from the category of finite dimensional complex Lie algebras to the category of complex associative algebras with unit. From now on we shall usually just write φ for $U(\varphi)$.

For a Lie algebra $\mathfrak{g}$ we define the opposite Lie algebra $\mathfrak{g}^{\mathrm{opp}}$ to be the vector space $\mathfrak{g}$ equipped with the bracket $[X,Y]^{\mathrm{opp}} := [Y,X]$.

For an associative algebra A with unit, we define the opposite algebra A^{opp} to be A as a complex linear space, but equipped with the product $(X,Y) \mapsto X \cdot Y := YX$.

(d) Show that the map $\iota : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto X^\vee := -X$ defines an isomorphism from $\mathfrak{g}$ onto $\mathfrak{g}^{\mathrm{opp}}$.

The associated map $U(\iota) : U(\mathfrak{g}) \to U(\mathfrak{g})^{\mathrm{opp}}$ is usually denoted by $u \mapsto u^\vee$.

(e) Show that for $X_1, \ldots, X_m \in \mathfrak{g}$ we have

$$(X_1 \cdots X_m)^\vee = (-1)^m X_m \cdots X_1.$$

The map $u \mapsto u^\vee$ is often called the canonical anti-automorphism of $U(\mathfrak{g})$.

Exercise 17. Let V be a finite dimensional complex linear space, and $A \in \mathrm{End}(V)$. We define the endomorphism δ_A of the tensor algebra $T(V)$ by

$$\delta_A(u) := \frac{d}{dt}\bigg|_{t=0} T(I+tA)(u), \qquad (u \in V).$$

(a) Show that δ_A preserves the gradation of $T(V)$.

(b) Show that δ_A is a derivation, i.e.,

$$\delta_A(u \otimes v) = \delta_A(u) \otimes v + u \otimes \delta_A(v), \qquad (u, v \in T(V)).$$

Now assume that $\mathfrak{g}$ is a finite dimensional complex Lie algebra, and that $A : \mathfrak{g} \to \mathfrak{g}$ is a derivation.

(c) Show that $\delta_A : T(\mathfrak{g}) \to T(\mathfrak{g})$ preserves the kernel of the canonical homomorphism $T(\mathfrak{g}) \to U(\mathfrak{g})$.

(d) Show that δ_A factors through a derivation $\bar{\delta}_A$ of the associative algebra $U(\mathfrak{g})$, and that this derivation extends A.

(e) Show that $\bar{\delta}_A$ is the unique derivation of $U(\mathfrak{g})$ which restricts to A on $\mathfrak{g}$. We will therefore also denote it by A.

We now assume that $A = \mathrm{ad}X$, where $X \in \mathfrak{g}$.

(f) Show that for all $u \in U(\mathfrak{g})$, we have $Au = Xu - uX$.

For obvious reasons we write $\mathrm{ad}(X)(u)$ for $A(u)$ and $[X,u]$ for $Xu - uX$. No confusion will arise from this!

Now assume that $\mathfrak{g}$ is the complexification of the Lie algebra $\mathfrak{g}_0$ of a Lie group G_0.

(g) Show that for all $X \in \mathfrak{g}_0$ and $u \in U(\mathfrak{g})$ we have

$$[X,u] = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp(tX))u.$$

(in particular, give an appropriate interpretation of the expression on the right-hand side of the equation).

Exercise 18. Let (π, V) be a finite dimensional representation of a real Lie group G. Then $\pi : G \to \mathrm{GL}(V)$ is a continuous homomorphism, hence smooth, and its derivative $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ defines a representation of $\mathfrak{g}$ in V, which by complex linear extension equips V with the structure of $\mathfrak{g}_{\mathbb{C}}$-module. With notation as in the previous exercise, show that

$$\pi(x) \circ H = \mathrm{Ad}(x)(H) \circ \pi(x) \qquad \text{on} \quad V$$

for all $H \in U(\mathfrak{g})$ and $x \in G$.

Exercise 19. Assume that $\mathfrak{g}$ is a complex Lie algebra.

(a) If $\mathfrak{g}$ is abelian, show that there exists a unique algebra homomorphism $\varphi : S(\mathfrak{g}) \to U(\mathfrak{g})$ such that $\varphi(X) = X$ for $X \in \mathfrak{g}$. Show that φ equals the symmetrizer map s.

(b) Show that the symmetrizer map $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ is an isomorphism of algebras if and only if $\mathfrak{g}$ is abelian.

Exercise 20.

(a) Let V be a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, having a cyclic highest weight vector. Show that V is irreducible.

(b) Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let R^+ be an associate system of positive roots and let V be a finite dimensional $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}^*$. Let $\alpha \in R^+$ and let $H_\alpha \in [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha]$ be such that $\alpha(H_\alpha) = 2$ and let $\mathfrak{s}_\alpha := \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha$ be the associated copy of $\mathfrak{sl}(2,\mathbb{R})$. Show that $\lambda(H_\alpha) \in \mathbb{N}$. Show that $\lambda \in \Lambda$.

Exercise 21. The purpose of this exercise is to give a suitable definition of the Laplace operator Δ for a Riemannian manifold M. We denote the metric by g and the associated Riemannian volume form (a density) by dV. There is no need to assume that M is oriented. The space of smooth vector fields on M is denoted by $\mathfrak{X}(M)$. Here it is convenient to work with complex valued vector fields; these are sections of the complexified tangent bundle $(TM)_\mathbb{C}$. For each $x \in M$, the inner product g_x is naturally extended to a complex bilinear form on $(T_xM)_\mathbb{C}$.

(a) Give the definition of a first order partial differential operator $D : C^\infty(M) \to \mathfrak{X}(M)$.

