Representation theory and applications in classical quantum mechanics

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1 Wigner's theorem

In quantum physics, the state space of a system is represented by the space of one-dimensional subspaces (rays) in a complex Hilbert space $\mathcal{H}$. If $v \in \mathcal{H} \setminus \{0\}$, the ray through v is denoted by

$$[v] := \mathbb{C}v.$$

The set of rays $[v]$, for $v \in \mathcal{H} \setminus \{0\}$, is called the projectivization of $\mathcal{H}$, and denoted by $\mathbb{P}(\mathcal{H})$. The canonical map $p : v \mapsto [v]$ is a surjection from $\mathcal{H} \setminus \{0\}$ onto $\mathbb{P}(\mathcal{H})$. The set $\mathbb{P}(\mathcal{H})$ is equipped with the quotient topology, i.e., the finest topology for which p is continuous. Thus, a set $U \subset \mathbb{P}(\mathcal{H})$ is open if its pre-image $p^{-1}(U)$ is open in $\mathcal{H} \setminus \{0\}$. We leave it to the reader to verify that $\mathbb{P}(\mathcal{H})$ becomes a Hausdorff space in this fashion and that p is open, i.e., p maps open sets to open sets.

Exercise 1.1 Let $S(\mathcal{H})$ denote the unit sphere in $\mathcal{H}$, equipped with the restriction topology. Show that the topology of $\mathbb{P}(\mathcal{H})$ coincides with the quotient topology for the natural surjective map $S(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$, $v \mapsto [v]$.

For purposes of quantum physics, the space $\mathbb{P}(\mathcal{H})$ is equipped with an additional structure. Given two vectors $v, w \in \mathcal{H} \setminus \{0\}$, the non-negative real number

$$(v, w) := \frac{|\langle v\,,\, w\rangle|^2}{\|v\|^2\|w\|^2}$$

depends on v and w through their rays $[v]$ and $[w]$. We shall therefore also denote it by $([v], [w])$. The number is called the *transition probability* from state $[w]$ to state $[v]$, because of the probabilistic nature of its interpretation in quantum mechanics, which we shall now explain. Let P_v denote the orthogonal projection $\mathcal{H} \to \mathbb{C}v$ and let P_w be defined similarly. Then P_v is a bounded self-adjoint operator in $\mathcal{H}$ with eigenvalues 1 and 0 and associated eigenspace decomposition $\mathcal{H} = \mathbb{C}v \oplus v^\perp$. The operator represents an observable with values 1 and 0. If the value 1 is measured, through an experiment devised to measure the observable

P_v, it is said that the system is found in the state $[v]$. The number $(p(v), p(w))$ is interpreted as the probability to find the system in state $[v]$, when measuring the observable P_v, when before measurement the system is known to be in state $[w]$ (such a state could be prepared through measurement of the observable P_w). The operator $P_v \circ P_w$ has rank 1 and therefore possesses a trace. Moreover, it is readily checked that

$$\operatorname{tr}(P_v \circ P_w) = ([v], [w]).$$

Exercise 1.2 We use the notation $\operatorname{End}(\mathcal{H})$ for the space of bounded linear operators of $\mathcal{H}$. If $\operatorname{End}(\mathcal{H})$ is equipped with the operator norm, then $v \mapsto P_v$ defines a continuous map $\mathcal{H} \setminus \{0\} \to \operatorname{End}(\mathcal{H})$, which factors to a continuous embedding $\mathbb{P}(\mathcal{H}) \hookrightarrow \operatorname{End}(\mathcal{H})$. Show that the topology of $\mathbb{P}(\mathcal{H})$ coincides with the restriction topology associated with the embedding $\mathbb{P}(\mathcal{H}) \hookrightarrow \operatorname{End}(\mathcal{H})$.

We shall now investigate the geometry of $\mathbb{P}(\mathcal{H})$. If $V \subset \mathcal{H}$ is a closed subspace, then V is a Hilbert space of its own right, and the inclusion map $V \to \mathcal{H}$ induces a continuous embedding $\mathbb{P}(V) \to \mathbb{P}(\mathcal{H})$. It is easily verified that the embedding is a homeomorphism onto a closed subset of $\mathbb{P}(\mathcal{H})$.

Let now e be a unit vector in $\mathcal{H}$. Then applying the mentioned construction to $V = e^\perp$ we obtain an embedding $\mathbb{P}(e^\perp) \to \mathbb{P}(\mathcal{H})$ via which we shall identify. Accordingly, the complement $U = U_e := \mathbb{P}(\mathcal{H}) \setminus \mathbb{P}(e^\perp)$ is an open neighborhood of $p(e)$. We will show that this open neighborhood is homeomorphic to $e^\perp$. Indeed, the set U consists of the rays $[y]$ with $y \in \mathcal{H}$ and $\langle y\,,\, e\rangle \neq 0$. Given such y we define

$$\chi([y]) = \frac{y - \langle y\,,\, e\rangle e}{\langle y\,,\, e\rangle}.$$

Clearly, the right-hand side of the above expression depends on y through its ray $[y]$ and belongs to $e^\perp$. It follows that χ is a well-defined map $U \to e^\perp$. Moreover, it is easy to check that χ is bijective, with inverse

$$\chi^{-1}(x) = [e + x] \qquad (x \in U).$$

It follows that $\chi = \chi_e$ is a homeomorphism from the open neighborhood $U = U_e$ of $[e]$ onto the Hilbert space $e^\perp$. We call (U_e, χ_e) the *affine chart* determined by the unit vector e. From the above observations we see that $\mathbb{P}(\mathcal{H})$ is a topological Hilbert manifold. If $\mathcal{H}$ is infinite dimensional, then $\mathcal{H}$ is isometrically isomorphic to $e^\perp$ so that the manifold $\mathbb{P}(\mathcal{H})$ can actually be modelled on $\mathcal{H}$.

Exercise 1.3 Investigate smoothness properties of the manifold $\mathbb{P}(\mathcal{H})$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. A morphism $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ is defined to be a continuous map that preserves the transition probabilities, i.e., $(Tx, Ty)_2 = (x, y)_1$ for all $x, y \in \mathbb{P}(\mathcal{H}_1)$. If the morphism T is a homeomorphism, then its inverse is a morphism as well. Such a morphism is called an isomorphism. An automorphism of $\mathbb{P}(\mathcal{H})$ is an isomorphism of $\mathbb{P}(\mathcal{H})$ onto itself. Equipped with composition, the set $\operatorname{Aut}(\mathbb{P}(\mathcal{H}))$ of automorphisms of $\mathbb{P}(\mathcal{H})$ is a group. We shall now investigate its structure.

The key result is the following theorem, due to Wigner [21]. If $\mathcal{H}_1, \mathcal{H}_2$ are complex Hilbert spaces, then an anti-linear map $\mathcal{H}_1 \to \mathcal{H}_2$ is defined to be an additive map such that $T(\lambda x) = \bar{\lambda} Tx$ for all $x \in \mathcal{H}_1$ and $\lambda \in \mathbb{C}$. An anti-unitary map is an anti-linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\langle Tx\,,\, Ty\rangle_2 = \langle y\,,\, x\rangle_1$ for all $x, y \in \mathcal{H}_1$.

Theorem 1.4 (Wigner's theorem) *Let $\mathcal{H}_1, \mathcal{H}_2$ be two complex Hilbert spaces of dimension at least two, with associated canonical projections $p_j : \mathcal{H}_j \setminus \{0\} \to \mathbb{P}(\mathcal{H}_j)$, for $j = 1, 2$. Let $e_j \in \mathcal{H}_j$ be unit vectors, for $j = 1, 2$. Let $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ be a continuous isomorphism that maps $[e_1]$ onto $[e_2]$. Then there exists a unique additive map $\widetilde{T} : \mathcal{H}_1 \to \mathcal{H}_2$ with the following properties.*

(a) $\widetilde{T} e_1 = e_2$.

(b) $p_2 \circ \widetilde{T} = T \circ p_1$.

Moreover, $\tilde{T}$ is bijective and either unitary or anti-unitary.

We will give the proof of this theorem in a number of steps. The first tool we need is the following. If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathbb{P}(\mathcal{H})$ then $x \perp y$ means that x and y are perpendicular as lines in $\mathcal{H}$. Clearly, this is equivalent to the condition that $(x, y) = 0$. If $B \subset \mathbb{P}(\mathcal{H})$ and $x \in \mathbb{P}(\mathcal{H})$, then $x \perp B$ means that $x \perp b$ for all $b \in B$. Moreover, $B^\perp := \{x \in \mathbb{P}(\mathcal{H}) \mid x \perp B\}$. Thus, $B^\perp$ consists of all rays in $\mathcal{H}$ that are perpendicular to all rays from B. By $\mathrm{csp}(B)$ we denote the closed linear span of $p^{-1}(B)$ in $\mathcal{H}$, and by $\mathrm{psp}(B)$ the image of this span in $\mathbb{P}(\mathcal{H})$. Then it is readily verified that

$$\mathrm{psp}(B) = (B^\perp)^\perp. \tag{1.1}$$

Lemma 1.5 *Let $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ be a morphism. If $B \subset \mathbb{P}(\mathcal{H}_1)$, then T maps $B^\perp$ into $T(B)^\perp$.*

Proof. This is an immediate consequence of the fact that T preserves the transition probabilities. □

Corollary 1.6 *Let $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ be an isomorphism and let $B \subset \mathbb{P}(\mathcal{H}_1)$. Then T maps* $\mathrm{psp}(B)$ *onto* $\mathrm{psp}(T(B))$.

Proof. Let $C_1 \subset \mathbb{P}(\mathcal{H}_1)$ and let $C_2 = T(C_1)$. Then $T(C_1^\perp) \subset C_2^\perp$ and $T^{-1}(C_2^\perp) \subset C_1^\perp$. Hence, $T(C_1^\perp) = C_2^\perp$. Applying this with $C_1 = B^\perp$, we obtain $T(B^{\perp\perp}) = T(B^\perp)^\perp$. The proof is finished by applying the above formula once more, this time with with $C_1 = B$, and by using (1.1). □

We now return to the proof of Wigner's theorem. Let notation be as in the formulation of the theorem. Then we observe that, for $j = 1, 2$, the affine coordinate patch $U_j := U_{e_j}$ consists of rays $[y]$ with $y \in \mathcal{H}_j \setminus e_j^\perp$, or equivalently of $\eta \in \mathbb{P}(\mathcal{H}_j)$ with $(\eta, [e_j]) \neq 0$. Thus, $U_j = \mathbb{P}(\mathcal{H}_j) \setminus [e_j]^\perp$. It follows by application of Lemma 1.5 that T maps U_1 bijectively onto U_2. We will investigate the bijective map $A : e_1^\perp \to e_2^\perp$ given by

$$A = \chi_2 \circ T \circ \chi_1^{-1},$$

where we have written $\chi_j := \chi_{e_j}$. Of course this map is completely determined by the formula

$$[e_2 + Ax] = T[e_1 + x],$$

for all $x \in e_1^\perp$. Making use of the fact that $([e_1 + x], [e_1]) = (1 + \|x\|^2)^{-1}$, and that T is a morphism, we see that

$$(1 + \|Ax\|^2)^{-1} = ([e_2 + Ax], [e_2]) = ([e_1 + x], [e_1]) = (1 + \|x\|^2)^{-1},$$

from which we deduce that

$$\|Ax\| = \|x\|, \qquad (x \in e_1^\perp). \tag{1.2}$$

Again making use of the fact that T is a morphism, we deduce that

$$\begin{aligned} \frac{|1 + \langle Ax\,,\, Ay\rangle|^2}{(1+\|Ax\|^2)(1+\|Ay\|^2)} &= ([e_2 + Ax], [e_2 + Ay]) \\ &= ([e_1 + x], [e_1 + y]) \\ &= \frac{|1 + \langle x\,,\, y\rangle|^2}{(1+\|x\|^2)(1+\|y\|^2)}, \end{aligned}$$

for all $x, y \in e_1^\perp$. Combining this with (1.2) we infer that

$$|1 + \langle Ax\,,\, Ay\rangle| = |1 + \langle x\,,\, y\rangle| \tag{1.3}$$

for all $x, y \in e_1^\perp$.

Let now $v \in e_1^\perp$ be a non-zero vector, and let $V = \mathbb{C}e_1 \oplus \mathbb{C}v$. Then $p_1(e_1 + \mathbb{C}v) = \mathbb{P}(V) \setminus [e_1]^\perp$. Moreover, since $\mathbb{P}(V) = \mathrm{psp}(\{[e_1], [e_1 + v]\})$, it follows from Corollary 1.6 that $T(\mathbb{P}(V)) = \mathrm{psp}(\{[e_2], [e_2 + Av]) = \mathbb{P}(e_2 + \mathbb{C}Av)$. Therefore

$$T(p_1(e_1 + \mathbb{C}v)) = \mathbb{P}(\mathbb{C}e_2 + \mathbb{C}Av) \setminus [e_2]^\perp = p(e_2 + \mathbb{C}Av).$$

From this we conclude that $A(\mathbb{C}v) \subset \mathbb{C}A(v)$. By continuity and bijectivity of T, hence of A, there exists a unique continuous bijective function $f_v : \mathbb{C} \to \mathbb{C}$ such that

$$A(zv) = f_v(z)Av, \qquad (z \in \mathbb{C}).$$

Assume now that v has norm 1. Since A is norm preserving, it follows that $\|A(zv)\| = |z|$ and $\|Av\| = 1$, so that

$$|f_v(z)| = |z| \qquad (z \in \mathbb{C}).$$

Moreover, it follows from (1.3) applied with $x = zv$ and $y = v$ that

$$|1 + f_v(z)| = |1 + z|.$$

It follows that the complex number $f_v(z)$ lies on the circle $C_1 \subset \mathbb{C}$ with center -1 and containing the point z; moreover, $f_v(z)$ also lies on the circle C_2 with center 0 and containing the point z. These circles C_1 and C_2 intersect in two points, z and $\bar{z}$. Therefore $f_v(z) \in \{z, \bar{z}\}$. It follows that $f_v : \mathbb{C} \to \mathbb{C}$ is a continuous function with $f_v(z) \in \{z, \bar{z}\}$ for all $z \in \mathbb{C}$. This implies that either $f_v = I$ or $f_v = \tau : z \mapsto \bar{z}$, by the lemma below.

Lemma 1.7 *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous injective function such that $f(z) \in \{z, \bar{z}\}$, for all $z \in \mathbb{C}$. Then f equals either the identity $I : z \mapsto z$ or the conjugation $\tau : z \mapsto \bar{z}$.*

Proof. The proof is an easy exercise, left to the reader. □

If w is a non-zero multiple of v then, clearly, $f_w = f_v$. Thus we see that $f_w \in \{I, \tau\}$, for all $w \in e_1^\perp \setminus \{0\}$.

Lemma 1.8 *Let $v \in e_1^\perp \setminus \{0\}$. Then $T[v] = [Av]$.*

Proof. We use that $f_v = I$ on $]0, \infty[$ and that T is continuous. Since $[v] = \lim_{\lambda\downarrow 0}[\lambda e_1 + v]$, it follows that

$$\begin{aligned} T[v] &= \lim_{\lambda\downarrow 0} T[e_1 + \lambda^{-1} v] \\ &= \lim_{\lambda\downarrow 0} [e_1 + f_v(\lambda^{-1}) A v] \\ &= \lim_{\lambda\downarrow 0} [\lambda e_1 + A v] \\ &= [Av]. \end{aligned}$$

This establishes the identity. □

Lemma 1.9 *Let $v, w \in e_1^\perp$. Then $\langle Av\,,\, Aw\rangle$ equals either $\langle v\,,\, w\rangle$ or $\langle w\,,\, v\rangle$.*

Proof. Since T preservers transition probabilities, $(T[v], T[w]) = ([v], [w])$. In view of the previous lemma this implies that $([Av], [Aw]) = ([v], [w])$. Since A is norm preserving, this implies that

$$\langle Av\,,\, Aw\rangle = \langle v\,,\, w\rangle.$$

On the other hand, from (1.3) with $x = v, y = w$ we have $|1 + \langle Av\,,\, Aw\rangle| = |1 + \langle v\,,\, w\rangle|$. It follows that $\langle Av\,,\, Aw\rangle$ lies on the intersection of the following two circles D_1 and D_2 in the complex plane; D_1 is the circle with center -1 containing $\langle v\,,\, w\rangle$ and D_2 is the circle with center 0 containing $\langle v\,,\, w\rangle$. The intersection consists of $\langle v\,,\, w\rangle$ and its conjugate $\langle w\,,\, v\rangle$. □

If $f \in \{I, \tau\}$, then we shall say that A is f-linear if $A(zv + w) = f(z)Av + Aw$ for all $v, w \in e_1^\perp$ and $z \in \mathbb{C}$.

Lemma 1.10 *There exists a unique function $f \in \{I, \tau\}$ such that A is f-linear. In particular, $f_v = f$, for all $v \in e_1^\perp \setminus \{0\}$.*

Proof. Assume that $v, w \in e_1^\perp$ are non-zero vectors with $v \perp w$ and let V denote the linear span of v, w. Then $V = (\{v, w\}^\perp)^\perp$. It follows from the previous lemma and the bijectivity of A that $A(V) = (\{Av, Aw\}^\perp)^\perp = \mathrm{span}(\{Av, Aw\})$. Let $z \in \mathbb{C}$ then it follows that

$$A(zv + w) = c(z)Av + d(z)Aw,$$

for unique continuous functions $c, d : \mathbb{C} \to \mathbb{C}$. Taking inner products with Aw and using Lemma 1.9, we obtain that $d(z) = 1$ for all z. From the injectivity of A it now follows that c is injective. Taking inner products with Av and using Lemma 1.9 once more, we obtain $c(z) \in \{z, \bar{z}\}$. By continuity and injectivity of c it follows that $c \in \{I, \tau\}$, see Lemma 1.7. Using the identity with $w = 0$ we see that $c = f_v$, so that

$$A(zv + w) = f_v(z)Av + Aw.$$

This identity with $z = 1$ implies that $A(v + w) = Av + Aw$ for $v, w \in e_1^\perp$ non-zero and perpendicular. Let $z \in \mathbb{C} \setminus \{0\}$. Then zv and zw are perpendicular non-zero vectors in $e_1^\perp$, so that

$$\begin{aligned} f_{v+w}(z)(Av + Aw) &= f_{v+w}(z)A(v + w) \\ &= A(zv + zw) \\ &= A(zv) + A(zw) \\ &= f_v(z)Av + f_w(z)Aw. \end{aligned}$$

We conclude that $f_v = f_{v+w} = f_w$ for all v and w as above. Hence, A is f_v-linear on the full linear span of v and w. The result now easily follows. □

Completion of the proof of Theorem 1.4. It follows from the fact that A is additive and norm-preserving, that A preserves the inner product on $e_1^\perp$. Indeed, consider the identity $\langle A(v+w)\,,\, A(v+w)\rangle = \langle v+w\,,\, v+w\rangle$.

