Extra Exercise 1

Let X be a topological space.

(a) Suppose that $\mathscr{S} = \{S_i \mid i \in I\}$ a locally finite collection of subsets of X. Show that the closure of the union $\cup\mathscr{S}$ is given by

$$\overline{\cup_{i\in I} S_i} = \cup_{i\in I} \bar{S}_i.$$

(b) Let now $\{\eta_i \mid i \in I\}$ be a locally finite collection of functions from $C(X)$. Show that the support of their sum $\eta := \sum_{i\in I} \eta_i$ is given by

$$\mathrm{supp}(\eta) \subset \cup_{i\in I} \mathrm{supp}(\eta_i).$$

If $\eta_i \geq 0$ for all $i \in I$, show that the inclusion becomes an equality.

Extra Exercise 2

Let $\{S_i \mid i \in I\}$ be a collection of subsets of a topological space. Show that the following assertions are equivalent.

(a) The collection $\{S_i \mid i \in I\}$ is locally finite.

(b) The collection $\{\bar{S}_i \mid i \in I\}$ is locally finite.

Extra Exercise 3

Let X be a topological space and $\mathscr{U} = \{U_i \mid i \in I\}$ an open covering of X. Show that the following assertions are equivalent.

(a) There exists a locally finite refinement $\mathscr{V}$ of $\mathscr{U}$.

(b) There exists a locally finite open covering $\mathscr{W} = \{W_i \mid i \in I\}$ of X such that $W_i \subset U_i$ for all $i \in I$.

Hint: for (a) $\Rightarrow$ (b): define $\mathscr{W}$ in terms of a suitable function $\varphi : \mathscr{V} \to I$.

Extra Exercise 4

Assume that X is locally compact Hausdorff and paracompact. Let $\mathscr{U} = \{U_i \mid i \in I\}$ be an open covering of X.

(a) Show that their exists a locally finite open covering $\mathscr{W}$ of X with the property that for every $W \in \mathscr{W}$ there exists a $U \in \mathscr{U}$ such that the closure $\bar{W}$ of W is compact and contained in U.

(b) By giving an example, show that there need not exist a locally finite open covering $\{W_i \mid i \in I\}$ such that for all $i \in I$ the closure $\bar{W}_i$ is compact and contained in U_i.

Extra Exercise 5

Let X be a topological space. Assume that for every open covering $\mathscr{U} = \{U_i \mid i \in I\}$ of X there exists a partition of unity $\{\eta_i \mid i \in I\}$ such that $\operatorname{supp}(\eta_i) \subset U_i$.

(a) Show that X is paracompact.

Now assume in addition that X is locally compact Hausdorff.

(b) Show that for every open cover $\mathscr{U} = \{U_i \mid i \in I\}$ of X there exists a partition of unity $\{\eta_j \mid j \in J\}$ subordinate to $\mathscr{U}$ such that $\operatorname{supp}(\eta_j)$ is compact for every $j \in J$. Show that in general such a partition of unity need not exist with $J = I$.