(b) For a smooth function $f \in C^\infty(M)$ the gradient vector field $\mathrm{grad} f \in \mathfrak{X}(M)$ is defined by

$$g_x(\mathrm{grad} f(x), \xi) = df(x)\xi, \qquad (x \in M, \xi \in T_xM).$$

Show that $\mathrm{grad} : C^\infty(M) \to \mathfrak{X}(M)$ is a first order partial differential operator.

(c) Give the definition of a first order partial differential operator $P : \mathfrak{X}(M) \to C^\infty(M)$.

(d) Show that there exists a unique first order partial differential operator $\mathrm{div} : \mathfrak{X}(M) \to C^\infty(M)$ such that

$$\int_M g_x(\mathrm{grad} f(x), v(x))\, dV = -\int_M f(x)\, \mathrm{div}\, v(x)\, dV,$$

for every $v \in \mathfrak{X}(M)$ and $f \in C_c^\infty(M)$. Hint: use local coordinates.

We define the Laplace operator on M to be the composed operator

$$\Delta := \mathrm{div} \circ \mathrm{grad} : \; C^\infty(M) \to C^\infty(M).$$

(e) Calculate $\mathrm{grad}, \mathrm{div}$ and Δ for $\mathbb{R}^n$ equipped with the Euclidean metric.

(f) Show that the Laplace operator Δ is a second order differential operator on M, which satisfies

$$\int_M \Delta f(x) g(x) dV = \int_M f(x) \Delta g(x)\, dV, \qquad (f, g \in C_c^\infty(M)).$$

Exercise 22. We now consider the setting of a Riemannian homogeneous space G/H. Here we assume that G is a unimodular Lie group and that H is a compact subgroup. This implies that G has a bi-invariant Haar measure, and that there exists an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Furthermore, $\mathfrak{q}$ can be equipped with an $\mathrm{Ad}(H)$-invariant positive definite inner product $\beta : \mathfrak{q} \times \mathfrak{q} \to \mathbb{R}$.

(a) Show that the inner product

$$g_x := dl_x([e])^{-1*}\beta$$

on $T_{[x]}(G/H)$ depends on x through its class $[x]$ in G/H. We will therefore also denote it by $g_{[x]}$. It has a unique extension to a non-degenerate bilinear form on $T_{[x]}(G/H)_{\mathbb{C}}$, which will be denoted by the same symbol.

(b) Show that $[x] \mapsto g_{[x]}$ defines a Riemannian structure on G/H which is G-invariant.

We denote by $C^\infty(G, \mathfrak{q}_{\mathbb{C}})$ the space of smooth functions $\varphi : G \to \mathfrak{q}_{\mathbb{C}}$ and by $C^\infty(G, H, \mathfrak{q}_{\mathbb{C}})$ the subspace consisting of functions $\varphi \in C^\infty(G, \mathfrak{q}_{\mathbb{C}})$ transforming according to the rule

$$\varphi(xh) = \mathrm{Ad}(h)^{-1}\varphi(x), \qquad (x \in G, h \in H).$$

(c) Given a vector field $v \in \mathfrak{X}(G/H)$ show that the function

$$\varphi_v : x \mapsto dl_x([e])^{-1} v([x]),\ \ G \to \mathfrak{q}_{\mathbb{C}}$$

belongs to $C^\infty(G, H, \mathfrak{q}_{\mathbb{C}})$.

(d) Show that the map $v \mapsto \varphi_v$ defines a linear isomorphism from $\mathfrak{X}(G/H)$ onto $C^\infty(G, H, \mathfrak{q}_{\mathbb{C}})$, and give a formula for the inverse map $\varphi \mapsto v_\varphi$.

(e) For $T = X \otimes Y \in T^2(\mathfrak{q}_{\mathbb{C}})$ we define the linear map $\bar{D}_T : C^\infty(G) \to C^\infty(G, \mathfrak{q}_{\mathbb{C}})$ by

$$\bar{D}_T f(x) = R_X f(x) Y.$$

Show that $T \mapsto \bar{D}_T$ extends to a linear map $T^2(\mathfrak{q}_{\mathbb{C}}) \to \mathrm{Hom}(C^\infty(G), C^\infty(G, \mathfrak{q}_{\mathbb{C}}))$ and that the extended map satisfies the transformation rule

$$R_h(\bar{D}_T f) = \mathrm{Ad}(h)^{-1} \bar{D}_{\mathrm{Ad}(h)T} R_h f.$$

(f) Now assume that $T \in T^2(\mathfrak{q}_{\mathbb{C}})^H$. Show that $\bar{D}_T$ restricts to a linear map $C^\infty(G/H) \to C^\infty(G, H, \mathfrak{q}_{\mathbb{C}})$, which corresponds to a first order partial differential operator $D_T : C^\infty(G/H) \to \mathfrak{X}(G/H)$ under the isomorphism $\mathfrak{X}(G/H) \simeq C^\infty(G, H, \mathfrak{q}_{\mathbb{C}})$.

(g) Let β^* be the dual of β on $\mathfrak{q}^*$. If $X_1, \ldots, X_n$ is an orthonormal basis for $\mathfrak{q}$, relative to β, show that (canonically)

$$\beta^* = \sum_{j=1}^{n} X_j \otimes X_j.$$

(h) Prove that $D_{\beta^*} = \mathrm{grad}$.