Let now $\tilde{T}$ be the unique f-linear extension of A to $\mathcal{H}_1$ with $\tilde{T}(e_1) = e_2$. Then $\tilde{T}$ satisfies all requirements.

Finally, we establish uniqueness. Let $\widehat{T} : \mathcal{H}_1 \to \mathcal{H}_2$ be additive and such that $\widehat{T}(e_1) = e_2$ and $p_2 \circ \widehat{T} = T \circ p_1$. Applying the latter identity to $e_1 + v$ with $v \in e_1^\perp$ we find that $p_2(e_2 + \widehat{T}x) = p_2(e_2 + Ax)$, whence $\widehat{T} = A = \tilde{T}$ on $e_1^\perp$. Let now λ be a non-zero complex number, and $x \in e_1^\perp$. Then $\widehat{T}(\lambda e_1 + x)$ has p_2-image $Tp_1(e_1 + \lambda^{-1}x) = p_2(e_1 + f(\lambda)^{-1}Ax) = p_2(f(\lambda)e_1 + x)$. Applying this with a non-zero x we find that $\widehat{T}(\lambda e_1 + x) = f(\lambda)e_1 + x$ so that, by additivity $\widehat{T}(\lambda e_1) = f(\lambda)\widehat{T}(e_1)$, for $\lambda \neq 0$. Hence $\widehat{T} = \tilde{T}$ on non-zero multiples of e_1 and on $e_1^\perp$. By additivity, this implies $\widehat{T} = \tilde{T}$. □

Let $\mathcal{H}$ be a complex Hilbert space. By $\mathrm{U}(\mathcal{H})$ we denote the group of unitary automorphisms of $\mathcal{H}$. We equip this group with the strong operator topology. Recall that a subbasis for this topology is given by the sets $N(x, O) = \{T \in \mathrm{U}(\mathcal{H}) | Tx \in O\}$, $x \in \mathcal{H}$ and O open in $\mathcal{H}$. This means that a sequence (T_j) in $\mathrm{U}(\mathcal{H})$ converges with limit T if and only if $T_j(x) \to T(x)$ for all $x \in \mathcal{H}$.

Exercise 1.11 Show that $\mathrm{U}(\mathcal{H})$ is a Hausdorff topological group. It may be convenient to first show that an equivalent subbasis for the topology is given by the sets $N(S, x, \varepsilon) = \{T \in \mathrm{U}^1(\mathcal{H}) \mid \|Tx - Sx\| < \varepsilon\}$, where $S \in \mathrm{U}^1(\mathcal{H})$, $x \in \mathcal{H}$ and $\varepsilon > 0$.

Let $\{e_\alpha\}$ be an orthonormal basis for $\mathcal{H}$ and define the operator $J\mathcal{H} \to \mathcal{H}$ by $J(\sum_\alpha c_\alpha e_\alpha) = \sum_\alpha \bar{c}_\alpha e_\alpha$. Then J is an anti-unitary automorphism of $\mathcal{H}$. Moreover, the set of all anti-unitary automorphisms of $\mathcal{H}$ is given by $\mathrm{U}^-(\mathcal{H}) = J\mathrm{U}(\mathcal{H})$. Equipped with the strong operator topology, the union $\mathrm{U}^1(\mathcal{H})$ is a Hausdorff topological group. Moreover, $\mathrm{U}(\mathcal{H})$ is an open and closed subgroup. If $T \in \mathrm{U}^1(\mathcal{H})$, then $q(T) : \mathbb{C}v \mapsto \mathbb{C}Tv = T(\mathbb{C}v)$ defines an automorphism of $\mathbb{P}(\mathcal{H})$. Via the map $z \mapsto zI$ we view the complex unit circle $\mathbb{T}$ as a subgroup of $\mathrm{U}(\mathcal{H})$.

Lemma 1.12 *The map $q : \mathrm{U}^1(\mathcal{H}) \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ is a surjective group homomorphism with kernel $\mathbb{T}$.*

Proof. That q is a group homomorphism is a straightforward consequence of the definitions. It is surjective by Wigner's theorem. Clearly $\mathbb{T} \subset \ker q$. If $U \in \ker q$, then $Ue = ze$ for a unique $z \in \mathbb{T}$. It follows that $z^{-1}Ue = e$ and $q(z^{-1}U) = I$, hence $U = zI \in \mathbb{T}$ by the uniqueness part of Wigner's theorem. □

We thus see that q induces an isomorphism $\mathrm{U}^1(\mathcal{H})/\mathbb{T} \simeq \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$. Accordingly, we equip $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ with the quotient topology, so that it becomes a Hausdorff topological group as well.

Exercise 1.13 The following setting is of importance for the theory of spin for the electron. The state space for this system is the projectivization of the two dimensional finite dimensional Hilbert space $\mathbb{C}^2$, equipped with the standard inner product. The projectivization $\mathbb{P}(\mathbb{C}^2)$ is one-dimensional complex projective space, equipped with the structure of transition probabilities.

(a) Show that the identity component $\mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2))_e$ is isomorphic to $G := \mathrm{SO}(3)$.

(b) Show that every non-trivial continuous homomorphism $\pi : \mathrm{SO}(3) \to \mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2))_e$ is an isomorphism.

(c) Let B denote the Killing form on $\mathfrak{g} := \mathfrak{so}(3)$. Then B is negative definite. Show that the associated orthogonal group $\mathrm{O}(\mathfrak{g}, -B)$ is isomorphic to $\mathrm{O}(3)$.

(d) Show that the map $\mathrm{Ad} : x \mapsto \mathrm{Ad}(x)$ defines an ismorphism from $\mathrm{SO}(3)$ onto $\mathrm{O}(\mathfrak{g}, -B)_e$. Show that the latter group equals $\mathrm{Aut}\,(\mathfrak{g})$.

(e) Show that the map $a \mapsto \mathbb{A}\mathrm{d}(a)$ defines an isomorphism from $\mathrm{SO}(3)$ onto $\mathrm{Aut}\,(\mathrm{SO}(3))$.

Lemma 1.14 *The sets $N(x, U) = \{T \in \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}) \mid Tx \in U\}$, with $x \in \mathbb{P}(\mathcal{H})$ and $U \subset \mathbb{P}(\mathcal{H})$ open, form a subbasis for the topology of* $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$.

In other words, the topology of $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ coincides with the weakest topology that makes all maps $\varphi \mapsto \varphi x$, $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H})) \to \mathbb{P}(\mathcal{H})$, for $x \in \mathbb{P}(\mathcal{H})$, continuous.
Proof. Let $x = p(v)$, for $v \in \mathcal{H} \setminus \{0\}$. Then $N(x, U)$ is the image under q of the set of $A \in \mathrm{U}^1(\mathcal{H})$ with $Av \in p^{-1}(U)$. The action map $\mathrm{U}^1(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ is continuous, hence $N(x, U)$ is open. Thus, the quotient topology on $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ is finer then the one defined by the given subbasis.

The second part of the original proof of this lemma was wrong. I thank Olaf Schnürer for making me aware of this. We finish this section with a correct proof, obtaining a few useful lemmas along the way.

Let $S(\mathcal{H})$ denote the unit sphere $\{v \in \mathcal{H} \mid \ \|v\| = 1\}$ in the Hilbert space $\mathcal{H}$. The following lemma basically asserts that $p : S(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ is a topological $\mathbb{T}$-bundle.

Lemma 1.15 *Let $x_0 \in \mathbb{P}(\mathcal{H})$. Then there exists an open neighborhood U of x_0 and a continuous map $\sigma : U \to S(\mathcal{H})$ such that $p \circ \sigma = I$ on U.*

Proof. Fix $e \in S(\mathcal{H})$ such that $[e] = x_0$. Let $U = U_e = \mathbb{P}(\mathcal{H}) \setminus \mathbb{P}(e^\perp)$ and let $\chi : U \to e^\perp$ be the affine chart at e. Define

$$\sigma(x) = \frac{e + \chi(x)}{\sqrt{1 + \|\chi(x)\|^2}}, \qquad (x \in U).$$

Then $\sigma : U \to S(\mathcal{H})$ is continuous, and $p\sigma(x) = [\sigma(x)] = [e + \chi(x)] = x$ for all $x \in U$. □

Lemma 1.16 *Let M be a Hausdorff topological space, and let $\varphi : M \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ be a map such that for all $x \in \mathbb{P}(\mathcal{H})$ the map $\varphi_x : m \mapsto \varphi(m)x$, $M \to \mathcal{H}$, is continuous. Then for every $m_0 \in M$ there exists an open neighborhood $\mathcal{O}$ of m_0 in M and a continuous map $\Phi : \mathcal{O} \to \mathrm{U}^1(\mathcal{H})$ such that $q \circ \Phi = \varphi$ on $\mathcal{O}$.*

Proof. Let $m_0 \in M$. Fix $e \in S(\mathcal{H})$. According to Lemma 1.15, applied to $x = \varphi(m_0)[e]$, there exists an open neighborhood U of $\varphi(m_0)[e]$ and a continuous section $\sigma : U \to S(\mathcal{H})$. By continuity of the map $m \mapsto \varphi(m)[e]$, there exists an open neighborhood $\mathcal{O}$ of m_0 in M such that $\varphi(\mathcal{O})[e] \subset U$.

The map $\varepsilon : \mathcal{O} \to S(\mathcal{H})$, $m \to \sigma(\varphi(m)[e])$, is continuous, and satisfies $p(\varepsilon(m)) = \varphi(m)[e]$. It follows from Wigner's theorem that there exists a unique map $\Phi : \mathcal{O} \to \mathrm{U}^1(\mathcal{H})$ with $\Phi(m)e = \varepsilon(m)$, $(m \in \mathcal{O})$, that lifts φ, i.e., the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{U}^1(\mathcal{H}) \\ {\scriptstyle \Phi} & \nearrow & \downarrow q \\ \mathcal{O} & \xrightarrow{\varphi} & \mathrm{Aut}\,(\mathbb{P}(\mathcal{H})). \end{array}$$

We will show that the map $\Phi : \mathcal{O} \to \mathrm{U}^1(\mathcal{H})$ is continuous on $\mathcal{O}$. For this it suffices to show that for every $w \in \mathcal{H}$ the map $\Phi_w : m \mapsto \Phi(m)w$ is continuous $\mathcal{O} \to \mathcal{H}$. We will first do this for $w = v \in e^\perp$. Then, as in the proof of Wigner's theorem, we have that

$$\varphi(m)[e+v] = [\varepsilon(m) + \Phi(m)v]. \tag{1.4}$$

Let $m_1 \in \mathcal{O}$. Then from Lemma 1.15 it follows that there exists an open neighborhood U_1 of $\varphi(m_1)[e+v]$ and a continuous section $\sigma_1 : U_1 \to S(\mathcal{H})$. There exists an open neighborhood $\mathcal{O}_1 \subset \mathcal{O}$ of m_1 in M such that $\varphi(\mathcal{O}_1)[e+v] \subset U_1$. It follows from (1.4) that $[\varepsilon(m) + \Phi(m)v] = p(a(m))$, with $a(m) = \sigma_1(\varphi(m)[e+v])$. From this in turn we see that

$$\Phi(m)v = \frac{a(m) - \langle a(m)\,,\, \varepsilon(m)\rangle \varepsilon(m)}{\langle a(m)\,,\, \varepsilon(m)\rangle},$$

for $m \in U_1$. It follows that Φ_v is continuous in a neighborhood of m_1. We have shown that Φ_v is continuous on $\mathcal{O}$, for every $v \in e^\perp$. For each $m \in \mathcal{O}$ there exists a unique $f_m \in \{1, \tau\}$, such that the map $\Phi(m)$ is f_m-linear. From

$$\Phi(m)(\lambda v) = f_m(\lambda)\Phi(m)v,$$

for $v \in e^\perp$ and $\lambda \in \mathbb{C}$, we infer that the map $m \mapsto f_m$ is continuous. Let now $w = \lambda e + v$ with $\lambda \in \mathbb{C}$, $v \in E^\perp$. Then it follows that

$$\Phi(m)w = f_m(\lambda)\varepsilon(m) + \Phi(m)v,$$

for $m \in \mathcal{O}$. We conclude that Φ_w is continuous on $\mathcal{O}$. □

We can now complete the proof of Lemma 1.14. Let M be the set $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$, equipped with the topology induced by the subbasis of sets $N(x, U)$. This topology is the weakest that makes all maps $T \mapsto Tx$, $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H})) \to \mathbb{P}(\mathcal{H})$, for $x \in \mathbb{P}(\mathcal{H})$, continuous. The identity map $\varphi : M \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ satisfies the hypothesis of the previous lemma. Let $T_0 \in M$. Then there exists an open neighborhood $\mathcal{O}$ of T_0 in M and a continuous lifting $\Phi : \mathcal{O} \to \mathrm{U}^1(\mathcal{H})$ of φ. From $\varphi = q \circ \Phi$ it follows that $\varphi : \mathcal{O} \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ is continuous for the quotient topology on the second space. We conclude that $\varphi : M \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ is continuous. Its inverse is continuous by the first part of the proof of Lemma 1.14. Hence, φ is a homeomorphism.

In the above we have in fact also proved the following result.

Lemma 1.17 *The map $q : \mathrm{U}^1(\mathcal{H}) \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ defines a topological principal fiber bundle with structure group $\mathbb{T}$.*

2 Elementary representation theory

In this section we discuss some of the basic notions of the representation theory of a Lie group G. We shall define representations in complete locally convex spaces (always assumed to be defined over $\mathbb{C}$, and to be Hausdorff). This class of topological linear spaces is sufficiently general as it contains Banach, hence Hilbert spaces, but also natural spaces such as $C(M)$ and $C_c(M)$, for M a locally compact Hausdorff space, and $C^\infty(M)$ and $C_c^\infty(M)$, for M a smooth manifold. Moreover, the class of locally convex spaces allows taking continuous linear duals, so that the natural spaces of Radon measures and distributions belong to this class.

More generally, if M is a smooth manifold, and $\mathcal{V}$ a Banach bundle over M, then the spaces $\Gamma^0(M, \mathcal{V})$ and $\Gamma_c^0(M, \mathcal{V})$ of continuous and compactly supported continuous sections are examples of such spaces, as well as their continuous linear duals. Moreover, the spaces $\Gamma^\infty(M, \mathcal{V})$ and $\Gamma_c^\infty(\mathcal{V})$ of smooth and compactly supported smooth functions are examples of such spaces, and so are their continuous linear duals.

We recall that a seminorm on a linear space V is a map $\nu : V \to \mathbb{C}$ with $\nu(\lambda x) = |\lambda|\nu(x)$ and $\nu(x + y) \leq \nu(x) + \nu(y)$, for all $\lambda \in \mathbb{C}$ and $x, y \in V$. We also recall that local convexity of the topology on a topological linear space V may be characterized by the requirement that a basis for the topology is given by the collection of open sets $B_\nu(a; r) := \{x \in V \mid \nu(x-a) < \varepsilon\}$ with ν ranging over the continuous seminorms on V, $a \in V$ and $\varepsilon > 0$.

The class of complete locally convex spaces is sufficiently restricted to allow integration in the following sense. If V a complete locally convex space and $f : \mathbb{R}^n \to V$ a compactly supported continuous function, then $\int f(x)\, dx$ has a well defined value in V as a limit of Riemann sums. Here the convexity and the completeness play a crucial role.

If V is a complete locally convex space, then by $\mathrm{Aut}\,(V)$ we denote the group of topological linear automorphisms of V.

Definition 2.1 A *continuous representation* of G in V is defined to be a group homorphism $\pi : G \to \mathrm{Aut}\,(V)$ such that the action map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous.

Example 2.2 Let G be a Lie group, and let $C^\infty(G)$ be the space of smooth functions on G, equipped with the locally convex topology of uniform convergence of all partial derivatives on compact subsets. The natural action of G on itself by left translation induces the left regular representation L of G in $C^\infty(G)$, defined by the formula $L_x\varphi(g) = \varphi(x^{-1}g)$, for $\varphi \in C^\infty(G)$ and $g, x \in G$. With a bit of effort it can be shown that this representation is continuous.

If (π_1, V_1) and (π_2, V_2) are continuous representations of G in complete locally convex spaces, then a continuous linear map $T : V_1 \to V_2$ is said to be *intertwining,* or G-*equivariant,* if $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$. The space of such maps is denoted by $\mathrm{Hom}_G(V_1, V_2)$. The representations π_1 and π_2 are said to be equivalent, notation

$$\pi_1 \simeq \pi_2,$$

if there exists a G-equivariant topological linear isomorphism from V_1 onto V_2.

Let (π, V) be a representation of the above type. A linear subspace $W \subset V$ is said to be G-*invariant* if $\pi(g)W \subset W$ for all $g \in G$. The representation π is said to be *irreducible* if 0 and V are the only *closed* invariant subspaces. If W is a closed invariant subspace, then W is a complete locally convex space for the restriction topology and the representation $\pi|_W$ of G in W defined by $\pi|_W(g) = \pi(g)|_W$ is continuous.

If (π_1, V_1) and (π_2, W_2) are continuous representations of G in complete locally convex spaces, then the direct sum $\pi_1 \oplus \pi_2$ is the representation of G in $V_1 \oplus V_2$ defined by $(\pi_1 \oplus \pi_2)(g) = \pi_1(g) \oplus \pi_2(g)$.

In the following we assume $\mathcal{H}$ to be a complex Hilbert space.

Definition 2.3 A *unitary* representation of G in $\mathcal{H}$ is a continuous representation π of G in $\mathcal{H}$ such that $\pi(g) : \mathcal{H} \to \mathcal{H}$ is a unitary map for every $g \in G$.

Two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are said to be *unitarily equivalent* if there exists a G-equivariant unitary isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$.

In view of the applications to quantum mechanics, the unitary representations will be of particular interest to us. The following criterion is often used to decide whether a given representation is unitary.

Lemma 2.4 *Let π be a representation of G in a Hilbert space $\mathcal{H}$ with the following properties.*

(a) *The map $\pi(g)$ is unitary for every $g \in G$.*

(b) *There exists a dense subset $V \subset \mathcal{H}$ such that $\lim_{g \to e} \pi(g)v = v$ for every $v \in V$.*

Then π is a unitary representation of G in $\mathcal{H}$.