(i) Given $T = X \otimes Y \in T^2(\mathfrak{q}_{\mathbb{C}})$ we define the map $\bar{\nabla}_T : C^\infty(G, \mathfrak{q}_{\mathbb{C}}) \to C^\infty(G)$ by

$$\bar{\nabla}_T \varphi(x) = \beta(R_X \varphi(x), Y).$$

Show that for $T \in T^2(\mathfrak{q}_{\mathbb{C}})^H$ the operator $\bar{\nabla}_T$ restricts to a linear operator $C^\infty(G, H, \mathfrak{q}_{\mathbb{C}}) \to C^\infty(G/H)$, which corresponds to a first order differential operator

$$\nabla_T : \mathfrak{X}(G/H) \to C^\infty(G/H).$$

(j) Prove that $\nabla_{\beta^*} = \mathrm{div}$.

(k) Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{q}$ as above. Show that for all $f \in C^\infty(G/H)$,

$$\Delta f = \sum_{j=1}^{n} R_{X_j}^2 f.$$

(l) Show that $L := \sum_{j=1}^{n} X_j^2 \in U(\mathfrak{g}_{\mathbb{C}})^H$ and that

$$\Delta = r(L).$$

Exercise 23. We assume that $\mathfrak{g}$ is a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, R the root system and R^+ a choice of positive roots. Without loss of generality you may assume that $\mathfrak{g} = \mathfrak{u}_{\mathbb{C}}$, with $\mathfrak{u}$ a compact semisimple Lie algebra, and that $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$, with $\mathfrak{t}$ a maximal torus in $\mathfrak{u}$. But this is not necessary.

(a) Argue that for all $\lambda, \mu \in \mathfrak{h}^*$ we have

$$\lambda + \mu \neq 0 \Rightarrow \mathfrak{g}_\lambda \perp \mathfrak{g}_\mu$$

(relative to the Killing form B of $\mathfrak{g}$).

(b) Show that one can find elements $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $B(X_\alpha, Y_\alpha) = 1$, for $\alpha \in R$.

(c) Show that the Killing form B restricts to a positive definite inner product on $\mathfrak{h}_{\mathbb{R}}$.

(d) Let $H_1, \ldots, H_r$ be a B-orthonormal inner product of $\mathfrak{h}_{\mathbb{R}}$. Describe the basis $\{H^j, X^\alpha, Y^\alpha\}$ of $\mathfrak{g}^*$ which is dual to the basis $\{H_j, X_\alpha, Y_\alpha\}$ of $\mathfrak{g}$ and show that the Killing form B is given by

$$B = \sum_{j=1}^{r} H^j \otimes H^j + \sum_{\alpha \in R^+} (X^\alpha \otimes Y^\alpha + Y^\alpha \otimes X^\alpha)$$

with respect to this basis.

(e) Describe the map $B : \mathfrak{g} \to \mathfrak{g}^*$ in terms of the above bases for $\mathfrak{g}$ and $\mathfrak{g}^*$.

(f) Show that the dualized Killing form is given by

$$B^* = \sum_{j=1}^{r} H_j \otimes H_j + \sum_{\alpha \in R^+} (X_\alpha \otimes Y_\alpha + Y_\alpha \otimes X_\alpha)$$

For $\alpha \in R$, let $H_\alpha \in \mathfrak{h}$ be the unique element of $\mathfrak{h}$ such that $H_\alpha \perp \ker \alpha$ relative to B and such that $\alpha(H_\alpha) = 2$. Warning: the span of the triple $H_\alpha, X_\alpha, Y_\alpha$ is isomorphic to the standard copy $\mathfrak{s}_\alpha$ of $\mathfrak{sl}(2, \mathbb{C})$, but the given triple need not be standard.

(g) Let $B_{\mathfrak{h}}$ denote the restriction of the Killing form to $\mathfrak{h} \times \mathfrak{h}$ Show that the associated map $B_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}^*$ satisfies

$$B_{\mathfrak{h}}(H_\alpha) = \frac{1}{2} B(H_\alpha, H_\alpha)\alpha.$$

Show that relative to the dual form $B^*_{\mathfrak{h}}$ we have

$$B(H_\alpha, H_\alpha) B^*_{\mathfrak{h}}(\alpha, \alpha) = 4.$$

From now on we shall use the notation $\langle \cdot \,, \cdot \rangle$ for both the linear form $B_{\mathfrak{h}}$ on $\mathfrak{h}$ and for the dual form $B^*_{\mathfrak{h}}$ on $\mathfrak{h}^*$. Then the above relation becomes

$$\langle H_\alpha \,, H_\alpha \rangle \langle \alpha \,, \alpha \rangle = 4, \qquad (\alpha \in R).$$

(h) Show that

$$B([X_\alpha, Y_\alpha], H) = \alpha(H), \qquad (H \in \mathfrak{h}),$$

and conclude that

$$[X_\alpha, Y_\alpha] = 2\frac{H_\alpha}{\langle H_\alpha\,, H_\alpha\rangle}.$$

(i) View the polynomial function $Q : \xi \mapsto B^*(\xi, \xi)$, $\mathfrak{g}^* \to \mathbb{C}$, as an element of $S^2(\mathfrak{g})$ and give an expression of Q in terms of the elements H_j, X_α and Y_α.

(j) Argue that the image $\Omega := s(Q)$ under the symmetrizer $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ belongs to $Z(\mathfrak{g})$. This element is called the **Casimir operator** of $\mathfrak{g}$. Express Ω in terms of the canonical images of H_j, X_α and Y_α in $U(\mathfrak{g})$.

(k) Calculate $'\gamma(\Omega)$ (relative to the positive system R^+).

(l) Show that the associated polynomial function on $\mathfrak{h}^*$ is in general not Weyl group invariant. (Give a simple counter example.)

(m) Show that the polynomial function $\gamma(\Omega) \in P(\mathfrak{h}^*)$ is given by

$$\gamma(\Omega, \lambda) = \langle\lambda\,, \lambda\rangle - \langle\delta\,, \delta\rangle, \qquad (\lambda \in \mathfrak{h}^*).$$

Check that this function is indeed invariant under W.

(n) Repeat (k), (l), (m) for the positive system $-R^+$ in place of R^+. Compare the obtained version of $\gamma(\Omega)$ to the previous one.

(p) Argue more generally that $\gamma(\Omega)$ is independent of the choice of positive roots.

Exercise 24. We now consider the situation of a connected compact semisimple Lie group U equipped with an involution σ. Let $B_\mathfrak{u}$ be the Killing form of the compact semisimple algebra $\mathfrak{u}$. Then $B_\mathfrak{u}$ is negative definite. Furthermore, $\mathfrak{g} = \mathfrak{u}_\mathbb{C}$ is a complex semisimple Lie algebra, whose (complex) Killing form B is equal to the complex bilinear extension of $B_\mathfrak{u}$.

We assume that K is an open subgroup of U^σ. Let $\mathfrak{q}$ be the -1-eigenspace of σ and let β be the restriction of $-B$ to $\mathfrak{q} \times \mathfrak{q}$. Then as in Exercise 22, the form β determines an invariant Riemannian structure on U/K, which determines a Laplace operator Δ.

Let $\mathfrak{t}$ be any maximal torus of $\mathfrak{u}$. Put $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$ and fix a choice R^+ of positive roots for $R := R(\mathfrak{g}, \mathfrak{h})$. Let Ω be the Casimir element of $Z(\mathfrak{g})$, defined as in Exercise 23.

(a) Let $p := \dim \mathfrak{u}$ and let $X_1, \ldots, X_p$ be an orthonormal basis of $\mathfrak{u}$ relative to the inner product $-B_\mathfrak{u}$. Show that

$$\Omega = -\sum_{j=1}^{p} X_j^2,$$

as an element of $U(\mathfrak{g})$.

(b) Show that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(U) & \xrightarrow{R_\Omega} & C^\infty(U) \\ \pi^* \uparrow & & \uparrow \pi^* \\ C^\infty(U/K) & \xrightarrow{-\Delta} & C^\infty(U/K) \end{array}$$

From this we conclude that $r(\Omega) = -\Delta$, as elements of $\mathbb{D}(U/K)$.

(c) Show that the eigenvalues of the Laplace operator Δ are of the form

$$s_\xi := -\langle \lambda_\xi \, , \, \lambda_\xi + 2\delta \rangle,$$

with $\xi \in U^\wedge$ such that $V_\xi^K \neq 0$ and with λ_ξ equal to the highest weight of ξ.

(d) Assume that $s \in \mathbb{C}$ is an eigenvalue of the Laplace operator Δ, with corresponding eigenspace

$$\mathscr{E}_s := \{ f \in C^\infty(U/K) \mid \Delta f = sf \}.$$

Show that $s \leq 0$ and that $\mathscr{E}_s$ is finite dimensional.

We now specialize to the situation that $U = \mathrm{SO}(3)$ and that σ equals the involution of $\mathrm{SO}(3)$ described in Exercise 15. We consider the standard action of $\mathrm{SO}(3)$ on S^2, the unit sphere in $\mathbb{R}^3$. The stabiliser of the standard basis vector $e_1 = (1,0,0)$ equals the connected component K of U^σ. In Exercise 15 we have seen that $K \simeq \mathrm{SO}(2)$.

(e) Show that the derivative at 0 of the map $\varphi : \mathfrak{q} \to S^2$, $X \mapsto (\exp X) \cdot e_1$ is given by

$$d\varphi(0) R_j = e_j \times e_1.$$

(f) Let b denote the Euclidean inner product on $T_{e_1} S^2 \simeq \{0\} \times \mathbb{R}^2$. Show that $d\varphi(0)^* b = \beta$, where β denotes the restriction of $-B/2$ to $\mathfrak{q}$.

We equip U/K with the U-invariant Riemannian metric induced by β.

(g) Show that the map $\psi : U/K \to S^2, \; xK \mapsto x \cdot e_1$ is an isometry.

If follows from (g) that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(S^2) & \xrightarrow{\Delta} & C^\infty(S^2) \\ \psi^* \downarrow & & \downarrow \psi^* \\ C^\infty(U/K) & \xrightarrow{\Delta_{U/K}} & C^\infty(U/K) \end{array}$$

Here Δ denotes the Laplace operator for the standard metric on the unit sphere S^2.

(h) Show that $\Delta = -\frac{1}{2} r(\Omega)$.

(i) Let δ_n be the irreducible representation of $\mathrm{SO}(3)$ of dimension $2n+1$. Show that its highest weight is given by $\lambda_n = n\alpha$, where α is the unique positive root (we assume that a maximal torus $\mathfrak{t} \subset \mathfrak{u}$ and a choice of positive roots has been fixed).

(j) Show that $\langle H_\alpha , H_\alpha \rangle = 2$ and that $\langle \alpha , \alpha \rangle = 2$.

(k) Show that the eigenvalues of Δ are given by $s_n := -n(n+1)$, and that the associated eigenspace equals

$$C^\infty(S^2)_{\delta_n}.$$

Argue that the eigenspace has dimension $2n+1$.

Exercise 25. We assume that K is a compact connected Lie group. The purpose of this exercise is to describe the image of $C^\infty(K)$ under Fourier transform.

We assume that $\mathfrak{t}$ is a maximal torus in K, $R = R(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, Λ the associated lattice of integral weights. We denote by Λ_K the sublattice consisting of $\lambda \in \Lambda$ such that there exists a character ξ_λ on $T = \exp(\mathfrak{t})$ such that $d\xi_\lambda(e) = \lambda$. Then it is known that Λ_K has finite index in Λ.