Proof. It suffices to prove that the action map $G \times \mathcal{H} \to \mathcal{H}$ is continuous. Let $g_0 \in G$ and $v_0 \in \mathcal{H}$. Then using (a) we infer that

$$\begin{aligned} \|\pi(g)v - \pi(g_0)v_0\| &\leq \|\pi(g)v - \pi(g)v_0\| + \|\pi(g)v_0 - \pi(g_0)v_0\| \\ &= \|v - v_0\| + \|\pi(g_0^{-1}g)v_0 - v_0\|, \end{aligned}$$

from which we see that it suffices to show that $\lim_{g \to e} \pi(g)v_0 = v_0$. We will do this by using (b). Let $\varepsilon > 0$. Then by density of V in $\mathcal{H}$, there exists a $v \in V$ such that $\|v_0 - v\| < \varepsilon/3$. Moreover, by (b) there exists a neighborhood U of e in G such that $\|\pi(g)v - v\| < \varepsilon/3$, for all $g \in U$. We now obtain, again by using (a), that

$$\|\pi(g)v_0 - v_0\| \leq \|\pi(g)v_0 - \pi(g)v\| + \|\pi(g)v - v\| + \|v - v_0\| < \varepsilon$$

for all $g \in U$. □

Remark 2.5 In particular it follows from the above result that a group homomorphism $\pi : G \to \mathrm{U}(\mathcal{H})$ defines a continuous representation if and only if it is continuous for the strong operator topology on $\mathrm{U}(\mathcal{H})$.

Example 2.6 Let dx be a choice of left invariant measure on G and let $L^2(G)$ be the associated space of square integrable functions. We define the left regular representation L of G in $L^2(G)$ by the same formula as in Example 2.2. Then L_g is a unitary map for every $g \in G$. Moreover, if $\varphi \in C_c(G)$, then $L_g\varphi \to \varphi$, uniformly with supports in a fixed compact set, as $g \to e$. Hence $L_g\varphi \to \varphi$ in $L^2(G)$. Applying the above lemma, we see that $(L, L^2(G))$ is a unitary representation.

3 Projective representations

We retain the notation of Section 1. Suppose that a quantum system is given with natural symmetries described by the action of a locally compact topological group G (all topological groups will be assumed to be Hausdorff from now on). Then one would expect G to act by automorphisms on the state space $\mathbb{P}(\mathcal{H})$ of the system. Thus, we are naturally led to the notion of a projective representation.

Definition 3.1 Let G be a topological group. A *projective representation* of G in $\mathcal{H}$ is defined to be a group homomorphism $\rho : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ such that the action map $(g, x) \mapsto \rho(g)x$, $G \times \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ is continuous.

Two projective representations π_1 and π_2 of G in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, are said to be *equivalent* if there exists a isomorphism $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ such that $\pi_2(g) \circ T = T \circ \pi_1(g)$ for all $g \in G$.

Let ρ be a projective representation of G as in the above definition, and let $x \in \mathbb{P}(\mathcal{H})$ and $O \subset \mathbb{P}(\mathcal{H})$ an open subset. Then by continuity of the action map, $\rho^{-1}(N(x, O))$ is open in G. It follows that $\rho : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ is a continuous group homomorphism.

If $\pi : G \to \mathrm{U}^1(\mathcal{H})$ is a (continuous) unitary representation, then clearly π induces the projective representation $q \circ \pi : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ of G. Wigner's theorem suggests the natural question when a continuous projective representation of G comes from a unitary one in the above fashion. Such a projective representation is said to *lift* to a unitary representation. We will first discuss a motivating example. After that, in the rest of this section, we will give a partial answer to the general question.

We will consider a particular projective representation $\pi : \mathrm{SO}(3) \to \mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2))_e$. Consider the following Hermitian 2×2 matrices, all with eigenvalues ± 1,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

these are the well-known Pauli spin matrices. The anti-Hermitian matrices $S_j := -i\sigma_j$ form an ordered basis of the Lie algebra $su(2)$, viewed as a real linear space. Their commutation relations are given by

$$[S_i, S_j] = 2\,\varepsilon_{ijk}\,S_k,$$

where ε_{ijk} is determined by $e_i \wedge e_j \wedge e_k = \varepsilon_{ijk}\, e_1 \wedge e_2 \wedge e_3$. The matrix of $\mathrm{ad} S_j$ with respect to this basis equals $2R_j$, where R_j is the matrix of $x \mapsto e_j \times x$, the infinitesimal rotation in $\mathbb{R}^3$ about the x_j-axis. It is thus seen that $X \mapsto \mathrm{mat}\,\mathrm{ad}X$ is a Lie algebra isomorphism from $su(2)$ onto $\mathfrak{so}(3)$. The map $\varphi : x \mapsto \mathrm{mat}\,\mathrm{Ad}(x)$ is the associated Lie group homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. We recall that φ is surjective, with kernel $\ker\varphi = \{-I, I\}$.

On the other hand, the inclusion of $\mathrm{SU}(2)$ into $\mathrm{U}(2)$ composed with the projection $q : \mathrm{U}(2) \to \mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2))_e$ defines a Lie group homomorphism $\psi : \mathrm{SU}(2) \to \mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2))_e$ which is surjective and has kernel $\{I, -I\}$. Thus, ψ is the projective representation of $\mathrm{SU}(2)$ which has the natural representation as a lift. For $\pi : \mathrm{SO}(3) \to \mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2))_e$ we take the unique homomorphism with $\pi \circ \varphi = \psi$, i.e., it makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{SU}(2) & \hookrightarrow & \mathrm{U}(2) \\ \varphi \downarrow & & \downarrow q \\ \mathrm{SO}(3) & \xrightarrow{\pi} & \mathrm{Aut}\,(\mathbb{P}(\mathbb{C}^2)). \end{array}$$

Exercise 3.2 Show that the projective representation π of SO(3) in $\mathbb{C}^2$ does not lift to a unitary representation. Show that every projective representation of the circle group $\mathbb{T}$ in $\mathbb{C}^2$ lifts to a unitary representation. Give an example of such a representation for which the lifting cannot map into SU(2).

The above representation π is essentially the only non-trivial projective representation of SO(3) in $\mathbb{C}^2$. Indeed, if ρ is a second such projective representation, then $\rho \circ \pi^{-1}$ is a non-trivial homomorphism of $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))_e$ into itself. By Exercise 1.13 it is an automorphism given by conjugation by an element $T \in \mathrm{Aut}\,(\mathbb{P}(\mathcal{H})_e)$. By Wigner's theorem, T is induced by an element of SU(2). Thus, we see that with respect to a suitable basis of $\mathbb{C}^2$, the projective representation ρ takes the same form as π.

We now turn to an application to quantum physics. Consider an electron in Euclidean three space $\mathbb{R}^3$. Let us assume that the state space associated with its spin is $\mathbb{P}(\mathcal{H})$, with $\mathcal{H}$ a finite dimensional Hilbert space. From experiment (bending in a magnetic field) we know that to every a in the unit sphere $S^2 \subset \mathbb{R}^3$ an observable σ_a is associated, called spin in the direction of a. This observable belongs to the space $\mathcal{S}$ of Hermitian operators of $\mathcal{H}$. The observable has two values, $+1$ or -1, corresponding to 'spin up or down' in the direction of a. The corresponding eigenspaces of σ_a in $\mathcal{H}$ are denoted by E_a^+ and E_a^- and the corresponding projections onto them by P_a^+ and P_a^-, respectively. Thus, $E_a^+ \perp E_a^-$ and

$$\sigma_a = P_a^+ - P_a^-. \tag{3.1}$$

The canonical projection $\mathrm{U}(\mathcal{H}) \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))_e$ has kernel $\mathbb{T}$, which commutes with all elements of $\mathrm{End}(\mathcal{H})$. Thus, $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))_e$ is a Lie group. Moreover, the linear action by conjugation of $\mathrm{U}^1(\mathcal{H})$ on $\mathcal{S}$ factors to a linear action of $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ on $\mathcal{S}$. We shall denote this action by $(A, S) \mapsto ASA^{-1}$.

As the configuration space of the system is invariant under rotation, there should exist a non-trivial projective representation $\pi : \mathrm{SO}(3) \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))_e$ such that

$$\sigma_{ga} = \pi(g)\sigma_a\pi(g)^{-1}, \qquad (g \in \mathrm{SO}(3)). \tag{3.2}$$

From this combined with (3.1) it follows in turn that $gE_a^{\pm} = E_{ga}^{\pm}$ (despite abuse of language, the meaning of this identity should be clear). As SO(3) acts transitively on S^2, it follows that all E_α^+ have the same dimension. Moreover, again by symmetry of the situation, $\sigma_{-a} = -\alpha_a$ so that $E_a^+ = E_{-a}^-$. Since spin never manifests itself through more than two values, it is reasonable to expect that $\dim E_a^+ = \dim E_a^- = 1$ for all $a \in S^2$. This implies that the space $\mathcal{H}$ is two-dimensional.

From what we proved before it follows that there exists essentially one non-trivial projective representation of $\mathcal{H}$. In particular, we may select an orthonormal basis of $\mathcal{H}$ with respect to which π attains the form introduced in the text preceding Exercise 3.2. From now on we shall work with this form. The natural action of SU(2) on $\mathbb{C}^2$ induces a transitive action of SU(2) on $\mathbb{P}(\mathbb{C}^2)$. Let T be the one-parameter subgroup of SU(2) generated by S_3, then T is the stabilizer of $[(1,0)]$, so that $\mathbb{P}(\mathbb{C}^2) \simeq \mathrm{SU}(2)/T$. The image of T in SO(3) under φ is the one-parameter subgroup R with infinitesimal generator $S_3 = -i\sigma_3$, which is also the isotropy subgroup of e_3 for the natural action of SO(3) on S^2. It follows that the natural submersion $g \mapsto \pi(g)[(1,0)]$ from SO(3) onto $\mathbb{P}(\mathbb{C}^2)$ factors to a covering $\tau : S^2 \simeq \mathrm{SO}(3)/R \to \mathbb{P}(\mathbb{C}^2)$ which is SO(3)-intertwining. Since $\mathbb{P}(\mathbb{C}^2)$ is diffeomorphic to a 2-sphere, it is simply connected, so that τ must be a diffeomorphism. For $a \in S^2$ we put $E_a := \tau(a)$, viewed as a one-dimensional subspace of $\mathbb{C}^2$. Since τ is intertwining, the associated Hermitian operators $\sigma_a = P_{E_a} - P_{E_a^\perp}$ satisfy the rule (3.2).

Exercise 3.3 Show that the matrix of σ_{e_j} is precisely the j-th Pauli spin matrix σ_j, for each $j = 1, 2, 3$.

We now return to the general question of lifting projective representations. Assume that G and B are topological groups. An extension of G by B is defined to be a short exact sequence of group homomorphisms

$$1 \to B \xrightarrow{j} \tilde{G} \xrightarrow{\varphi} G \to 1 \tag{3.3}$$

with φ continuous and with j a homeomorphism onto the closed normal subgroup $\ker\varphi$ of $\tilde{G}$. If $j(B)$ is central in $\tilde{G}$, the extension is called central. The above central extension is called trivial if the exact sequence splits, i.e., there exists a continuous homomorphism $\psi : G \to \tilde{G}$ such that $\varphi \circ \psi = I_G$. It is readily seen that this implies that the map $\Psi : B \times G \to \tilde{G}, (b, g) \mapsto j(b)\psi(g)$ is an isomorphism of topological groups. In particular, if the central extension is trivial, then $\widetilde{G}$ is isomorphic to the direct product of B and G.

Let now $\rho : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ be a continuous projective representation of G. We define the topological group $\rho^*(\mathrm{U}^1(\mathcal{H}))$ by pulling back the quotient map $\mathrm{U}^1(\mathcal{H}) \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ by ρ. Thus, $\rho^*(\mathrm{U}^1(\mathcal{H}))$ is defined as the fibered product

$$\rho^*(\mathrm{U}^1(\mathcal{H})) = G \times_\rho \mathrm{U}^1(\mathcal{H}) := \{(g, A) \in G \times \mathrm{U}^1(\mathcal{H}) \mid \rho(g) = q(A)\},$$

which is readily seen to be a closed subgroup of $G \times \mathrm{U}^1(\mathcal{H})$ (use that ρ is continuous). Clearly projection onto the first factor induces a surjective continuous group homomorphism $\rho^*(\mathrm{U}^1(\mathcal{H})) \to G$, with kernel equal to the natural image of $\mathbb{T}$. Since $\mathbb{T}$ is compact, the continuous embedding $\mathbb{T} \to \rho^*(\mathrm{U}^1(\mathcal{H}))$ is a closed map, hence a homeomorphism onto a closed normal subgroup of $\widetilde{G}$. It follows that

$$1 \to \mathbb{T} \to \rho^*(\mathrm{U}^1(\mathcal{H})) \to G \to 1 \tag{3.4}$$

is a central extension of G by the circle group $\mathbb{T}$. We denote the associated monomorphism by j and the associated epimorphism by φ.

Lemma 3.4 *The natural epimorphism $\varphi : \rho^*(\mathrm{U}^1(\mathcal{H})) \to G$ is open, hence induces an isomorphism $\rho^*(\mathrm{U}^1(\mathcal{H}))/\mathbb{T} \simeq G$.*

Proof. Let $\mathcal{O}$ be an open subset of $\rho^*(\mathrm{U}^1(\mathcal{H}))$ and let $(g_0, A_0) \in \mathcal{O}$. Then there exist open neighborhoods Ω of g_0 in G and O of A_0 in $\mathrm{U}^1(\mathcal{H})$ such that $\rho^*(\mathrm{U}^1(\mathcal{H})) \cap \Omega \times O \subset \mathcal{O}$. Since $q : \mathrm{U}^1(\mathcal{H}) \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ is an open map, $q(O)$ is open in $\mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ and contains $q(A_0)$. By continuity of ρ there exists an open neighborhood Ω_0 of g_0 in Ω such that $\rho(\Omega_0) \subset p(O)$. Put $\mathcal{O}_0 = \rho^*(\mathrm{U}^1(\mathcal{H})) \cap \Omega_0 \times O$. Then $(g_0, A_0) \in \mathcal{O}_0 \subset \mathcal{O}$. Moreover, if $g \in \Omega_0$, then $\rho(g) \in p(O)$ hence $\rho(g) = p(A)$ for some $A \in O$. It follows that $(g, A) \in \mathcal{O}_0$. Hence Ω_0 is contained in $\varphi(\mathcal{O}_0)$. It follows that every point of the subset $\varphi(\mathcal{O})$ of G is interior. □

Proposition 3.5 *The following two conditions are equivalent.*

(a) *The projective representation $\rho : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ admits a lifting $\pi : G \to \mathrm{U}^1(\mathcal{H})$.*

(b) *The central extension (3.4) is trivial.*

Proof. First assume that ρ admits a lifting $\pi : G \to \mathrm{U}^1(\mathcal{H})$. Then the map $\psi = (I, \pi)$ defines a continuous group homomorphism $G \to \rho^*(\mathrm{U}^1(\mathcal{H}))$, which splits the exact sequence (3.4). Hence the central extension is trivial.

Conversely, assume that the central extension is trivial, and let $\psi : G \to \rho^*(\mathrm{U}^1(\mathcal{H}))$ be a splitting for the sequence (3.4). Let $p_2 : \rho^*(\mathrm{U}^1(\mathcal{H})) \to \mathrm{U}^1(\mathcal{H})$ be the map induced by the projection onto the second coordinate. Then $\pi := p_2 \circ \psi : G \to \mathrm{U}^1(\mathcal{H})$ is a continuous group homomorphism. It now follows from Lemma 2.4 that π is a continuous unitary representation. Clearly, π defines a lifting for ρ.

Theorem 3.6 *If G is a locally compact group, then the group $\rho^*(\mathrm{U}^1(\mathcal{H}))$ is locally compact as well. If G is a Lie group, then $\rho^*(\mathrm{U}^1(\mathcal{H}))$ has a compatible structure of Lie group.*

Proof. If H is a topological group and $K \subset H$ is a closed normal subgroup, then K and H/K locally compact implies that H is locally compact. Thus, the first assertion follows by application of Lemma 3.4.

For the proof of the second assertion we use part of the solution to Hilbert's fifth problem, due to Gleason, Montgomery, Zippin, as formulated below.

Theorem 3.7 *Let H be a locally compact separable topological group. Then H has a compatible structure of Lie group if and only if there exists an open neighborhood of the identity element that contains no non-trivial subgroup of H.*

Proof. If H has a compatible structure of Lie group, let $\mathfrak{h}$ be the associated Lie algebra and $\exp : \mathfrak{h} \to H$ the exponentional map. Let Ω be a bounded open neighborhood of 0 in $\mathfrak{h}$ such that $\exp|_\Omega$ is a diffeomorphism onto an open subset of e. Let $\Omega_0 = \frac{1}{2}\Omega$. We claim that $\exp(\Omega_0)$ contains no non-trivial subgroup. For indeed, let K be such a non-trival subgroup. Let $k \in K \setminus \{0\}$, then $k = \exp X$ for a unique $X \in \Omega_0$. There exists a maximal $n_X \in \mathbb{N}$ such that $X_n := 2^n X \in \Omega_0$. Now $2X_n \in \Omega \setminus \Omega_0$ and since exp is injective on Ω it follows that $k^{2n+1} = \exp(2X_n) \notin \exp(\Omega_0)$. It follows that k^{2n+1} is an element of K not in $\exp(\Omega_0)$, contradiction. This establishes the necessity of the condition for H to have a compatible structure of Lie group.

The sufficiency of the condition is a deep result, see [17] for details. □

Completion of the proof of Theorem 3.6. The group $\mathbb{T}$ is a Lie group hence has a neighborhood O_1 of 1 that contains no non-trivial subgroup. Since the embedding $j : \mathbb{T} \to \rho^*(\mathrm{U}^1(\mathcal{H}))$ is a homeomorphism onto a compact subgroup, there exists an open neighborhood O_2 of e in $\rho^*(\mathrm{U}^1(\mathcal{H}))$ such that $j^{-1}(O_2) = O_1$. Assume that G is a Lie group. Then there exists an open neighborhood O_3 of e that contains no non-trivial subgroup. Let $O = O_2 \cap \varphi^{-1}(O_3)$. If H is a subgroup of $\rho^*(\mathrm{U}^1(\mathcal{H}))$ contained in O, then $\varphi(H)$ is a subgroup of G contained in O_3, hence trivial. It follows that $H \subset \ker\varphi = \mathrm{im}\, j$, hence $j^{-1}(H)$ is a subgroup of $\mathbb{T}$ contained in O_1. It follows that $j^{-1}(H)$ is trivial. Hence, H is trivial as well. □

In the following we assume that G and B are Lie groups, and that $\tilde{G}$ is a Lie group extension of G by B. Thus, we have an associated exact sequence of Lie group homomorphisms (3.3). Taking derivatives of the homomorphisms in the sequence at the identity elements of the groups involved, we obtain a short exact sequence of Lie algebras

$$0 \to \mathfrak{b} \xrightarrow{j_*} \widetilde{\mathfrak{g}} \xrightarrow{\varphi_*} \mathfrak{g} \to 0. \tag{3.5}$$

This sequence is said to split if there exists a Lie algebra homomorphism $\eta : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ such that $\varphi \circ \eta = I_{\mathfrak{g}}$. In this case the extension is called trivial and the splitting induces an isomorphism of $\widetilde{\mathfrak{g}}$ with the direct sum Lie algebra $\mathfrak{b} \oplus \mathfrak{g}$.