Let R^+ be a choice of positive roots, Λ^+ be the associated set of dominant weights, and $\Lambda_K^+ = \Lambda_K \cap \Lambda^+$. Then it is known that via highest weight theory, $\hat{K}$ corresponds to Λ_K^+. It follows that for each $\lambda \in \Lambda_K$ there is a unique irreducible representation $\delta_\lambda \in \hat{K}$ whose infinitesimal character equals $\chi_\lambda : Z(\mathfrak{k}) \to \mathbb{C}$, $Z \mapsto \gamma(Z, \lambda)$. (Beware: in this notation λ is the infinitesimal character of δ_λ, which is different from the highest weight).

(a) Argue that $\delta_\lambda \sim \delta_\mu \iff W\lambda = W\mu$.

For $f \in L^2(K)$ and $\lambda \in \Lambda_K$ we define the Fourier transform $\tilde{f}(\lambda)$ by

$$\tilde{f}(\lambda) = \delta_\lambda(f) \in \mathrm{End}(V_{\delta_\lambda}).$$

(b) If $f \in C^\infty(G)$ and $Z \in \mathbb{Z}(\mathfrak{k})$, show that for every $\lambda \in \Lambda_K$,

$$(L_Z f)\tilde{} = \gamma(Z, \lambda)\tilde{f}.$$

(c) If $f \in C^\infty(G)$, show that for every $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that

$$\|\tilde{f}(\lambda)\|_{\mathrm{HS}} \leq C_N(1 + \|\lambda\|)^{-N}.$$

Now assume that $f \in L^2(K)$ and that for every $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that the estimate in (b) is valid. The purpose is to show that then f is a smooth function.

For each $R \in \mathbb{N}$ we define the smooth function $\varphi_R \in \mathscr{R}(K)$ by

$$\varphi_R = \sum_{\substack{\lambda \in \Lambda_K^+ \\ \|\lambda\| \leq R}} \dim(\delta_\lambda)\chi_{\delta_\lambda}.$$

(d) Show that $\varphi_R * f \to f$ in $L^2(K)$, for $R \to \infty$. In the following we assume that R_j is an increasing sequence of positive real numbers, with $R_j \to \infty$ for $j \to \infty$.

(e) Show that for every $X \in \mathfrak{t}$ there exist constants $C > 0$ and $N > 0$ such that

$$\|\delta_\lambda(X)\|_{\text{op}} \leq C(1 + \|\mathfrak{g}\|)^N, \qquad (\lambda \in \Lambda_K^+).$$

What is the best N you can find?

(f) Show that a similar estimate is valid for every $X \in \mathfrak{k}$.

(g) Show that a similar estimate is valid for every $u \in U(\mathfrak{g})$.

(h) Show that for every $u \in U(\mathfrak{g})$ the sequence $L_u(\varphi_{R_j} * f)$ $(j \in \mathbb{N})$ is Cauchy in $C(K)$, equipped with the sup-norm.

(i) Let $p \in \mathbb{N}$. Show that $f \in C^p(K)$ and that $\varphi_{R_j} * f \to f$ in $C^p(K)$, equipped with the usual Banach topology.

Exercise 26. We consider the one dimensional complex projective space $\mathbb{P}^1(\mathbb{C})$. As a set it is defined as the set of one dimensional subspaces $\mathbb{C}v$ of $\mathbb{C}^2$, for $v \in \mathbb{C}^2 \setminus \{0\}$. It has a unique structure of complex analytic manifold for which the maps $z \mapsto \mathbb{C}(z,1)$ and $w \mapsto \mathbb{C}(1,w)$ define holomorphic open embeddings $\mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ (the inverses of affine charts).

(a) Show that $\mathrm{SL}(2,\mathbb{C})$ has a natural holomorphic action on $\mathbb{P}^1(\mathbb{C})$.

(b) Show that this action is transitive.

(c) We consider the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, equipped with the obvious structure of complex analytic manifold. Show that the map $\varphi : \widehat{\mathbb{C}} \to \mathbb{P}^1(\mathbb{C})$ defined by

$$\varphi(z) = \mathbb{C}(z,1), \ (z \in \mathbb{C}), \quad \text{and} \quad \varphi(\infty) = \mathbb{C}(0,1)$$

is a holomorphic diffeomorphism.

(d) Through the above diffeomorphism we transfer the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ to an action on $\widehat{\mathbb{C}}$. Show that the action of an element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

on $\widehat{\mathbb{C}}$ is given by

$$g \cdot z = \frac{az+b}{cz+d}.$$

Note that the appearing fractional linear transformation is thus interpreted as a biholomorphic invertible transformation of the Riemann sphere, which sends $-d/c$ to ∞, and which sends ∞ to a/c.

The action of $\mathrm{SL}(2,\mathbb{C})$ restricts to an action of $\mathrm{SL}(2,\mathbb{R})$ on $\widehat{\mathbb{C}}$ which we shall now consider.

(e) Show that the stabilizer in $\mathrm{SL}(2,\mathbb{R})$ of the imaginary unit i equals $K := \mathrm{SO}(2)$.

We consider the standard Iwasawa subgroups $A = \{a_t \mid t \in \mathbb{R}\}$, $N = \{n_x \mid x \in \mathbb{R}\}$, where

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \text{and} \qquad n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

(f) Calculate $n_x a_t \cdot i$. Show that $\mathrm{SL}(2,\mathbb{R}) \cdot i$ contains the upper half plane H consisting of the points $z \in \mathbb{C}$ with $\mathrm{Im}\, z > 0$.