We now have the following result.

Lemma 3.8 *Let (3.3) be a short exact sequence of Lie groups, with G a simply connected group. Then (3.3) splits if and only if the associated short exact sequence (3.5) of Lie algebras splits.*

Proof. If the sequence of groups splits, let $\psi : G \to \tilde{G}$ define a splitting. Then $\psi_* = T_e\psi$ defines a splitting of the sequence of Lie algebras.

Conversely, assume that the sequence (3.5) splits, and let $\eta : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ be a splitting homomorphism. Since G is simply connected, there exists a unique Lie group homomorphism $G \to \widetilde{G}$ with derivative $\psi_* = \eta$. Since G is connected, $\psi_* \circ \varphi_* = I_{\mathfrak{g}}$ implies that $\psi \circ \varphi = I_G$. □

In the proof of the above lemma we have used the following well known and important result, for which we refer to [8]. [1]

Proposition 3.9 *Let G, H be Lie groups, and assume that G is simply connected. Then for every homomorphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$ there exists a unique homomorphism $F : G \to H$ of Lie groups, such that $f = T_eF$.*

One of the consequences of this result for a simply connected group Lie group G is the following. Any representation of its Lie algebra $\mathfrak{g}$ in a finite dimensional space V lifts to a continuous (hence smooth) representation of G in V.

In view of Lemma 3.8 it is a good idea to investigate the possible central extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{b}$. Two central extensions $(\widetilde{\mathfrak{g}}, j, \varphi)$ and $(\widetilde{\mathfrak{g}}', j', \varphi')$ are called equivalent if and only if there exists a Lie algebra isomorphism $\alpha : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}'$ such that $\alpha \circ j = j'$ and $\varphi' \circ \alpha = \varphi$. The set of equivalence classes of such central extension can be related to the second Lie algebra cohomology $H^2(\mathfrak{g}, \mathfrak{b})$, where $\mathfrak{b}$ is viewed as a trivial $\mathfrak{g}$ module.

Let $\wedge^k\mathfrak{g}^*$ denote the k-th exterior power of $\mathfrak{g}^*$. In the usual way we identify $\wedge^k\mathfrak{g}^* \otimes \mathfrak{b}$ with the space of alternating real multilinear maps $\mathfrak{g}^k \to \mathfrak{b}$. We define the map $d_k : \wedge^k\mathfrak{g}^* \otimes \mathfrak{b} \to \wedge^{k+1}\mathfrak{g}^* \otimes \mathfrak{b}$ by

$$d_k\omega(X_0, X_1, \ldots, X_k) = \sum_{i<j}^{k} (-1)^{i+j+1}\omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),$$

for $k \geq 0$ and by $d_k = 0$ for $k < 0$. It is readily checked that $d_{k+1}d_k = 0$ for all $k \geq 0$ (this involves the Jacobi identity). The k-th cohomology group $H^k(\mathfrak{g}, \mathfrak{b})$ is defined to be the real vector space

$$H^k(\mathfrak{g}, \mathfrak{b}) \simeq \ker d_k / \operatorname{im} d_{k-1}.$$

Note that $H^k(\mathfrak{g}, \mathfrak{b}) \simeq H^k(\mathfrak{g}, \mathbb{R}) \otimes \mathfrak{b}$.

Let now $(\tilde{\mathfrak{g}}, j, \varphi)$ be a central extension of $\mathfrak{g}$ by $\mathfrak{b}$. Let $\xi : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ be any linear map such that $\psi \circ \xi = I_{\mathfrak{g}}$. If $X, Y \in \mathfrak{g}$, then $[\xi(X), \xi(Y)] - \xi([X, Y])$ has image 0 in $\mathfrak{g}$, hence belongs to

[1] If you like, we could also discuss the result in one of the practice sessions

ker$\varphi = j(\mathfrak{b})$. It follows that there exists a unique alternating bilinear map $\omega_\xi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{b}$ such that

$$j(\omega_\xi(X, Y)) = [\xi(X), \xi(Y)] - \xi([X, Y]),$$

for all $X, Y \in \mathfrak{g}$. Clearly, ω_ξ is trivial if and only if ξ is a Lie algebra homomorphism. By application of the Jacobi identity it follows that $d\omega_\xi = 0$. Thus, ω_ξ defines a class in $H^2(\mathfrak{g}, \mathfrak{b})$. Let now $\eta : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ be a second linear map such that $\varphi \circ \eta = I_{\mathfrak{g}}$. Then it follows that $\lambda := \xi - \eta$ maps $\mathfrak{g}$ into $j(\mathfrak{b})$, so that $[\lambda(X), \lambda(Y)] = 0$, and it follows that $[\xi(X), \xi(Y)] = [\eta(X), \eta(Y)]$ for all $X, Y \in \mathfrak{g}$. Hence, $j((\omega_\xi - \omega_\eta)(X, Y)) = -\lambda([X, Y])$ for all $X, Y \in \mathfrak{g}$, from which we read off that $\omega_\eta - \omega_\xi \in \operatorname{im} d_1$. It follows that the class of ω_ξ in $H^2(\mathfrak{g}, \mathfrak{b})$ is independent of the particular choice of ξ. We therefore denote this class by $\omega(\tilde{\mathfrak{g}}, j, \varphi)$. It is readily checked that the class $\omega(\tilde{\mathfrak{g}}, j, \varphi)$ depends on the central extension through its equivalence class.

Proposition 3.10 *The map $(\tilde{\mathfrak{g}}, j, \varphi) \mapsto \omega(\tilde{\mathfrak{g}}, j, \varphi)$ defines an bijection from the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{b}$ onto $H^2(\mathfrak{g}, \mathfrak{b})$.*

Proof. We already showed that the map factors to a map ω from the set $\mathcal{C}$ of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{b}$ to $H^2(\mathfrak{g}, \mathfrak{b})$. Let λ be a closed form in $\wedge^2\mathfrak{g} \otimes \mathfrak{b}$. As a linear space we define $\widetilde{\mathfrak{g}}_\lambda = \mathfrak{b} \oplus \mathfrak{g}$. Moreover, we define j_λ to be the embedding of $\mathfrak{b}$ on the first factor, and we define φ_λ to be the projection onto the second factor. We define the antisymmetric bilinear map $[\,\cdot\,,\,\cdot\,] : \tilde{\mathfrak{g}}_\lambda \times \tilde{\mathfrak{g}}_\lambda \to \mathfrak{g}_\lambda$ by the requirements that it equals zero on $\mathfrak{b} \times \tilde{\mathfrak{g}}_\lambda$ and that for $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ it is given by $[X, Y]_\lambda = [X, Y] + \lambda(X, Y)$. Then it is not difficult to check that $[\,\cdot\,,\,\cdot\,]$ satisfies the Jacobi identities, hence defines a Lie algebra structure on $\tilde{\mathfrak{g}}_\lambda$. Clearly, $(\mathfrak{g}_\lambda, j_\lambda, \varphi_\lambda)$ is a central extension of $\mathfrak{g}$ by $\mathfrak{b}$. If $\mu \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{b}$ represents the same cohomology class as λ then $\mu - \lambda = d\xi$ for a linear map $\xi : \mathfrak{g} \to \mathfrak{b}$. It now follows that $(X, Y) \mapsto (X + \xi(Y), Y)$ defines an equivalence from the central extension $\widetilde{\mathfrak{g}}_\lambda$ onto the central extension $\widetilde{\mathfrak{g}}_\mu$. Accordingly, the map $\lambda \mapsto \widetilde{\mathfrak{g}}_\lambda$ induces a map from $H^2(\mathfrak{g}, \mathfrak{b})$ to the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{b}$. We leave it to the reader to verify that this map is a two-sided inverse of the map induced by ω. □

Corollary 3.11 *Let $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then every central extension of $\mathfrak{g}$ by $\mathfrak{b}$ is trivial.*

Proof. This follows immediately from the above proposition. However, it may be illustrative to repeat the proof in the present particular setting, since Corollary 3.11 is all we will need in the sequel. Let $(\tilde{\mathfrak{g}}, j, \varphi)$ be a central extension. Select a linear map $\xi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\varphi \circ \xi = I_{\mathfrak{g}}$. Define $\omega_\xi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{b}$ as above. Then $d\omega_\xi = 0$ hence, $\omega = d\lambda$ for a linear map $\lambda : \mathfrak{g} \to \mathfrak{b}$. Then $\omega_\xi(X, Y) = \lambda([X, Y])$ for all $X, Y \in \mathfrak{g}$. It is now readily seen that $\psi := \xi + j \circ \lambda$ is a Lie algebra homomorphism $\mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\varphi \circ \psi = I_{\mathfrak{g}}$. It follows that the central extension is trivial. □

The following consequence will be of fundamental importance of our discussion of projective representations of the Poincaré group.

Corollary 3.12 *Let G be a simply connected Lie group whose Lie algebra $\mathfrak{g}$ satisfies $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then every projective representation $\rho : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ has a lifting to a continuous unitary representation $\pi : G \to \mathrm{U}(\mathcal{H})$.*

Proof. It follows from combining the above results that ρ has a lifting to a continuous representation $\pi : G \to \mathrm{U}^1(\mathcal{H})$. The group $\pi^{-1}(\mathrm{U}(\mathcal{H}))$ is open and closed in G, hence equals G. It follows that π maps G into $\mathrm{U}(\mathcal{H})$. □

Proposition 3.13 *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then $H^2(\mathfrak{g}, \mathbb{R}) = 0$.*

Proof. Since $\mathfrak{g}$ is semisimple, its Killing form B is non-degenerate. Let $\omega \in \wedge^2\mathfrak{g}^*$ be closed. Then there exists a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\omega(X, Y) = B(X, \varphi(Y))$ for all $X, Y \in \mathfrak{g}$. Since ω is closed, it follows that φ is a derivation. Since ad is a Lie algebra isomorphism from $\mathfrak{g}$ onto the Lie algebra of derivations of $\mathfrak{g}$, it follows that $\varphi = \mathrm{ad}(Z)$ for some $Z \in \mathfrak{g}$. It follows that $\omega(X, Y) = B(X, [Z, Y]) = B([X, Y], Z) = \lambda([X, Y]$, where $\lambda = B(\,\cdot\,, Z)$. We conclude that $\omega = d\lambda$ so that the cohomology class of ω is trivial. □

Corollary 3.14 *Let G be a simply connected real semisimple Lie group. Then every projective representation of G in a complex Hilbert space $\mathcal{H}$ lifts to a unitary representation of G in $\mathcal{H}$.*

Exercise 3.15 The group $\mathrm{PGL}(n, \mathbb{C})$ is defined to be the group of all diffeomorphisms of $\mathbb{P}(\mathbb{C}^n)$ that are induced by an element of $\mathrm{GL}(n, \mathbb{C})$. Let G be a simply connected Lie group with $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Show that every Lie group homomorphism $G \to \mathrm{PGL}(n, \mathbb{C})$ lifts to a Lie group homomorphism $G \to \mathrm{GL}(n, \mathbb{C})$.

Remarks. The final goal of this exposition is to classify the irreducible projective representations of the connected Poincaré group, $G = \mathrm{SO}_e(3, 1) \ltimes \mathbb{R}^4$. [2] We will show that its Lie algebra has zero second cohomology. However, we cannot immediately apply the material of the present section since the Poincaré group is not simply connected. This can be remedied by passing to the universal covering group of G.

In general, if G is a connected Lie group, then the universal covering $(\widetilde{G}, \tilde{e})$ of the pointed space (G, e) has a unique structure of Lie group that turns the covering $p : \widetilde{G} \to G$ into a Lie group homomorphism. The Lie group $\widetilde{G}$ is called the universal covering group of the Lie group G.

For the connected Poincaré group G, we will show that its universal covering group is given by $\widetilde{G} = \mathrm{SL}^2(2, \mathbb{C}) \ltimes \mathbb{R}^4$. If π is a projective representation of the connected Poincaré group, then $\pi \circ p$ is a projective representation of its universal covering group $\widetilde{G}$. In view of what we discussed above, it lifts to a unitary representation of $\widetilde{G}$.

4 Projection valued Borel measures

In this section we shall introduce the notion of a projection valued measure, and discuss some of its properties. We begin by recalling some elementary results from measure theory.

Let X be a set. We recall that a *σ-algebra* or σ-field on X is a collection $\mathcal{A}$ of subsets of X satisfying the following conditions

(a) $\emptyset$ belongs to $\mathcal{A}$;

(b) for every set $A \in \mathcal{A}$, the complement $A' = X \setminus A$ also belongs to $\mathcal{A}$;

(c) for every sequence (A_n) in $\mathcal{A}$ the union $\cup_n A_n$ belongs to $\mathcal{A}$.

We observe that from (a) and (b) it follows that $X \in \mathcal{A}$. Moreover, from (b) and (c) it follows that every countable intersection of sets from $\mathcal{A}$ belongs to $\mathcal{A}$ again. In particular, finite unions and intersections of sets from $\mathcal{A}$ belong to $\mathcal{A}$ again.

[2] at least those that are physically relevant

In this section we assume that M is a locally compact Hausdorff space. The space M is equipped with the so-called Borel σ-algebra $\mathcal{B}$; this is the σ-algebra generated by the collection of all open subsets of M. The sets of $\mathcal{B}$ will be called the (Borel) measurable subsets of M. A function $f : M \to \mathbb{C}$ is called (Borel) measurable if the pre-image $f^{-1}(S)$ of every measurable subset $S \subset \mathbb{C}$ is measurable in M. For this it obviously suffices that the pre-image of every open subset of $\mathbb{C}$ is Borel measurable in M.

By $\mathcal{M}_b(M)$ we denote the space of bounded measurable functions, equipped with the supremum norm $\|\cdot\|_\infty$. It is well known that $\mathcal{M}_b(M)$ is a Banach space. Clearly, it contains the space $C_b(M)$ of bounded continuous functions as a closed subspace. The continuous linear dual $C_b(M)'$ is therefore a Banach space as well. An element $\xi \in C_b(M)'$ is said to *vanish at infinity,* if for every $\varepsilon > 0$ there exists a compact subset $K \subset M$ such that

$$|\xi(f)| \leq \varepsilon \|f\|_\infty$$

for all $f \in C_b(M)$ with $f = 0$ on K. The space of such functionals is denoted by $C_b(M)'_\circ$. We leave it to the reader to check that this is a closed subspace of $C_b(M)'$, hence a Banach space of its own right. In the following we will give a characterization of the space $C_b(M)'_\circ$ in terms of measure theory, in the spirit of the Riesz representation theorem.

By a (complex valued) set function on $\mathcal{B}$ we shall mean a function $\mu : \mathcal{B} \to \mathbb{C}$. It is said to be *bounded* if the set $\mu(\mathcal{B})$ of values is a bounded subset of $\mathbb{C}$. The set function μ is called *additive,* if

$$\mu(U \cup V) = \mu(U) + \mu(V)$$

for all disjoint sets $U, V \in \mathcal{B}$. Note that the condition implies that $\mu(\emptyset) = 0$.

For an additive set function $\mu : \mathcal{B} \to \mathbb{C}$ its *variation* is defined to be the set function $\mathcal{B} \to [0, \infty]$, defined by

$$v(\mu, E) = \sup \sum_{j=1}^{k} |\mu(A_j)|, \qquad (E \in \mathcal{B}),$$

where the supremum is taken over all finite sequences (A_j) of mutually disjoint measurable subsets of E. The variation can be shown to be additive. It is important because of its application in estimates involving the set function μ.

Lemma 4.1 *Let μ be a bounded additive set function on $\mathcal{B}$. Then the variation $v(\mu)$ is a bounded additive set funciton on $\mathcal{B}$ with non-negative values. Moreover, for every $E \in \mathcal{B}$,*

$$\sup_{\substack{U \in \mathcal{B} \\ U \subset E}} |\mu(U)| \leq v(\mu, E) \leq 4 \sup_{\substack{U \in \mathcal{B} \\ U \subset E}} |\mu(U)|.$$

Proof. We refer to [9], III.1.4, Lemma 5, for a proof. The factor 4 appears because in the estimation μ is split into a real and imaginary part. Each of these is written as a difference of non-negative parts by means of a so-called Hahn decomposition. All 4 terms can be estimated in terms of μ. □

In particular, if μ is bounded additive, then its total variation $|\mu| := v(\mu, M)$ is bounded. A bounded additive set function μ induces a linear functional $I_\mu : C_b(M) \to \mathbb{C}$, given by integration:

$$I_\mu(f) := \int_M f d\mu, \qquad (f \in C_b(M)).$$

We shall now give a characterization of this integral which will turn out to be useful in the sequel. By a *simple function* on M we mean a measurable function $\varphi : M \to \mathbb{C}$ whose image is finite. Let $\alpha_1, \ldots, \alpha_n$ be the complex numbers in the image of φ, then the pre-images $U_i = \varphi^{-1}(\alpha_i)$ are mutually disjoint measurable subsets of M whose union is M; moreover,

$$\varphi = \sum_i \alpha_i 1_{U_i}. \tag{4.1}$$

Conversely, any function that can be expressed as a sum of this form, with $\alpha_i \in \mathbb{C}$ and U_i Borel measurable subsets of M, is simple. The space $\Sigma(M)$ of simple measurable functions is a linear subspace of $\mathcal{M}_b(M)$ which is dense for the subnorm. Indeed, let $f \in \mathcal{M}_b(M)$ be real-valued and let $\varepsilon > 0$. Let $R > \|f\|_\infty$. Then the interval $[-R, R]$ may be written as a finite union of disjoint intervals I_j of length strictly smaller than ε. We fix $\alpha_j \in I_j$ and put $U_j := \varphi^{-1}(I_j)$. Then the function $\varphi = \sum_j \alpha_j 1_{U_j}$ is simple, and $\|f - \varphi\|_\infty < \varepsilon$. This can be extended to complex valued functions by splitting in real and imaginary parts (which are easily seen to be measurable).

For a simple function of the form (4.1), with U_j disjoint, the integral $I_\mu(\varphi)$ is defined by

$$I_\mu(\varphi) = \sum_j \alpha_j \mu(U_j).$$

It is readily verified that this definition if independent of the particular expression (4.1). Moreover, $I_\mu : \Sigma(M) \to \mathbb{C}$ is linear.

It readily follows from the expression (4.1) with U_i mutually disjoint that

$$|I_\mu(\varphi)| \leq \|\varphi\|_\infty v(\mu, M) = |\mu| \|\varphi\|_\infty. \tag{4.2}$$

This implies that I_μ extends to a continuous linear functional on $\mathcal{M}_b(M)$ with operator norm at most $|\mu|$. The space $ba(M)$ of bounded additive set functions on $\mathcal{B}$, equipped with the norm $|\mu|$ is a Banach space. It follows that $\mu \mapsto I_\mu$ is a continous map between Banach spaces.

An additive set function $\mu : \mathcal{B} \to \mathbb{C}$ is said to be *regular* if for every measurable set $A \in \mathcal{B}$ and every $\varepsilon > 0$ there exist a closed set C and an open set O with $C \subset A \subset O$ such that for every measurable set $S \subset O \backslash C$ the estimate $|\mu(S)| \leq \varepsilon$ is valid. The space of regular bounded additive set functions, denoted $\mathrm{rba}(M)$, is a closed subspace of $\mathrm{ba}(M)$, hence Banach space of its own right. In case μ is regular, it can be shown that $\|I_\mu\| = |\mu|$. Moreover, the following result is valid.

Theorem 4.2 *Let M be a locally compact Hausdorff space (or more generally, a normal space). The map $\mu \mapsto I_\mu|_{C_b(M)}$ is an isometric isomorphism from the space* $\mathrm{rba}(M)$ *of regular bounded additive set functions on $\mathcal{B}$ onto the continuous linear dual $C_b(M)'$ of the space of bounded continuous functions on M.*

Proof. We refer to [9], IV.6.2, Thm. 2, for details. □

For compact spaces the situation is better. A set function $\mu : \mathcal{B} \to \mathbb{C}$ is called *countably additive* if for every countable sequence $(U_k)_{k\geq 1}$ of mutually disjoint sets from $\mathcal{B}$,

$$\mu(\cup_{k=1}^\infty U_k) = \sum_{k=1}^\infty \mu(U_k).$$

The sum on the right-hand side is required to be absolutely convergent. A (complex valued) *Borel measure* is defined to be a countably additive set function $\mathcal{B} \to \mathbb{C}$. The space $\mathrm{rca}(M)$ of regular bounded Borel measures on $\mathcal{B}$ is known to be a closed subspace of $\mathrm{rba}(M)$, hence a Banach space.

Theorem 4.3 (The Riesz representation theorem) *Let M be compact. The map $\mu \mapsto I_\mu|_{C(M)}$ defines an isometry from the space $\mathrm{rca}(M)$ of regular bounded Borel measures on M onto $C(M)'$, the continuous linear dual of the space of continuous functions on M.*

Proof. We refer to [9], IV.6.2, Thm. 3. □

In particular, from combining Theorems 4.2 and 4.3, we see that $\mathrm{rba}(M) = \mathrm{rca}(M)$. This fact, due to Alexandroff, is in fact used in the proof of the Riesz representation theorem. We shall now describe an analogue of the Riesz representation theorem for a locally compact Hausdorff space M.

An regular bounded Borel measure μ is said to vanish at infinity, if for every $\varepsilon > 0$ there exists a compact set $K \subset M$ such that the total variation of μ over $M \setminus K$ satisfies the estimate

$$v(\mu, M \setminus K) < \varepsilon.$$

The space $\mathrm{rca}(M)_\circ$ of such measures is a closed linear subspace of $\mathrm{rca}(M)$, hence a Banach space.

Theorem 4.4 *Let M be a locally compact Hausdorff space. Then the isometry $\mu \mapsto I_\mu$ of Theorem 4.2 restricts to an isometry from $\mathrm{rca}(M)_\circ$ onto $C_b(M)'_\circ$.*

Proof. The result follows by combining Theorems 4.2 and 4.3. I have not found a reference for this result in the literature. In a future improved version of these notes, a proof will be added. □

At this point it is perhaps a good idea to mention the following easy result.

Lemma 4.5 *Let M be locally compact and σ-compact (a countable union of compact sets). Then every non-negative finite Borel measure vanishes at infinity.*

Proof. If $U \subset V$ are Borel sets, then V is the disjoint union of the Borel sets U and $V \setminus U$, whence $\mu(V) = \mu(U) + \mu(V \setminus U) \geq \mu(U)$. It follows that μ is monotonous. It is now readily checked that $v(\mu, E) = \mu(E)$ for every $E \in \mathcal{B}$.

There exists an increasing sequence of compact sets $(K_j)_{j \geq 1}$ such that $\cup_j K_j = M$. Put $E_0 = \emptyset$ and define the sequence E_j inductively by $E_j = K_j \setminus E_{j-1}$. Then M is the disjoint union of the Borel sets E_j, hence $\sum_{k \geq 1} \mu(E_k) = \mu(M) < \infty$. It follows that $\mu(M \setminus K_n) = \sum_{j \geq n} \mu(E_j) \to 0$. The result now follows from the fact that $v(\mu, M \setminus K_n) = \mu(M \setminus K_n)$, as stated above. □

With these facts of measure theory collected, we now proceed with the definition of projection valued measures.

Definition 4.6 Let M be a locally compact Hausdorf space and $\mathcal{H}$ a Hilbert space. A *projection-valued Borel measure* in $\mathcal{H}$ based on M is a map P from the Borel σ-algebra $\mathcal{B}$ on M to the set of orthogonal projections in $B(\mathcal{H})$ with the following properties,

(a) for all $U, V \in \mathcal{B}$, $P(U \cap V) = P(U)P(V)$;

(b) $P(M) = I$;

(c) P is countably additive, i.e., for each countable sequence (U_n) of mutually disjoint measurable subsets,

$$P(\cup_n U_n) = \sum_n P(U_n).$$

The sum on the right-hand side is required to converge in the strong operator topology.

The requirement in (c) of convergence in the strong operator topology means that $\sum_n P(U_n)x$ should converge in $\mathcal{H}$, for any $x \in \mathcal{H}$. It follows from (c) that P is finitely additive, by letting $U_n = 0$ after some index. From combining (c) and (b) we see that $P(M) = P(M) + P(\emptyset)$, so that $P(\emptyset) = 0$. If we combine this with (a), we see that $P(U)$ and $P(V)$ have perpendicular images if U and V are disjoint measurable sets.

Let $x, y \in \mathcal{H}$. Then it follows from the above definition that

$$\mu_{x,y} : U \mapsto \langle P(U)x\,,\, y\rangle$$

defines a countably additive bounded Borel measure.

Lemma 4.7 *Let P be a projection valued measure based on M. Then, for every $x \in \mathcal{H}$ and each Borel set $U \subset M$,*

$$\mu_{x,x}(U) = \|P(U)x\|^2.$$

In particular, $\mu_{x,x}$ has all its values in $[0, 1]$.

Proof. $\mu_{x,x}(U) = \langle P(U)x\,,\, x\rangle = \langle P(U)x\,,\, P(U)x\rangle = \|P(U)x\|^2$. □

Definition 4.8 The projection valued Borel measure P in $\mathcal{H}$ is said to be *regular* if for all $x, y \in \mathcal{H}$, the Borel measure $\mu_{x,y} : U \mapsto \langle P(U)x\,,\, y\rangle$ is regular. It is said to *vanish at infinity* if for all $x, y \in \mathcal{H}$ the Borel measure $\mu_{x,y}$ vanishes at infinity.

Lemma 4.9 *Let P be a projection-valued Borel measure in $\mathcal{H}$ based on a locally compact and σ-compact space M. Then P vanishes at infinity.*

Proof. It follows from Lemmas 4.7 and 4.5 that the Borel measure $\mu_{x,x}$ vanishes at infinity, for every $x \in \mathcal{H}$. Using that

$$\mu_{x+y,x+y} = \mu_{x,x} + \mu_{y,y} + 2\mu_{x,y}, \tag{4.3}$$

we see that P vanishes at infinity. □

Lemma 4.10 *Let P be a projection valued Borel measure in $\mathcal{H}$, based on M.*

(a) *The measure P is regular if and only if for every Borel measurable subset $E \subset M$ every $x \in \mathcal{H}$ and every $\varepsilon > 0$ there exists a closed subset C of E and an open neighborhood O of E in M, such that*

$$\|P(O \setminus C)x\| < \varepsilon.$$

(b) *The measure P vanishes at infinity if and only if for every $x \in \mathcal{H}$ and every $\varepsilon > 0$ there exists a compact subset $K \subset M$ such that $\|P(M \setminus K)x\| < \varepsilon$.*

Proof. It follows from the displayed condition in (a) and positivity of $\mu_{x,x}$ that $|\mu_{x,x}(S)| < \varepsilon^2$ for every measurable subset S of $O \setminus C$. Hence, $\mu_{x,x}$ is regular. It now follows by application of (4.3) that $\mu_{x,y}$ is regular for all $x, y \in \mathcal{H}$. Thus, the condition is sufficient. Its necessity follows by application of Lemma 4.7. The proof of the equivalence in (b) proceeds along similar lines. □

Let P be a projection valued Borel measure on M, with values in $\mathcal{H}$. Then for $f \in \mathcal{M}_b(M)$, $x, y \in \mathcal{H}$, we define

$$\int_M f(m)\langle dP(m)x\,,\,y\rangle := \int_M f d\mu_{x,y}.$$

where $\mu_{x,y}$ is defined as above. By Lemma 4.7, $\mu_{x,x}$ is a non-negative measure, so that for $x = y$ the expression in the above equation is absolutely majorized by $\|f\|_\infty \mu_{x,x}(\operatorname{supp} f) \leq \|f\|_\infty \|x\|^2$. Using that $\mu_{x+y,x+y} = \mu_{x,x} + \mu_{y,y} + 2\mu_{x,y}$, we see that

$$\begin{aligned} \left|\int_M f(m)\langle dP(m)x\,,\,y\rangle\right| &\leq \frac{1}{2}(\|x+y\|^2 + \|x\|^2 + \|y\|^2)\|f\|_\infty \\ &\leq (\|x\| + \|y\|)^2 \|f\|_\infty. \end{aligned}$$

From this estimate with x, y in the unit sphere, it follows that there exists a unique continuous linear operator $I_P(f) : \mathcal{H} \to \mathcal{H}$ with the property that

$$\langle I_P(f)x\,,\,y\rangle = \int_M f(m)\langle dm(P)x\,,\,y\rangle, \qquad (x, y \in \mathcal{H}).$$

Moreover, the operator norm of $I_P(f)$ is bounded by $4\|f\|_\infty$. It follows that $f \mapsto I_P(f)$ defines a continuous linear operator $\mathcal{M}_b(M) \to B(H)$, the latter space being equipped with the operator norm. We note that for the simple function φ given by (4.1),

$$I_P(\varphi) = \sum_i \alpha_i P(U_i).$$

In view of the density of $\Sigma(M)$ in $\mathcal{M}_b(M)$ and the continuity of the map $f \mapsto I_P(f)$, this expression completely determines the definition of I_P on bounded measurable functions.

Lemma 4.11 *The map $P \mapsto I_P$ is injective from the space of regular projection valued Borel measures on M to the space $B(C_b(M), B(H))$ of bounded linear operators $C_b(M) \to B(H)$.*

Proof. This follows from the above discussion, by application of Theorem 4.2. □

Depending on our viewpoint, we will use one of the following ways to denote I_P,

$$P(f) := \int_M f dP := I_P(f), \qquad (f \in \mathcal{M}_b(M)).$$

Lemma 4.12 *Let P be a projection valued measure in $\mathcal{H}$, based on M. Then the continuous linear map $f \mapsto P(f)$, $\mathcal{M}_b(M) \to B(\mathcal{H})$ is a star homomorphism; i.e., for all $f, g \in C_b(M)$,*

(a) $P(\bar{f}) = P(f)^*$,

(b) $P(fg) = P(f)P(g)$.

In addition,

(c) $P(1_M) = I_{\mathcal{H}}$.

Proof. Assertion (c) is an immediate consequence of definitions. We will give the proof of (b). The proofs of the remaining assertion (a) proceeds along the same line. If $f = \sum_i \alpha_i U_i$ and $g = \sum_j \beta_j V_j$ are simple functions, then

$$fg = \sum_{i,j} \alpha_i \beta_j 1_{U_i \cap V_j},$$

from which it readily follows that

$$P(fg) = \sum_{i,j} \alpha_i \beta_j P(U_i) P(V_j) = P(f)P(g).$$

Thus, (b) holds for f and g in the space $\Sigma(M)$. The latter space is dense in $\mathcal{M}_b(M)$; moreover, $(f,g) \to fg$ and $(A,B) \to AB$ are continuous maps $\mathcal{M}_b(M) \times \mathcal{M}_b(M) \to \mathcal{M}_b(M)$ and $B(H) \times B(H) \to B(H)$, respectively. It follows that assertion (b) is valid for all bounded measurable functions f and g as well. □

A linear map $L : C_b(M) \to B(\mathcal{H})$ satisfying conditions (a) and (b) of the above lemma with L instead of P is called a *star homomorphism.* The map is said to vanish at infinity if for every $x \in \mathcal{H}$ and each $\varepsilon > 0$ there exists a compact subset $K \subset M$ such that for all $f \in C_b(M)$ with $f = 0$ on K, the estimate $\|L(f)x\| \leq \varepsilon \|f\|_\infty$ is valid.

Proposition 4.13 *Let $L : C_b(M) \to B(\mathcal{H})$ be a continuous star homomorphism with $L(1) = I$, and vanishing at infinity. Then there exists a unique regular projection valued Borel measure P in $\mathcal{H}$, based on M, such that $L(f) = \int_M f dP$, for all $f \in C_b(M)$, as above. Moreover, P vanishes at infinity.*

Proof. First of all, there exists a constant $C > 0$ such that $\|L(f)\| \leq C\|f\|_\infty$ for all $f \in C_b(M)$. We will now establish the result by application of Theorem 4.4. For $x, y \in \mathcal{H}$ we define the continuous linear functional $L_{x,y} : C_b(M) \to \mathbb{C}$ by $f \mapsto \langle L(f)x\,,\, y\rangle$. Since L vanishes at infinity, $L_{x,y} \in C_b(M)'_\circ$, so by Theorem 4.4 there exists a unique regular, bounded Borel measure $\mu_{x,y}$, which vanishes at infinity, such that $L_{x,y}(f) = \int_M f d\mu_{x,y}$ for all $f \in C_b(M)$. Let U be a Borel measurable subset of M. Then $\mu_{x,y}(U)$ can be arbitrarily well approximated by $L_{x,y}(f)$ with $f \in C_c(M)$ such that $0 \leq f \leq 1$. Since $L_{x,y}(f)| \leq C\|f\|_\infty \|x\|\|y\|$, it follows that $\mu_{x,y}(U) \leq C\|x\|\|y\|$. Since $L_{x,y}$ depends on $(x,y) \in \mathcal{H} \times \bar{\mathcal{H}}$ in a bilinear fashion, it follows that $\mu_{x,y}(U)$ depends on (x,y) in the same fashion. We conclude that there exists a continuous linear operator $P(U) : \mathcal{H} \to \mathcal{H}$ of operator norm at most C, such that $\langle P(U)x\,,\, y\rangle = \mu_{x,y}(U)$ for all $x, y \in \mathcal{H}$. By the mentioned approximation combined with $L(f)^* = L(\bar{f}) = L(f)$ for real valued f, we infer that $P(U)^* = P(U)$.

Let $x, y \in \mathcal{H}$ and let $V \subset M$ be a Borel measurable set. We will first show that, for all $f \in C_b(M)$,

$$\int_V f d\mu_{x,y} = \int_M f d\mu_{P(V)x,y}. \tag{4.4}$$

Let $\varepsilon > 0$. Then by regularity of the measures, combined with vanishing at infinity, there exist a compact set $K \subset V$ and an open set $O \supset V$ such that for every $S \subset O \setminus K$ the estimates

$|\mu_{x,y}(S)| < \varepsilon$ and $|\mu_{x,L(\bar f)y}(S)| < \varepsilon$ are valid. Let $g \in C_c(O)$ be such that $0 \le g \le 1$ and $g = 1$ on K. Then $g - 1_V$ has values in $[-1,1]$ and support contained in $O \setminus K$, whence

$$| \int_V f\, d\mu_{x,y} - \int_M fg\, d\mu_{x,y} \,| < \varepsilon\, \|f\|_\infty\,, \qquad \text{and} \tag{4.5}$$

$$|\ \mu_{x,P(\bar f)y}(V) - \int_M g\, d\mu_{x,P(\bar f)y} \,| < \varepsilon. \tag{4.6}$$

We now observe that

$$\int_M fg\, d\mu_{x,y} = \langle L(fg)x\,,\, y\rangle = \langle L(f)L(g)x\,,\, y\rangle = \int_M g\, d\mu_{x,P(\bar f)y}.$$

Combining this equality with the estimates (4.5) and (4.6), we find that

$$\left| \int_V f\, d\mu_{x,y} - \int_M f\, d\mu_{P(V)x,y} \right| < \varepsilon(\|f\|_\infty + 1).$$

This is true for every $\varepsilon > 0$, whence (4.4).

Let U be a second measurable subset of M and let $\varepsilon > 0$. We may select a compact subset K' and an open subset O' of M such that $K' \subset U \subset O'$ and such that, for every measurable subset $S \subset O' \setminus K'$,

$$\mu_{x,y}(S) < \varepsilon, \quad \text{and} \quad \mu_{P(V)x,y}(S) < \varepsilon.$$

Let $f \in C_c(O)$ be such that $0 \le f \le 1$ and $f = 1$ on K'. Then $f - 1_U$ has values in $[-1,1]$ and support contained in $O' \setminus K'$, whence

$$\left| \int_M f\, d\mu_{P(V)x,y} - \int_U d\mu_{P(V)x,y} \right| < \varepsilon. \tag{4.7}$$

On the other hand, $V \cap \operatorname{supp}(f - 1_U) \subset O' \setminus K'$ and we find that

$$\left| \int_V f\, d\mu_{x,y} - \int_V 1_U\, d\mu_{x,y} \right| < \varepsilon. \tag{4.8}$$

Combining these two estimates with the equality (4.4) and the observation that the second integral in (4.7) equals $\mu_{P(V)x,y}(U) = \langle P(U)P(V)x\,,\, y\rangle$, whereas the second integral in (4.8) equals $\mu_{x,y}(U \cap V) = \langle P(U \cap V)x\,,\, y\rangle$, we obtain that

$$|\langle P(U \cap V)x\,,\, y\rangle - \langle P(U)P(V)x\,,\, y\rangle| < 2\varepsilon.$$

This is true for all $x, y \in \mathcal{H}$ and all $\varepsilon > 0$. We conclude that P satisfies condition (a) of Definition 4.6. From this condition with $U = V$, it follows that $P(U)$ is a projection operator, for every measurable subset $U \subset M$.