(g) Calculate the stabilizer of 0 in $\mathrm{SL}(2,\mathbb{R})$ and show that $\mathrm{SL}(2,\mathbb{R}) \cdot 0 = \widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.

(h) Argue that the $\mathrm{SL}(2,\mathbb{R})$ action on $\widehat{\mathbb{C}}$ has three orbits: $H_+, \widehat{\mathbb{R}}$ and $-H$.

(k) Use $\mathrm{SL}(2,\mathbb{R}) \cdot i = H$ to conclude that $\mathrm{SL}(2,\mathbb{R}) = NAK$.

Exercise 27. The purpose of this exercise is to use the identification $H \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ of the previous exercise, in order to calculate the hyperbolic metric on the upper half plane H.

Let $\mathfrak{p}$ denote the subspace of symmetric matrices in $\mathfrak{sl}(2,\mathbb{R})$ We shall use the linear isomorphism $\varphi : \mathbb{R}^2 \mapsto \mathfrak{p}$ given by

$$\varphi(x,y) = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}$$

(a) Show that the map $\Phi : (x,y) \mapsto \exp[\varphi(x,y)] \cdot i$ defines a diffeomorphism from $\mathbb{R}^2$ onto H. By calculating partial derivatives, show that its total derivative $d\Phi(0,0)$ equals the map $\mathbb{R}^2 \to \mathbb{C}$, $(x,y) \mapsto 2y + 2xi$.

(b) Show that for $k \in \mathrm{SO}(2)$ we have

$$k\,\Phi(x,y)\,k^{-1} = \Phi(k^2 \cdot (x,y))$$

In the following we will denote by $dx \otimes dx + dy \otimes dy$ the standard metric on the Euclidean space $\mathbb{R}^2$. Its restriction to H is said to be the standard Euclidean metric on H and denoted by β.

(a) Show that

$$dl_k(i)^* \beta_i = \beta_i.$$

Hint: use Φ.

(b) For $x \in \mathrm{SL}(2,\mathbb{R})$ we define g_x to be the inner product on $T_{x \cdot i}(H) \simeq \mathbb{R}^2$, given by

$$g_x := dl_x(i)^{-1*} \beta_i.$$

Show that g_x depends on x through its image $x \cdot i$. We shall therefore also write $g_{x \cdot i}$. Show that $z \mapsto g_z$ defines a Riemannian metric on H for which $\mathrm{SL}(2,\mathbb{R})$ acts by isometries. This metric is called the hyperbolic metric on H.

(c) Show that
$$g_z = y^{-2}\beta_z, \qquad (z = x+iy \in H).$$
Hint: first, prove this for $z = a_t \cdot i$.

(d) Let $t \in \mathbb{R}$ and calculate the hyperbolic distance $d(a_t \cdot i, i)$ of the point $a_t \cdot i$ to the origin i.

Exercise 28. We assume that G is a connected real semisimple Lie group with finite center, that K is a maximal compact subgroup (hence comes from a Cartan decomposition) and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the associated Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the associated root system, and $W \subset \mathrm{GL}(\mathfrak{a}^*)$ the associated Weyl group generated by the reflections $s_\alpha \in \mathrm{GL}(\mathfrak{a}^*)$.

(a) Show that through the contragredient action, W may be realized as a subgroup of $\mathrm{GL}(\mathfrak{a})$.

We write $N_K(\mathfrak{a})$ for the normalizer of $\mathfrak{a}$ in K and $Z_K(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in K. Thus,
$$N_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)(\mathfrak{a}) = \mathfrak{a}\}, \quad Z_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)|_{\mathfrak{a}} = 0\}.$$

(b) Show that for every $w \in W$ there exists a $k \in N_K(\mathfrak{a})$ such that $w = \mathrm{Ad}(k)|_{\mathfrak{a}}$.

(c) Show that the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$ equals the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

(d) Show that $Z_K(\mathfrak{a})$ is a normal subgroup of $N_K(\mathfrak{a})$ and that $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is a finite group. The natural map $N_K(\mathfrak{a}) \to \mathrm{GL}(\mathfrak{a})$, $k \mapsto \mathrm{Ad}(k)|_{\mathfrak{a}}$ induces an embedding of $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ into $\mathrm{GL}(\mathfrak{a})$ via which we shall view $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ as a subgroup of the latter group.

(e) Show that $W \subset N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Remark. It can be shown that the inclusion in (e) is an equality.

Exercise 29. Let G be a connected real semisimple Lie group with finite center. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and K the associated maximal compact subgroup of G.

The purpose of this exercise is to show that all maximal abelian subspaces of $\mathfrak{p}$ are conjugate under $\mathrm{Ad}(K)$.

(a) Let $H \in \mathfrak{a}^{\mathrm{reg}}$ (regular means that $\alpha(H) \neq 0$ for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$). Let $X \in \mathfrak{p}$ and consider the function $f : K \to \mathbb{R}$ given by
$$f(k) = B(\mathrm{Ad}(k)X, H).$$
Show that f has a stationary point $k_0 \in K$.

(b) Show that $B([U, \mathrm{Ad}(k)X], H) = 0$ for all $U \in \mathfrak{k}$.

(c) By using root spaces, show that $[\mathfrak{k}, H] = \mathfrak{a}^\perp \cap \mathfrak{p}$.

(d) Show that $\mathrm{Ad}(k_0)X \in \mathfrak{a}$.

(e) Let $\mathfrak{b} \subset \mathfrak{p}$ be a maximal abelian subspace, and let $X \in \mathfrak{b}^{\mathrm{reg}}$. Show that $\mathfrak{b} = Z_{\mathfrak{g}}(X) \cap \mathfrak{p}$.

(f) Show that $\mathrm{Ad}(k_0)\mathfrak{b} = \mathfrak{a}$.

Exercise 30. Assume that G is a connected real semisimple Lie group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $\theta : G \to G$ be the associated global Cartan involution.

(a) Let $H \subset G$ be a closed subgroup which is θ invariant. Show that

$$H = (H \cap K) \exp(\mathfrak{h} \cap \mathfrak{p}).$$

and that the map $(k, X) \mapsto k \exp(X)$ is a diffeomorphism from $(H \cap K) \times (\mathfrak{h} \cap \mathfrak{p})$ onto H.

(b) Let $X \in \mathfrak{a}$ and let Z be its centralizer in G. Show that $Z = Z \cap K) \exp(\mathfrak{z} \cap \mathfrak{p})$.

Exercise 31. Let G be a connected real semisimple Lie group, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an infinitesimal Iwasawa decomposition.

Let $X \in \mathfrak{a}$ and let H be its centralizer in G. Show that

$$H = (H \cap K) A (H \cap N).$$

Hint: first compare with the infinitesimal Iwasawa decomposition. Warning: in general, H is not connected.

Exercise 32. Let G be a Lie group and H a closed subgroup.

(a) Show that the cotangent bundle $T^*(G/H)$ is an equivariant vector bundle on G/H.

(b) Describe a natural diffeomorphismn

$$(G \times (\mathfrak{g}/\mathfrak{h})^*)/H \to T^*(G/H).$$

(c) Show that the exterior bundle $\wedge T^*(G/H)$ is equivariant.

(d) Describe a natural diffeomorphism

$$(G \times \wedge(\mathfrak{g}/\mathfrak{h})^*)/H \to T^*(G/H).$$

(e) Prove that the space $\Gamma(G, \wedge^k T^*(G/H))^G$ of G-invariant sections of the bundle is linearly isomorphic to the space $[\wedge^k (\mathfrak{g}/\mathfrak{h})^*]^H$ of $\overline{\mathrm{Ad}}(H)$-invariants in $\wedge^k (\mathfrak{g}/\mathfrak{h})^*$.

(f) Formulate a generalization of the result in (e) to an arbitrary equivariant vector bundle $p : \mathscr{V} \to G/H$.

Exercise 33. **(Frobenius reciprocity)** Let K be a compact Lie group and H a closed subgroup. Let (xi, V_ξ) be a finite dimensional continuous representation of H.

(a) Let $\mathrm{ev}_e : C(K : H : \xi) \to V_\xi$ be defined by $f \mapsto f(e)$. Show that for every finite dimensional representation (π, V) of K the map $\varepsilon : T \mapsto \mathrm{ev}_e \circ T$ gives a linear isomorphism

$$\mathrm{Hom}_K(V, C^\infty(K : H : \xi)) \xrightarrow{\simeq} \mathrm{Hom}_M(V, V_\xi).$$

This is more conveniently expressed as

$$\mathrm{Hom}_K(\pi, \mathrm{ind}_H^K(\xi)) \;\simeq\; \mathrm{Hom}_M(\pi|_M, \xi), \quad \text{naturally.}$$

One says that the forgetful functor $\pi \rightsquigarrow \pi|_M$ is left-adjoint to the induction functor $\xi \rightsquigarrow \mathrm{ind}_H^K(\xi)$.

(b) Show that for every irreducible $\delta \in \widehat{K}$ the isotypical component $C(K : H : \xi)[\delta]$ has finite dimension.

(c) Give the decomposition of $C(K : H : \xi)_K$ into isotypical components, and prove the correctness of your result.

(d) Argue that the functions in $C(K : H : \xi)_K$ are smooth.

Exercise 34. **(Infinitesimal character of principal series)** The purpose of this exercise is to show that representations of the principal series have an infinitesimal character, and to determine this infinitesimal character.

We assume that G is a real semisimple Lie group with finite center, that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of its Lie algebra, that $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace, $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ the associated root system, Σ^+ a choice of positive roots, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and

$$G = KAN$$

the associated Iwasawa decompositions.

Furthermore, we write $M = Z_K(\mathfrak{a})$, define $P = MAN$, and assume that (ξ, V_ξ) is an irreducible unitary representation of M. We denote by

$$\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$$

the restriction of the left regular representation of G on $C^\infty(P : \xi : \lambda)$ (the smooth version of the principal series representation).

The center $Z(\mathfrak{m}_\mathbb{C})$ of $U(\mathfrak{m}_\mathbb{C})$ acts on V_ξ by an infinitesimal character which we denote by χ_ξ.

(a) Show that
$$U(\mathfrak{g}_\mathbb{C}) = U(\mathfrak{m}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}) \oplus (\mathfrak{n}_\mathbb{C} U(\mathfrak{g}_\mathbb{C}) + U(\mathfrak{g}_\mathbb{C})\bar{\mathfrak{n}}_\mathbb{C}).$$

(b) Given $Z \in Z(\mathfrak{g}_\mathbb{C})$ we denote by ${}^{\backprime}\mu(Z)$ the projection of Z on $U(\mathfrak{m}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C})$ according to the above decomposition. Show that
$${}^{\backprime}\mu(Z) \in U(\mathfrak{a}_\mathbb{C})Z(\mathfrak{m}_\mathbb{C}) \quad \text{and} \quad Z - {}^{\backprime}\mu(Z) \in \mathfrak{n}_\mathbb{C} U(\mathfrak{g}_\mathbb{C})\bar{\mathfrak{n}}_\mathbb{C}.$$

(c) If $\varphi \in C^\infty(P : \xi : \lambda)$, show that
$$L_Z\varphi(e) = [\xi \otimes (\lambda + \rho)]({}^{\backprime}\mu(Z))\varphi(e).$$
Show that $\pi_{\xi,\lambda}$ has an infinitesimal character $\chi_{\xi,\lambda}$. Hint: observe that $U(\mathfrak{a}_\mathbb{C})Z(\mathfrak{m}_\mathbb{C})$ equals the center of $U(\mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C})$.