It remains to establish the countable additivity of P. For all $x, y \in \mathcal{H}$, the map $U \mapsto \langle P(U)x\,,\, y\rangle$ is a measure, hence countably additive. It follows that P is finitely additive. Let (U_n) be a decreasing sequence of measurable subsets of M such that $\cap_n U_n = \emptyset$. Then for each $x \in \mathcal{H}$ we have $\|P(U_n)x\|^2 = \langle P(U_n)x\,,\, x\rangle = \mu_{x,x}(U_n)$, hence $P(U_n)x \to 0$ as $n \to \infty$. It follows that $P(U_n) \to 0$ in the strong operator topology on $B(\mathcal{H})$. Combining this with the finite additivity, we conclude that P satisfies condition (c) of Definition 4.6. $\square$

5 Spectral theorems

In this section we collect some background on spectral theorems that we shall need in the rest of the exposition. Our basic tool is the Gelfand-Naimark theory of commutative C^*-algebras with unit (also called Banach $*$-algebras with unit).

A *Banach algebra* is an associative algebra B over the field $\mathbb{C}$, equipped with a norm $\|\cdot\|$ for which B is complete, and which satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in B$. A Banach algebra is called unital if it possesses a unit e for the multiplication.

Assume that B is a commutative unital Banach algebra. Then the spectrum $\mathrm{spec}\,(x)$ of an element $x \in B$ is defined to be the set of $\lambda \in \mathbb{C}$ for which $x - \lambda e$ has no inverse in B. By the use of the geometric series for $(e-y)^{-1}$, $(y \in B)$, it is not difficult to show that $\mathrm{spec}\,(x)$ is a closed subset of disk in $\mathbb{C}$ of radius $\|x\|$, centered at 0. The spectral norm $\|x\|_{\mathrm{sp}}$ of an element $x \in B$ is defined by

$$\|x\|_{\mathrm{sp}} := \sup\{|\lambda| \mid \lambda \in \mathrm{spec}\,x\}.$$

It follows from the above discussion that

$$\|x\|_{\mathrm{sp}} \leq \|x\|, \qquad (x \in B). \tag{5.1}$$

By application of elementary complex function theory in one variable, it follows that $\lambda \mapsto (x - \lambda e)^{-1}$ is a holomorphic B-valued function on $\mathbb{C} \setminus \mathrm{spec}\,(x)$. From this in turn it follows that the spectral norm can be expressed in terms of the norm on B by

$$\|x\|_{\mathrm{sp}} = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Let $\mathrm{spec}\,(B)$ denote the set of continuous algebra homomorphisms $B \to \mathbb{C}$ (the map $\xi \mapsto \ker\xi$ gives a one-to-one correspondence with the space of maximal ideals of B, whence the notation). We equip the set $\mathrm{spec}\,(B)$ with the weakest topology for which all evaluation maps $\hat{x} : \xi \mapsto \xi(x)$, for $x \in B$, are continuous. It can be shown that for all $x \in B$,

$$\|\hat{x}\|_\infty = \|x\|_{\mathrm{sp}}; \tag{5.2}$$

see [9] for a proof.

An *involution* on a Banach algebra B (possibly without unit) is a skew-linear map $x \mapsto x^*$ such that such that $e^* = e$, $x^{**} = x$, $(xy)^* = y^* x^*$ and $\|x^*\| = \|x\|$, for all $x, y \in B$.

A *C^*-algebra* is a Banach algebra equipped with an involution $x \mapsto x^*$ such that $\|xx^*\| = \|x\|^2$ for all $x \in B$.

Example 5.1 An important example of a C^*-algebra with unit is the algebra $B(\mathcal{H})$ of bounded linear operators in a Hilbert space $\mathcal{H}$, equipped with the operator norm, and with the involution $T \mapsto T^*$, where T^* denotes the adjoint of T.

Let M be a compact Hausdorff space, and $C(M)$ the space of continuous functions $M \to \mathbb{C}$, equipped with the supnorm $\|\cdot\|$. Equipped with pointwise multiplication, $C(M)$ is a commutative Banach algebra with unit. Equipped with the involution $f \mapsto \bar{f}$ induced by complex conjugation, it is a commutative C^*-algebra with unit. The following theorem, due to Gelfand and Naimark, is the main result of the theory of commutative C^*-algebras.

Theorem 5.2 *Let B be a commutative C^*-algebra with unit. Then $\hat{B}$ is compact. Moreover, the map $x \mapsto \hat{x}$ is an isometric isomorphism of unital C^*-algebras from B onto $C(\mathrm{spec}\,(B))$.*

For applications the following lemma is important. We say that a set $S \subset B$ generates the (unital) C^*-algebra B if B is the smallest closed (unital) C^*-subalgebra of B containing S.

Proposition 5.3 *Let B be a commutative unital C^*-algebra, generated by a single element a. Then the evaluation map $\mathrm{ev}_a : \mathrm{spec}\,(B) \to \mathbb{C}$, $\xi \mapsto \xi(a)$ establishes a homeomorphism from $\mathrm{spec}\,(B)$ onto $\mathrm{spec}\,(a)$.*

Corollary 5.4 *Let B be a commutative unital C^*-algebra, generated by a single element a. Then there exists a unique isomorphism of commutative unital C^*-algebras $B \to C(\mathrm{spec}\,(a))$ that maps a onto the function $\lambda \mapsto \lambda$.*

Proof. Uniqueness follows from the fact that x generates. Existence follows from combining Theorem 5.2 and Proposition 5.3, as follows. Let $\Phi : x \mapsto \hat{x}$ be the isomorphism of the theorem. Let $\mathrm{ev}_a : \mathrm{spec}\,(B) \to \mathrm{spec}\,(a)$ be the homeomorphism of the proposition. It induces an isomorphism of commutative unital C^*-algebras $\mathrm{ev}_a^* : C(\mathrm{spec}\,(a)) \to C(\mathrm{spec}\,(B))$, $\varphi \mapsto \varphi \circ \mathrm{ev}_a$. The map $\mathrm{ev}_a^{*-1} \circ \Phi$ is an ismorphism $B \to C(\mathrm{spec}\,(a))$. Let $f : \lambda \mapsto \lambda$, then $\mathrm{ev}_a^*(f) : \mathrm{spec}\,(B) \to \mathbb{C}$ is given by $\xi \mapsto f(\xi(a)) = \xi(a) = \hat{a}(\xi) = \Phi(a)(\xi)$. We conclude that $f = \mathrm{ev}_a^{*-1} \circ \Phi(a)$. □

The above results leads to the spectral resolution theorem for normal bounded operators in Hilbert space.

Let $\mathcal{H}$ be a Hilbert space, and $B(H)$ the C^*-algebra of bounded linear endomorphisms of $\mathcal{H}$. An operator $T \in B(H)$ is called *normal* if it commutes with its adjoint.

Let T be normal. Then the unital C^*-algebra generated by T is commutative. By Proposition 5.3, $\mathrm{spec}\,(B) \simeq \mathrm{spec}\,(T)$.

Theorem 5.5 (The spectral theorem for bounded normal operators) *Let T be a normal bounded operator in the Hilbert space $\mathcal{H}$. Then there exists a unique projection valued measure P in $\mathcal{H}$, based on $\mathrm{spec}\,(T)$, such that*

$$T = \int \lambda \, dP(\lambda). \tag{5.3}$$

Proof. Let B be generated by T as in the text before the theorem. From Corollary 5.4 it follows that there exists a unique isomorphism from B onto $C(\mathrm{spec}\,(T))$ that maps T onto the function $\lambda \mapsto \lambda$. In view of the results of the first section, the inverse of this isomorphism determines a projection valued measure with (5.3). For the proof of uniqueness, we refer to [10]. □

The unique projection valued measure of the above theorem is called the spectral measure of T. The expression (5.3) is called the spectral resolution of T. The following results are not hard to prove, and left to the reader.

If T is self-adjoint then T is normal, $\mathrm{spec}\, T$ is real, and the above result is the spectral theorem for a bounded self-adjoint operator. On the other hand, if T is unitary, then T is normal and $\mathrm{spec}\, T$ is contained in the unit circle.

If P is a projection valued measure based on a compact subset C of $\mathbb{C}$, then (5.3) defines a bounded normal operator and P is its spectral measure. In particular, P is zero on the complement of $\operatorname{spec}(T)$. Moreover, T is self-adjoint if and only if $\operatorname{spec}(T) \subset \mathbb{R}$ and T is unitary if and only if $\operatorname{spec}(T)$ is contained in the unit circle.

It is very important to have the spectral theorem for bounded normal operators, since also the spectral theorem for unbounded self-adjoint operators in $\mathcal{H}$ can be derived from it. We give the formulation of this result, but refer to [10] for details.

By a densely defined operator in $\mathcal{H}$ we mean a linear map $T : D_T \to \mathcal{H}$ with D_T, the domain of T, a dense subset of $\mathcal{H}$. A densily defined operator is called *symmetric* if $\langle Tv\,,\, w\rangle = \langle v\,,\, Tw\rangle$ for all $v, w \in D_T$. It is called *closed* if its graph is a closed subset of $\mathcal{H} \oplus \mathcal{H}$.

A self-adjoint operator in $\mathcal{H}$ is defined to be a densily defined operator that is closed and symmetric. The following result can be proved by applying Theorem 5.5 to the so-called resolvent operator. We refer to [10] for details.

Theorem 5.6 (Spectral theorem for self-adjoint operators) *Let T be a self-adjoint operator in $\mathcal{H}$. Then there exists a unique projection-valued measure P based on $\mathbb{R}$, such that*

$$Tv = \int_{\mathbb{R}} \lambda \, dP(\lambda)v$$

for all $v \in D_T$. Moreover, D_T consists of the elements $v \in \mathcal{H}$ with

$$\int \lambda^2 \, \langle dP(\lambda)v\,,\, v\rangle < \infty.$$

6 Smooth vectors in a representation

If $(\pi, \mathcal{H})$ is a unitary representation and $W \subset \mathcal{H}$ a closed invariant subspace, then its orthocomplement $W^\perp$ is also closed and invariant. Moreover, $\mathcal{H} = W \oplus W^\perp$ as an orthogonal direct sum. It is now clear that π is unitarily equivalent to $\pi|_W \oplus \pi_{W^\perp}$. Moreover, the orthogonal projection onto W, denoted P_W, is an element of $\mathrm{End}_G(\mathcal{H})$. The following lemma gives different characterizations of irreducibility for unitary representations. Its simple proof is left to the reader.

Lemma 6.1 *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then the following conditions are equivalent.*

(a) *The representation π is irreducible.*

(b) *The only G-equivariant orthogonal projections in $\mathcal{H}$ are 0 and I.*

(c) *The representation π is not unitarily equivalent to the direct sum of two unitary representations.*

The next lemma is deeper, as it involves the spectral resolution for bounded self-adjoint operators.

Lemma 6.2 *Let $(\pi, \mathcal{H})$ be a unitary representation for G. Then π is irreducible if and only if $\mathrm{End}_G(\mathcal{H}) = \mathbb{C}I$.*

Proof. Let $T \in \mathrm{End}_G(\mathcal{H})$. We first assume that T is self-adjoint. Let $E \mapsto P(E)$ be the spectral resolution for T. Then by uniqueness of the spectral resolution, $P(E) \in \mathrm{End}_G(\mathcal{H})$, for every Borel subset $E \subset \mathbb{C}$. It follows that $P(E) \in \{0, I\}$. This implies the existence of a unique point $\lambda \in \mathbb{C}$ such that $P(\mathbb{C} \setminus \{\lambda\}) = 0$. Hence $P(\{\lambda\}) = I$ and $T = \lambda I$.

Let now $T \in \mathrm{End}_G(\mathcal{H})$ be arbitrary. Then $T + T^*$ and $iT - iT^*$ are bounded self-adjoint and G-equivariant, hence scalar. It follows that T is scalar as well. □

The reader may wonder why it is not sufficient to restrict attention to the representation theory in, for instance, Hilbert spaces, instead of the more general locally convex spaces. The reason for this is that for a proper development of the theory it is crucial to look at the subspace of so-called C^∞ vectors. This subspace is a locally convex space in a natural way, as we will now see.

Let V be a complete locally convex space and $\omega \subset \mathbb{R}^n$ an open subset. A function $f : \Omega \to V$ is said to be smooth if all partial derivatives of f exist and are continuous. It is then easily seen that the order of taking the partial derivatives is immaterial, so that the familiar multi-index notation $\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$ makes sense. The space $C^\infty(\Omega, V)$ of smooth functions $\Omega \to V$ equipped with the topology of uniform convergence of all $\partial^\alpha f$ on compact subsets of Ω is a locally convex space, which is Fréchet as soon as V is Fréchet. Using partitions of unity one may extend the definition to obtain the locally convex space $C^\infty(M, V)$ for M a smooth manifold.

Let now (π, V) be a continuous representation of a Lie group G in the complete locally convex space V. A vector $v \in V$ is called smooth for π if the map $g \mapsto \pi(g)v$ belongs to $C^\infty(G, M)$. The space of smooth vectors in V is denoted by V^∞. It may be equipped with the finest locally convex topology that makes the natural embedding $V^\infty \subset C^\infty(G, V)$ continuous.

Clearly, V^∞ is a G-invariant subspace of V. Moreover, it carries the natural representation of $\mathfrak{g}$ (or $\mathfrak{g}$-module structure) obtained by differentiation,

$$Xv = \frac{d}{dt}\bigg|_{t=0} \pi(\exp tX)v,$$

for $v \in V^\infty$ and $X \in \mathfrak{g}$. It thus naturally becomes a $U(\mathfrak{g})$-module, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, the complexification of $\mathfrak{g}$. The above mentioned topology on V^∞ may now also be characterized by the fact that it is determined by the following collection of seminorms,

$$v \mapsto \max_{X \in S} p(Xv),$$

with $S \subset U(\mathfrak{g})$ a finite subset and p a continuous seminorm on V.

We shall now introduce the space of Gårding vectors in V. Let dx be a choice of left Haar measure on G. For $\varphi \in C_c(G)$ we define the operator $\pi(\varphi) : V \to V$ by

$$\pi(\varphi) := \int_G \varphi(x)\pi(x)\, dx. \tag{6.1}$$

It is readily checked that the operator $\pi(\varphi)$ is continuously linear, and that the map $(\varphi, v) \mapsto \pi(\varphi)v$ is a continous linear map $C_c(G) \times V \to V$.

The space of continuous *Gårding vector* in V is defined to be the linear span of al vectors of the form $\pi(\varphi)v$, for $v \in V$ and $\varphi \in C_c(G)$. We agree to denote this space by V_0.

Lemma 6.3 *The space V_0 is dense in V.*

Proof. Let $v \in V$, ν a continuous seminorm on V and $\varepsilon > 0$. Fix an open neighborhood U of e in G such that $\nu(\pi(x)v - v) < \frac{1}{2}\varepsilon$ for all $x \in U$. Fix $f \in C_c(U)$ with $f \geq 0$ and $\int_G f(x)\, dx = 1$. Then

$$\begin{aligned} \nu(\pi(f)v - v) &= \nu(\int_G f(x)[\pi(x)v - v]\, dx) \\ &\leq \int_U f(x)\nu(\pi(x)v - v)\, dx < \varepsilon. \end{aligned}$$

□

The space of *smooth* Gårding vectors in V is defined to be the linear span of all vectors of the form $\pi(\varphi)v$, for $v \in V$ and $\varphi \in C_c^\infty(G)$. We agree to denote this space by $V_{\circ\circ}$.

Lemma 6.4 *The space $V_{\circ\circ}$ is contained in V^∞ and dense in V.*

Proof. By left invariance of the Haar measure, it follows that $\pi(g)\pi(\varphi)v = \pi(L_g\varphi)v$. The map $g \mapsto L_g\varphi$ is smooth $G \to C_c(G)$, and the map $\varphi \mapsto \pi(\varphi)v$ is continuously linear $C_c(G) \to V$. It follows that $g \mapsto \pi(g)\pi(\varphi)v$ is smooth. Thus, $V_{\circ\circ} \subset V^\infty$. Density is established as in the proof of Lemma 6.3, with f a smooth function. □

We end this section with some remarks on the operators $\pi(\varphi)$ defined by (6.1), with dx a choice of left Haar measure on G. By invariance of the measure, we have, for $g \in G$ and $\varphi \in C_c(G)$,

$$\pi(L_g\varphi) = \pi(g)\pi(\varphi).$$

For two functions $\varphi, \psi \in C_c(G)$, we define the convolution product $\varphi * \psi \in C_c(G)$ by

$$\varphi * \psi(x) = \int_G \varphi(y)(L_y\psi)(x) = \int_G \varphi(y)\psi(y^{-1}x)\, dy.$$

Then by interchange of integration, it follows that

$$\pi(\varphi * \psi) = \pi(\varphi)\pi(\psi).$$

In the rest of this section, we will assume that the representation π is a unitary representation in a Hilbert space $\mathcal{H}$. Then by estimation under the integral sign, it follows that the operator norm $\|\pi(\varphi)\|_{\rm op}$ is bounded by the L^1 norm $\|\varphi\|_1$ of φ. Therefore, the map $\varphi \mapsto \pi(\varphi)$ extends to a continuous linear map $L^1(G) \to B(\mathcal{H})$, of operator norm at most 1.

If $\varphi, \psi \in C_c(G)$, then by estimation under the integral sign, combined with Fubini's theorem, it follows that

$$\|\varphi * \psi\|_1 \leq \|\varphi\|_1 \|\psi\|_1.$$

This shows that the convolution product extends to a continuous bilinear product on $L^1(G)$ and that $(L^1(G), *)$ is a Banach algebra without unit. Moreover, by density of $C_c(G)$ in $L^1(G)$ and continuity, it follows that the map $\varphi \mapsto \pi(\varphi)$ is an algebra homomorphism from the Banach algebra $L^1(G)$ (without unit) to the Banach algebra $B(H)$.

7 The Heisenberg group

In this section we consider the well-known position and momentum operators. They satisfy the Heisenberg commutation relations. These determine a Lie algebra, the Heisenberg algebra, which in turns determines a simply connected Lie group, the so-called Heisenberg group. We will use the theory developed in the previous sections to classify all irreducible unitary representations of this group. In particular we will discuss the Stone-von Neumann theorem. Finally, the space of Schwartz functions and the Fourier transform will seen to be natural objects in the frame work of the Heisenberg group.

First of all, we consider the Hilbert space $L^2(\mathbb{R}^n)$, relative to Lebesgue measure. We denote by Recall that a Schwartz function on $\mathbb{R}^n$ is a smooth function $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\nu_{k,N}(f) := \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |\partial^\alpha f(x)| < \infty$$

for all $k, N \in \mathbb{N}$. We denote the space of all such Schwartz functions by $\mathcal{S}(\mathbb{R}^n)$. Equipped with the topology induced by the indicated semi-norms $\nu_{k,N}$ this space becomes a Fréchet space. It contains $C_c^\infty(\mathbb{R}^n)$ hence is dense in $L^2(\mathbb{R}^n)$.

The position operators

$$q_j := x_j : f \mapsto x_j f$$

are continuous operators on $\mathcal{S}(\mathbb{R}^n)$, for $1 \leq j \leq n$. They are symmetric with respect to the L^2-inner product, and have self-adjoint extensions. The self-adjoint extension of x_j has the following spectral resolution Q_j. Let $\pi_j : \mathbb{R}^n \to \mathbb{R}$ denote projection on the j-th coordinate. Let U be a Borel measurable subset of $\mathbb{R}$, then the projection $Q_j(U)$ is defined by $Q_j(U)f = 1_{p_j^{-1}(U)} f$. It is for this reason that the state space is said to be based on position.

The momentum operators

$$p_j = -i\partial_j : f \mapsto \frac{1}{i} \frac{\partial f}{\partial x_j}$$

are also continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$. Moreover, they allow unique extensions to unbounded self-adjoint operators on $L^2(\mathbb{R}^n)$. Their spectral projections Q_j can be described as follows. We define the Fourier transform $\mathcal{F}$ of a Schwartz function f to be the function $\mathcal{F}f : \mathbb{R}^n \to \mathbb{C}$ given by

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\, e^{-i\langle \xi\, , x \rangle}\, dx.$$

It is well known that $\mathcal{F}$ is a topological linear ismorphisms of $\mathcal{S}$ onto itself that is isometric with respect to the L^2-norms. Therefore, it has a unique extension to an isometric isomorphism from $L^2(\mathbb{R}^n)$ onto itself. We recall that

$$\mathcal{F} \circ D_j = \xi_j \circ \mathcal{F}, \qquad \mathcal{F} \circ x_j = D_j \circ \mathcal{F}.$$

It follows that $D_j = \mathcal{F}^{-1} \circ \xi_j \circ \mathcal{F}$, so that the spectral resolution of p_j is given by

$$P_j(U) = \mathcal{F}^{-1} \circ Q_j \circ \mathcal{F}.$$

Thus we see that $\mathcal{F}$ defines a unitary isomorphism from $L^2(\mathbb{R}^3)$ onto momentum based state space.

We note that the position and momentum operators satisfy the following Heisenberg commutation relations on $\mathcal{S}(\mathbb{R}^n)$,

$$[p_k, p_l] = 0, \quad [q_k, q_l] = 0, \quad [p_k, q_l] = -i\delta_{kl} I.$$

It follows that the operators ip_k, iq_l, I span a $2n+1$-dimensional Lie subalgebra $\mathfrak{h}$ of the algebra $\mathrm{End}(\mathcal{S}(\mathbb{R}^n))$ of continuous linear operators, equipped with the commutator brackets. Alternatively, this Lie algebra may be described as follows. Let V be the real linear space spanned by the ip_k. Let W be the real linear space spanned by the iq_l. The spaces V and W are in duality by the pairing determined by $\langle ip_k\,,\, iq_l\rangle = \delta_{kl}$. Accordingly, we identify W with the dual V^* of V. The Heisenberg algebra can now be described as the direct sum of vector spaces

$$\mathfrak{h} = V \oplus V^* \oplus \mathbb{R}I,$$

with bracket determined by the requirement that I is central, V and V^* abelian, and finally,

$$[v, v^*] = \langle v\,,\, v^*\rangle I \qquad (v \in V,\ v^* \in V^*).$$

We shall now describe the associated simply connected Lie group. It is most easily found by considering the unitary one parameter groups generated by the Hermitian operators p_k and q_l. For $a \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$ we define $T_a f : x \mapsto f(a+x)$. Then one readily sees that

$$e^{itp_k} = T_{te_k}.$$

Let $e^1, \ldots, e^n$ denote the basis of $(\mathbb{R}^n)^*$ dual to the standard one on $\mathbb{R}^n$. Then

$$e^{itq_l} : f \mapsto e^{ite^l} f.$$

It follows from this that

$$e^{isp_k} e^{itq_l} e^{-isp_k} = e^{-ist\delta_{kl}} e^{itq_l} = e^{-i\langle isp_k\,,\, itq_l\rangle} e^{itp_l}.$$

This suggests to define the Heisenberg group as

$$\mathrm{H} = V \times V^* \times \mathbb{T},$$

with product rule

$$(x_1, \xi_1, \tau_1)(x_2, \xi_2, \tau_2) = (x_1 + x_2, \xi_1 + \xi_2, e^{i\langle x_2\,,\, \xi_1\rangle} \tau_1 \tau_2).$$

Exercise 7.1 We use the basis ip_k to obtain a linear isomorphism $V \to \mathbb{R}^n$ via which we shall identify. Accordingly, we identify $V^* \simeq (\mathbb{R}^n)^*$ and $\mathrm{H} \simeq \mathbb{R}^n \times (\mathbb{R}^n)^* \times \mathbb{T}$. Using this identification, we define a map $\pi = \pi_{\mathrm{H}} : \mathrm{H} \to L^2(\mathbb{R}^n)$ by

$$\pi(x, \xi, \tau) = \tau e^{\xi} T_y.$$

(a) Check that $\pi = \pi_{\mathrm{H}}$ defines a unitary representation of H in $L^2(\mathbb{R}^n)$. The present particular realization of π is called the the Schrödinger realization.

(b) Show that the space of smooth vectors for this representation equals $\mathcal{S}(\mathbb{R}^n)$.

(c) Show that H has Lie algebra $\mathfrak{h}$.

Later in this exposition we shall indicate how the irreductible unitary representations of the Heisenberg group may be classified. In particular we will prove the following result.

Theorem 7.2 (Stone-Von Neumann theorem). *There exists precisely one irreducible unitary representation π of* H *such that $\pi(\tau) = \tau I$ for every $\tau \in \mathbb{T}$. This representation is unitarily equivalent to the representation π_{H} defined above.*

The Stone-Von Neumann theorem implies that there is essentially onely one way to realize the commutation relations of the Heisenberg group by means of a group of unitary operators in Hilbert space.

For further material on the Heisenberg group and in particular its relation to Harmonic analysis in $\mathbb{R}^n$, we refer the reader to informative paper [11] of R. Howe.

8 Unitary representations of abelian groups

In this section we describe the theory of unitary representations of a commutative Lie group A. We begin with an easy observation.

Lemma 8.1 *The irreducible unitary representations of A are all* 1*-dimensional.*

Proof. Use Lemma 6.2. □

Clearly the above lemma is valid for locally compact abelian groups. The same is true for the other results of this section. We refer to [14] for details.

We will be satisfied with understanding the results for abelian Lie groups, since this is what is needed in the applications we have in mind. For the same reason, we may and will assume that A is a *connected* commutative Lie group. This has the advantage that it enables us to use the classical theory of Fourier series and integrals, which we will now recall.

Let $\mathbb{T}$ be the unit circle group in $\mathbb{C}$. Then it follows from the lemma that the unitary dual of A, i.e., the set of equivalence classes of irreducible unitary representations, is identified with the space

$$\widehat{A} = \mathrm{Hom}(A, \mathbb{T})$$

of (smooth) unitary characters of A. If χ is such a character, the associated representation in $\mathbb{C}$ is given by $A \times \mathbb{C} \to \mathbb{C}, (a, z) \mapsto \chi(a)z$. We shall now describe the set $\widehat{A}$.

Since A is connected, the exponential map $\exp : \mathfrak{a} \to A$ is a surjective group homomorphism with a discrete kernel. This discrete kernel is of the form $\Gamma = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_p$ with $\gamma_1, \ldots, \gamma_p$ a sequence of elements in $\mathfrak{a}$ that are linearly independent over $\mathbb{R}$. Accordingly, $A \simeq \mathfrak{a}/\Gamma \simeq (\mathbb{R}/\mathbb{Z})^p \times \mathbb{R}^q$, with $p, q \in \mathbb{N}$, $p + q = \dim A$.

If $\chi \in \widehat{A}$, then $\chi \circ \exp = e^{\chi_*}$, where $\chi_* : \mathfrak{a} \to i\mathbb{R}$ is the derivative of χ at the identity element e. We leave it to the reader to verify that the map $\chi \mapsto \chi_*$ is a bijection from $\widehat{A}$ onto the subset $\Gamma^\vee$ of elements $\nu \in i\mathfrak{a}^*$ with $\nu(\Gamma) \subset 2\mathbb{Z}\pi i$. We agree to equip this subset with the topology by restricting that of the finite dimensional real vector space $i\mathfrak{a}^*$. Moreover, $\widehat{A}$ is equipped with the topology for which the map $\chi \mapsto \chi_*$ becomes a homeomorphism. Note that this topology may also be characterized as the topology of uniform convergence on compact subsets of A.

The representation theory of the group A allows a simple description in terms of spectral measures, as follows.

Theorem 8.2 *Let π be a unitary representation of A in a Hilbert space $\mathcal{H}$. Then there exists a uniquely determined projection-valued measure $P = P_\pi$ in $\mathcal{H}$, based on $\widehat{A}$, such that for all*

$f \in L^1(A)$,

$$\pi(f) = \int_{\widehat{A}} \hat{f}\, dP. \tag{8.1}$$

The map $\pi \mapsto P_\pi$ is a bijection from the collection of unitary representations of A in $\mathcal{H}$ onto the collection of projection-valued measures in $\mathcal{H}$, based on $\widehat{A}$.

Finally, two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitarily equivalent if and only if the associated projection-valued measures P_{π_1} and P_{π_2} are unitarily equivalent.

The proof of this result is based on the analysis of Banach algebras discussed in Section 5. We will use the rest of this section to present it.

We start by recalling some facts from the harmonic analysis of the group A. As we have assumed that A is a connected commutative Lie group, so that $A \simeq \mathbb{Z}^p \times \mathbb{R}^q$, these facts follows from the classical theory of Fourier series and Fourier integrals. This theory has an appropriate generalization to abelian locally compact groups, see e.g. [14].

We equip A with a choice of a translation invariant smooth density da. Given a function $\varphi \in L^1(A)$, we define its Fourier transform $\hat{\varphi} : \widehat{A} \to \mathbb{C}$ by

$$\hat{\varphi}(\chi) = \int_A \varphi(a)\overline{\chi(a)}\, da.$$

Then $\hat{\varphi}$ is a continuous function on $\widehat{A}$ tending to zero at infinity. Let $C_\circ(\widehat{A})$ be the space of such functions. Then $C_\circ(\widehat{A})$ is the closure of $C_c(\widehat{A})$ in the Banach space $C_b(\widehat{A})$ of bounded continuous functions, equipped with the sup-norm $\|\cdot\|_\infty$.

From the end of Section 5 we recall that $L^1(A)$, equipped with the convolution product is a Banach algebra without unit. It is commutative since A is commutative. The Fourier transform $f \mapsto \hat{f}$ is a continous homomorphism from the Banach algebra $L^1(A)$ into the Banach algebra $C_\circ(\widehat{A})$.

By the following standard trick, we embed $L^1(A)$ isometrically into a commutative Banach algebra B with unit. Let δ_e be an element not in $L^1(A)$. We define B as the linear space $\mathbb{C}\delta_e \oplus L^1(A)$ and equip it with the unique algebra structure with unit δ_e that extends the convolution algebra structure of $L^1(A)$. Thus, $\delta_e * \varphi = \varphi * \delta_e = \varphi$ for all $\varphi \in L^1(G)$, and we see that here δ_e may be interpreted as the Dirac function. We equip B with the norm $\|\lambda\delta_e + \varphi\| := |\lambda| + \|\varphi\|_1$. Then B is a Banach algebra. For $\chi \in \widehat{A}$, the linear map $\xi_\chi : B \to \mathbb{C}$ given by $\xi_\chi(\delta_e) = 1$ and by $\xi_\chi(\varphi) = \hat{\varphi}(\chi)$ for $\varphi \in L^1(A)$, is a continuous character. Moreover, the linear map $\xi_\circ : B \to \mathbb{C}$ determined by

$$\xi_\circ|_{L^1(A)} = 0, \qquad \xi_\circ(\delta_e) = 1$$

is a continuous character as well.

Lemma 8.3 *The map $\chi \mapsto \xi_\chi$ is a homeomorphism from $\widehat{A}$ onto* $\mathrm{spec}\,(B) \setminus \{\xi_\circ\}$.

Proof. For a proof of this lemma, which is reasonably straightforward, we refer the reader to [14]. □

In view of the above lemma, we may view $\mathrm{spec}\,(B)$ as the one-point compactification of $\widehat{A}$. Accordingly, we may identify $C_\circ(\widehat{A})$ as the closed subalgebra without unit of the Banach algebra $C(\mathrm{spec}\,(B))$, consisting of all functions vanishing at the point $\xi_\circ$. Finally, we may extend the Fourier transform to a continuous homomorphism $B \to C(\text{spec}\,(B))$ of unital Banach algebras, by requiring that $\hat{\delta}_e = 1$.

We define the involution $*$ on $L^1(A)$ by

$$\varphi^*(a) = \overline{\varphi(a^{-1})}, \qquad (a \in A),$$

and extend it to an involution of B by requiring that $\delta_e^* = \delta_e$. Then the Fourier transform defines a $*$-homomorphism from B to $C(\text{spec}\,(B))$.

Lemma 8.4 *Let $f : B_1 \to B_2$ be a homorphism of unital Banach algebras. Then for every $x \in B_1$, the set* $\text{spec}\,(f(x))$ *is contained in* $\text{spec}\,(x)$. *In particular, it follows that*

$$\|f(x)\|_{\text{sp}} \le \|x\|_{\text{sp}}.$$

Proof. Let $\lambda \notin \text{spec}\,(x)$. Then $x - \lambda e$ has an inverse y. Since f is an algebra homomorphism $f(x) - \lambda e = f(x - \lambda e)$ has inverse $f(y)$. It follows that $\lambda \notin \text{spec}\,(f(x))$, whence the result. □

We will now start with the proof of Theorem 8.2. Let π be a unitary representation of A in $\mathcal{H}$. At the end of Section 5 we have seen that $\varphi \mapsto \pi(\varphi)$ defines a homomorphism of Banach algebras $L^1(A) \to B(\mathcal{H})$. It is readily verified that this map is a homomorphisms for the given involutions. It may be extended to a continuous $*$-homomorphism from B into $B(\mathcal{H})$, by requiring that $\pi(\delta_e) = I$. Here we note that although $(L^1(A), *)$ is not a C^*-algebra, the closure of the image $\pi(B)$ is a commutative C^*-algebra with unit. This has the following consequence.

Lemma 8.5 *For all $\varphi \in B = \mathbb{C}\delta_e \oplus L^1(A)$ we have*

$$\|\pi(\varphi)\| \le \|\hat{\varphi}\|_\infty.$$

Proof. Let $\varphi \in B$. Since the closure of $\pi(B)$ in $B(\mathcal{H})$ is a commutative unital C^*-algebra, we may apply Theorem 5.2 to it, and obtain

$$\|\pi(\varphi)\| = \|\pi(\varphi)^\wedge\|_\infty = \|\pi(\varphi)\|_{\text{sp}}.$$

Since $\pi : B \to B(\mathcal{H})$ is a homomorphism of unital Banach algebras, it follows by applicaton of Lemma 8.4 that

$$\|\pi(\varphi)\|_{\text{sp}} \le \|\varphi\|_{\text{sp}} = \|\hat{\varphi}\|_\infty.$$

The proof is completed by combining these two estimates. □

Lemma 8.6 *There is a unique continuous linear map $L : C(\text{spec}\,(B)) \to B(\mathcal{H})$ such that $\pi(\varphi) = L(\hat{\varphi})$ for all $\varphi \in C(\text{spec}\,(B))$.*

Proof. From classical Fourier theory, it follows that the Fourier transform $\mathcal{F} : \varphi \mapsto \hat{\varphi}$ from B to $C(\text{spec}\,(B))$ is injective. Let $V \subset C(\text{spec}\,(B))$ be its image, equipped with the restriction topology. Then the inverse transform $\mathcal{F}^{-1} : V \to B(\mathcal{H})$ is continuous by the estimate of Lemma 8.5. Moreover, the map L must on V be given by the formula $L = \pi \circ \mathcal{F}^{-1}$. Thus, for existence and uniqueness of L is suffices to show that V is dense in $C(\text{spec}\,(B))$.

For this we observe that by classical Fourier theory, $\mathcal{F}(L^1(A))$ is dense in $C_\circ(\hat{A})$. Moreover, $\mathcal{F}(\mathbb{C}\delta_e) = \mathbb{C}1$. Since $C(\text{spec}\,B) = \mathbb{C}1 \oplus C_\circ(\hat{A})$, it follows that $V = \mathcal{F}(B)$ is dense in $C(B)$. □

Lemma 8.7 *The map $L : C(B) \to B(\mathcal{H})$ of Lemma 8.6 is a star homomorphism with $L(1) = I$.*

Proof. As in the proof of the Lemma 8.6, we use the notation $\mathcal{F}$ for the Fourier transform $\varphi \mapsto \hat{\varphi}$ from B to $C(\mathrm{spec}\,(B))$. The mentioned lemma asserts that the following diagram commutes

$$\begin{array}{ccc} & C(\mathrm{spec}\,(B)) & \\ {\scriptstyle\mathcal{F}} \nearrow & & \searrow {\scriptstyle L} \\ B & \xrightarrow{\pi} & B(\mathcal{H}). \end{array}$$

The maps π and $\mathcal{F}$ are star homomorphisms. Since $\mathcal{F}$ has dense image and L is continuous, it follows that L is a star homomorphism. Finally, $L(1) = L(\mathcal{F}(\delta_e)) = \pi(\delta_e) = I$. □

Completion of the proof of Theorem 8.2. According to Proposition 4.13, the map L determines a projection valued measure P in $\mathcal{H}$, based on $\mathrm{spec}\, B$. We will first show that $P(\{\xi_\circ\}) = 0$.