Our next goal is to derive a simpler formula for the infinitesimal character $\chi_{\xi,\lambda}$.

(d) Show that $U(\mathfrak{a}_\mathbb{C})Z(\mathfrak{m}_\mathbb{C})$ can be identified with the space $P(\mathfrak{a}_\mathbb{C}^*) \otimes Z(\mathfrak{m}_\mathbb{C})$ of polynomial functions $\mathfrak{a}_\mathbb{C}^* \to Z(\mathfrak{m}_\mathbb{C})$. Accordingly, show that
$$\chi_{\xi,\lambda}(Z) = \chi_\xi({}^{\backprime}\mu(Z)(\lambda + \rho)), \qquad (Z \in Z(\mathfrak{g}_\mathbb{C}),$$
where χ_ξ denotes the infinitesimal character of ξ.

We are now going to use the Harish-Chandra isomorphism for $Z(\mathfrak{m}_\mathbb{C})$ to describe the infinitesimal character χ_ξ. We fix a maximal torus $\mathfrak{t}$ in the (compact) Lie algebra $\mathfrak{m}$.

(e) Show that $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, i.e., $\mathfrak{h}$ is maximal subject to the conditions that a) it be abelian, and 2) for every $X \in \mathfrak{h}$, the endomorphism $\mathrm{ad}(X)_{\mathbb{C}} \in \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ diagonalizes.

(f) Let R be the root system of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Show that for every $\alpha \in R$ we have

$$\alpha|_{\mathfrak{a}} \neq 0 \Rightarrow \alpha|_{\mathfrak{a}} \in \Sigma.$$

The direct sum decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ allows us to identify $\mathfrak{t}_{\mathbb{C}}^*$ with the elements of $\mathfrak{h}_{\mathbb{C}}^*$ vanishing on $\mathfrak{a}$. Likewise, $\mathfrak{a}_{\mathbb{C}}^*$ may be identified with the subspace of $\mathfrak{h}_{\mathbb{C}}^*$ consisting of the functionals vanishing on $\mathfrak{t}$.

(g) Show that $R_{\mathfrak{m}} := \{\alpha \in R \mid \alpha|_{\mathfrak{a}} = 0\}$ can be identified with the root system of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{m}_{\mathbb{C}}$, and that

$$\mathfrak{g}_{\mathbb{C}\alpha} = \mathfrak{m}_{\mathbb{C}\alpha}, \qquad (\alpha \in R_{\mathfrak{m}}).$$

(h) Show that there exists a choice of positive roots R^+ for R such that

$$\alpha \in R^+ \setminus R_{\mathfrak{m}} \Rightarrow \alpha|_{\mathfrak{a}} \in \Sigma^+.$$

Such a positive system R^+ is said to be compatible with the system Σ^+. We assume it fixed from now on.

(i) Show that $R_{\mathfrak{m}}^+ := R^+ \cap R_{\mathfrak{m}}$ is a positive system for $R_{\mathfrak{m}}$.

(j) Let $\mathfrak{g}_{\mathbb{C}}^+$ be the sum of the root spaces $\mathfrak{g}_{\mathbb{C}\alpha}$ for $\alpha \in R^+$, and let $\mathfrak{m}_{\mathbb{C}}^+$ be the similar sum of the root spaces $\mathfrak{m}_{\mathbb{C}\beta}$ for $\beta \in R_{\mathfrak{m}}^+$. Show that

$$\mathfrak{g}_{\mathbb{C}}^+ = \mathfrak{m}_{\mathbb{C}}^+ \oplus \mathfrak{n}_{\mathbb{C}}.$$

(k) Let δ be half the sum of the roots in R^+, and let $\delta_{\mathfrak{m}}$ be half the sum of the positive roots in $R_{\mathfrak{m}}^+$. Show that

$$\delta = \delta_{\mathfrak{m}} + \rho.$$

(l) Let $`\gamma_{\mathfrak{m}} : Z(\mathfrak{m}_{\mathbb{C}}) \to S(\mathfrak{t}_{\mathbb{C}})$ be the algebra homomorphism determined by

$$Z - `\gamma_{\mathfrak{m}}(Z) \in \mathfrak{m}_{\mathbb{C}}^+ U(\mathfrak{m}_{\mathbb{C}}) \mathfrak{m}_{\mathbb{C}}^-, \qquad (Z \in Z(\mathfrak{m}_{\mathbb{C}})).$$

Show that the Harish-Chandra isomorphism $\mathfrak{g}_{\mathfrak{m}} : Z(\mathfrak{m}_{\mathbb{C}}) \to S(\mathfrak{t}_{\mathbb{C}})^{W(\mathfrak{m},\mathfrak{t})}$ is given by

$$\gamma_{\mathfrak{m}}(Z, \Lambda) = `\gamma_{\mathfrak{m}}(Z, \Lambda + \delta_M).$$

Warning: this formula has δ_M in place of the usual $-\delta_M$ because of a different choice of a positive system in the definition of $`\gamma_{\mathfrak{m}}$.

(m) Let $\Lambda_\xi \in \mathfrak{t}_{\mathbb{C}}^*$ be such that $\chi_\xi = \gamma_{\mathfrak{m}}(\,\cdot\,, \Lambda_\xi)$. Show that

$$\chi_{\xi+\lambda}(Z) = \gamma(Z, \Lambda_\xi + \lambda), \qquad (Z \in Z(\mathfrak{g})).$$