Let $\{U_n\}$ be a decreasing sequence of open neighborhoods of e in A with $U_n \to \{e\}$. For each U_n we may select a function $f_n \in C_c(U_n)$ with L^1-norm 1. Then as in the proof of Lemma 6.3 it follows that $\pi(f_n) \to I$ for the strong operator topolgy on $B(\mathcal{H})$, i.e., pointwise. On the other hand, it is readily seen that $\mathcal{F}(f_n) \to 1$, locally uniformly on $\hat{A}$.

Let $x, y \in \mathcal{H}$. Then $\mu_{x,y}(\,\cdot\,) = \langle P(\,\cdot\,)x\,,\, y\rangle$ is a regular Borel measure on B. Moreover,

$$\int_B \mathcal{F} f_n \, d\mu_{x,y} = \langle L(\mathcal{F} f_n)x\,,\, y\rangle = \langle \pi(f_n)x\,,\, y\rangle \to \langle x\,,\, y\rangle,$$

as $n \to \infty$. On the other hand,

$$\int_B \mathcal{F} f_n \, d\mu_{x,y} \to \mu_{x,y}(\widehat{A}) = \langle P(\widehat{A})x\,,\, y\rangle.$$

It follows that $P(\widehat{A}) = I$, so that

$$P(\{\xi_\circ\}) = P(B) - P(\widehat{A}) = I - I = 0.$$

This implies that P restricts to a regular projection valued Borel measure on $\widehat{A}$. Moreover, it satisfies the required identity (8.1) for every $f \in L^1(A)$.

We will now establish uniqueness. If Q is a second regular projection valued Borel measure on $\widehat{A}$ satisfying the identity (8.1), then it follows that $P = Q$ on $\mathcal{F}(L^1(A))$, hence on $C_\circ(\widehat{A})$ by continuity and density. Since $\widehat{A}$ is σ-compact, any projection valued measure on $\widehat{A}$ vanishes at infinity. Hence, $P = Q$ on $C_b(\widehat{A})$, and uniqueness follows.

To prove that $\pi \mapsto P_\pi$ is injective, we will show that the representation π can be retrieved from P. Let (f_n) be a sequence in $C_c(A)$ as above. Then for every $a \in A$ and $x \in \mathcal{H}$

$$\pi(a)x = \lim_{n\to\infty} \pi(a)\pi(f_n)x = \lim_{n\to\infty} \pi(L_a f_n)x = \lim_{n\to\infty} P(\mathcal{F}(L_a f_n))x,$$

where L denotes the left regular representation of A in $C_c(A)$. We define the function $\mathrm{i}(a) \in C_b(\widehat{A})$ by

$$\mathrm{i}(a)(\chi) = \chi(a), \qquad (\chi \in \widehat{A}).$$

Then $\mathcal{F}(L_a f_n) = \mathrm{i}(a)\mathcal{F}(f_n)$ is a bounded sequence in $C_b(\widehat{A})$ that converges locally uniformly on $\widehat{A}$ to the function $\mathrm{i}(a)$. Since P vanishes at infinity, it follows that $P(\mathcal{F}(L_a\varphi_n))v \to P(i(a))v$, and we conclude that $\pi(a) = P(i(a))$.

For surjectvity, assume that a projection valued measure P in $\mathcal{H}$ based on $\widehat{A}$ be given. Then P vanishes at infinity, hence defines a continuous linear $*$-homomorphism $C_b(\widehat{A}) \to B(\mathcal{H})$. On the other hand, $a \mapsto \mathrm{i}(a)$ defines a linear map $A \to \widehat{A}$ with

$$\mathrm{i}(ab) = \mathrm{i}(a)\mathrm{i}(b), \quad \mathrm{i}(a^{-1}) = \overline{\mathrm{i}(a)} \quad \text{and} \ \ \mathrm{i}(e) = 1. \tag{8.2}$$

For $a \in A$ we define the bounded operator $\pi(a)$ in $\mathcal{H}$ by

$$\pi(a) := P(\mathrm{i}(a)). \tag{8.3}$$

Then it follows from (8.2) that $\pi(ab) = \pi(a)\pi(b)$, $\pi(a^{-1}) = \pi(a)^*$ and $\pi(e) = I$. We see that $\pi(a) := P(i(a))$ defines a representation of A in $\mathcal{H}$ by unitary maps. To show that it is continuous let (a_n) be a sequence in A tending to e. Then $i(a_n)$ is a sequence in $C_b(\widehat{A})$, uniformly bounded by 1 and converging to 1, locally uniformly on $\widehat{A}$. This implies that $P(i(a_n))x \to x$ for every $x \in \mathcal{H}$. We conclude that $\lim_{a \to e} P(a)x = x$. By Lemma 2.4 it follows that π is a continuous unitary representation of A in $\mathcal{H}$.

We claim that $P = P_\pi$. For this we note that for $\psi \in C_c(A)$, the function $\hat{\psi}$ belongs to $C_\circ(\widehat{A})$. This implies that $a \mapsto i(a)\hat{\psi}$ is a continuous bounded function $A \to C_\circ(A)$. It now follows that for every $\varphi \in L^1(A)$,

$$\int_A \varphi(a)\mathrm{i}(a)\hat{\psi}\, da = \hat{\varphi}\hat{\psi},$$

as an integral in the Banach space $C_\circ(\widehat{A})$. Applying P to both sides of this equation, we find

$$\int_A \varphi(a)P(\mathrm{i}(a))P(\hat{\psi})\, da = P(\hat{\varphi})P(\hat{\psi}),$$

the integral being convergent in $B(\mathcal{H})$, equipped with the operator norm. We conclude that

$$\pi(\varphi)P(\hat{\psi}) = P(\hat{\varphi})P(\hat{\psi}).$$

We now substitute $\psi = f_n$, and take the limit for $n \to \infty$. Since $\hat{f}_n$ is a bounded sequence in $C_b(\widehat{A})$ which converges locally uniformly to 1, it follows that $P(\hat{f}_n) \to I$ in $B(\mathcal{H})$. We conclude that π and P are related as in (8.1). By the already established uniqueness, it now follows that $P = P_\pi$.

We have now proved that the map $\pi \mapsto P_\pi$ is a bijection between the unitary representations of A in $\mathcal{H}$ and the projection valued measures in $\mathcal{H}$ based on $\widehat{A}$. It remains to prove the final assertion. Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be unitary representations of A and let P_1 and P_2 be the associated spectral projections. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary isomorphism. Then $\rho(\,\cdot\,) := T^{-1}\pi_1(\,\cdot\,)T$ is a unitary representation of $\mathcal{H}$. By uniqueness, the associated projection-valued measure is given by $P_\rho(\,\cdot\,) = T^{-1} \circ P_1(\,\cdot\,) \circ T$. From the fact that the map $\pi \mapsto P_\pi$ is bijective it follows that $\rho = \pi_2$ if and only if $P_\rho = P_2$. We conclude that π_1 and π_2 are unitarily equivalent if and only if P_1 and P_2 are unitarily equivalent. Moreover, the equivalences can be realized by the same isometric isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$. $\square$

9 Induced representations

We assume that H is a closed subgroup of a Lie group G. If ξ is a continuous (hence smooth) representation of H in a finite dimensional linear space V, we denote by $\mathcal{V}$ the associated smooth vector bundle

$$\mathcal{V} = G \times_H V$$

over the coset manifold G/H. We briefly recall the definition. The group H acts smoothly, properly and freely on $G \times V$ by the rule

$$h \cdot (x, v) = (xh^{-1}, \xi(h)v). \tag{9.1}$$

As a manifold, $\mathcal{V}$ is defined to be the quotient for this action. The projection $G \times V \to G$ on the first component factors to a smooth map $\pi : \mathcal{V} \to G/H$. Let p denote the canonical projection $G \to G/H$. Then we have the following commutative diagram,

$$\begin{array}{ccc} G \times V & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \pi \\ G & \xrightarrow{p} & G/H. \end{array} \tag{9.2}$$

There is a unique way to equip $\pi : \mathcal{V} \to G/H$ with a compatible structure of vector bundle such that in the above diagram, the trivial bundle $G \times V \to G$ is the pull-back $p^*(\mathcal{V})$ of $\mathcal{V}$ under p, as we will now show.

If $a \in G/H$ and $x \in p^{-1}(a)$, then the map $j_x : V \to \mathcal{V}, v \mapsto [(x, v)]$ is a smooth diffeomorphism from V onto the fiber $\mathcal{V}_a := \pi^{-1}(a)$. If y is a second point from $p^{-1}(a)$, then $y = xh$ for a unique $h \in H$, and it is readily seen that

$$j_y = j_x \circ \xi(h). \tag{9.3}$$

Hence, the fiber $\mathcal{V}_a$ has a unique structure of linear space for which all maps j_x, $x \in p^{-1}(a)$ are linear isomorphisms.

We claim that, equipped with this structure of linear space on the fibers, $\mathcal{V}$ is a smooth vector bundle. This is seen as follows. Let O be an open set in G/H over which the principal bundle $p : G \to G/H$ admits a trivialization. This trivialization gives rise to a section $s : O \to G$. In terms of this section we define $\tilde{s} : O \times V \to \pi^{-1}(O)$ by $\tilde{s}(x, v) = [(s(x), v)]$. Then it is readily seen that $\tilde{s}$ is a smooth bijection with inverse equal to the smooth map $\pi^{-1}(O) \to O \times V$ induced by $p \times I : p^{-1}(O) \times V \to O \times V$. From this we see that $\mathcal{V}$ is a smooth vector bundle over O, the map $\tilde{s}^{-1}$ being a smooth trivialization. The construction shows that $G \times V$ is the pull-back of $\mathcal{V}$ under p.

Finally, we observe that the requirement that $G \times V = p^*(\mathcal{V})$ forces the above maps j_x to be linear isomorphisms, hence determines the vector bundle structure of $\mathcal{V}$ uniquely.

Definition 9.1 A *homogeneous smooth vector bundle* on G/H is a smooth vector bundle $\pi : E \to G/H$, equipped with a smooth action of G such that, for all $g \in G$, the action by g maps each fiber E_a, $(a \in G/H)$, linearly to the fiber E_{ga}.

Let V be as above. The smooth action of G on $G \times V$ given by $g \cdot (x, v) = (gx, v)$ factors to a smooth action of G on the associated vector bundle $\mathcal{V}$. It is readily seen that with this action, the bundle $\mathcal{V}$ is homogeneous.

Conversely, if $\mathcal{W}$ is any homomogeneous smooth vector bundle (of finite rank) on G/H, then the fiber $W := \mathcal{W}_{eH}$ carries a smooth H-representation. The map $G \times W \to \mathcal{W}$, $(g, w) \mapsto gw$ factors to an isomorphism of vector bundles $G \times_H W \simeq \mathcal{W}$, exhibiting $\mathcal{W}$ as an associated vector bundle.

In the language of categories, the functors $V \mapsto G \times_H V$ and $\mathcal{W} \mapsto \mathcal{W}_{eH}$ establish an equivalence of categories between the category of continuous finite dimensional representations of H and the category of smooth homogeneous vector bundles on G/H.

Let $\Gamma(\mathcal{V}) = \Gamma(G/H, \mathcal{V})$ denote the space of all sections of the vector bundle $\mathcal{V}$ (without any further assumption like continuity). The actions of G on G/H and $\mathcal{V}$ give rise to a representation L of G in $\Gamma(\mathcal{V})$ given by

$$[L_g s](x) = gs(g^{-1}x). \tag{9.4}$$

This representation is called the representation induced by the representation ξ of H and also denoted by $\mathrm{ind}_H^G(\xi)$. At a later stage we will look at various continuous subrepresentations of this representation.

From the diagram (9.2), in which the trivial bundle $G \times V$ on G is realized as the pull-back bundle $p^*(\mathcal{V})$, one obtains a natural the following alternative characterization of the space of sections of the bundle $\mathcal{V}$. Let $\tilde{p} : G \times V \to \mathcal{V}$ be the natural map. If $x \in G$, then $\tilde{p}$ is a linear ismorphism from $\{x\} \times V$ onto $\mathcal{V}_{p(x)}$. Hence, if $s \in \Gamma(\mathcal{V})$, there is a unique section $p^*(s) \in \Gamma(G \times V)$ such that the following diagram commutes

$$\begin{array}{ccc} G \times V & \xrightarrow{\tilde{p}} & \mathcal{V} \\ {\scriptstyle p^*(s)} \uparrow & & \uparrow {\scriptstyle s} \\ G & \xrightarrow{p} & G/H. \end{array} \tag{9.5}$$

As usual, we identify the section $p^*(s)$ of $G \times V$ with the map $p^*(s)_2 : G \to V$. Then $s \mapsto p^*(s)_2$ is a linear isomorphism from $\Gamma(\mathcal{V})$ onto the space $\mathcal{F}(G, \xi)$ of functions $\varphi : G \to V$ that transform according to the rule

$$\varphi(xh) = \xi(h)^{-1}\varphi(x), \tag{9.6}$$

for all $x \in G$ and $h \in H$. Moreover, via this isomorphism, the representation $\mathrm{ind}_H^G \xi$ transfers to the restriction of the left regular representation. This justifies the use of the symbol L in (9.4) in hindsight.

Remark 9.2 At this point we would like to point out a tricky point. The space $\Gamma^0(\mathcal{V})$ of continuous sections is an invariant subspace of the space of all sections and corresponds to the space $C(G, \xi)$ of continuous functions $G \to V$ transforming according to the rule (9.6). However, if G/H is not compact, then the restriction of π^ξ to $\Gamma^0(G/H, \mathcal{V})$ is not continuous. This problem occurs already for the trivial line bundle, in which case $\Gamma_0(\mathcal{V})$ equals $C(G/H)$, equipped with the left regular representation. The problem is that there exists a continuous function $\varphi : G/H \to \mathbb{C}$ such that $\lim_{g \to e} L_g \varphi = \varphi$ is not valid (this means that the continuity of φ lacks uniformity).

On the other hand, it can be shown that the restriction of π^ξ to the space $\Gamma_c^0(\mathcal{V})$ of compactly supported continuous sections, equipped with the usual locally convex topology, is continuous.

It can also be shown that the restriction of π^ξ to the space $\Gamma^\infty(\mathcal{V})$ of smooth sections, equipped with the usual locally convex topology, is continuous. At this point we shall not investigate these results any further, as we will not need them in the sequel.

In the rest of this section, we shall discuss the concept of *unitary induction.* Again we assume that G is a Lie group, and H a closed subgroup, but this time we start with a unitary representation ξ of H in a Hilbert space V (which we allow to be infinite dimensional). In the construction that is to follow we need the concept of a continuous *Hilbert bundle* on a locally compact Hausdorff topological space M.

First, we define a continuous family of Banach spaces over M to be a Hausdorff space $\mathcal{V}$, together with a surjective continuous map $\pi : \mathcal{V} \to M$, such that each fiber $\mathcal{V}_m = \pi^{-1}(m)$, for $m \in M$, is equipped with a the structure of a Banach space compatible with the restriction topology. An isomorphism of two such fibrations $\pi : \mathcal{V} \to M$ and $\pi' : \mathcal{V} \to M$ is a homeomorphism $f : \mathcal{V} \to \mathcal{V}'$ such that f maps the fiber $\mathcal{V}_m$ linearly onto $\mathcal{V}'_m$, for all $m \in M$. A family of Banach spaces is said to be trivial if it is isomorphic to a product $M \times B$, with B a fixed Banach space.

Definition 9.3 A *Banach bundle* on M is defined to be a continuous family of Banach spaces $\pi : \mathcal{V} \to M$ that is locally trivial, i.e., for every $a \in M$ there exists an open neighborhood U of a such that the family $\pi_U : \mathcal{V}_U \to U$ is trivial; here we have written $\mathcal{V}_U := \pi^{-1}(U)$ and $\pi_U = \pi|_{\pi^{-1}(U)}$.

A *Hilbert bundle* on M is defined to be a Banach bundle $\pi : \mathcal{V} \to M$ equipped with a compatible Hilbert space structure on each fiber $\mathcal{V}_a$, for $a \in M$, such that it is locally trivial in the following sense. For each $a \in M$ there exists an open neighborhood U of a, a Hilbert space $\mathcal{H}$ and an isomorphism of Banach bundles $\tau : \mathcal{V}_U \to U \times \mathcal{H}$ that restricts to an isometric isometry from the fiber $\mathcal{V}_x$ onto $\mathcal{H}$, for every $x \in U$.

We shall now go through the construction of the beginning of this section, but this time with ξ a unitary representation of H in a Hilbert space V. As a topological space, $\mathcal{V}$ is defined to be the quotient of $G \times V$ by the continuous H-action given by (9.1). This space is readily checked to be Hausdorff. The same arguments as before give rise to the commutative diagram (9.2), where this time all maps but the bottom one are just continuous. The assertion now is that $\mathcal{V}$ carries a unique structure of Hilbert bundle, for which the trivial Hilbert bundle $G \times V$ becomes the pull-back under p. From (9.3) we see that each fiber $\mathcal{V}_a$ carries a unique structure of Hilbert space such that all maps j_x, for $x \in p^{-1}(a)$, are isometric isomorphisms. This turns $\mathcal{V}$ into a continuous family of Banach spaces. Local triviality is established by the arguments following (9.3).

We may now define a representation L of G in the space $\Gamma^0_c(\mathcal{V})$ of compactly supported continuous sections by the rule (9.4). As our next step, we would like to define a pre-Hilbert structure on the space $\Gamma^0_c(\mathcal{V})$ of compactly supported continuous sections of the bundle $\mathcal{V}$ for which this representation becomes unitary. If $\varphi, \psi \in \Gamma^0_c(\mathcal{V})$, we may define a function $\langle \varphi \, , \, \psi \rangle_{\mathcal{V}} \in C_c(G/H)$ by the formula $\langle \varphi \, , \, \psi \rangle_{\mathcal{V}}(x) = \langle \varphi(x) \, , \, \psi(x) \rangle_x$. In case G/H has no invariant density, we have no canonical way to integrate this function over G/H. Our remedy for this is to apply a twist with the bundle $\mathcal{D}^{1/2}$ of half densities on G/H.

The bundle of half densities is a homogeneous line bundle on G/H of its own right. Let α be a positive real number. We recall that a complex α-density on a real linear space of finite dimension n is a map $\omega : \wedge^n V \to \mathbb{C}$ such that $\omega(\lambda x) = |\lambda| \omega(x)$ for all $x \in \wedge^n V$. If $\eta \in \wedge^n V^* \simeq (\wedge^n V)^*$ is a non-zero alternating n-form on V then $\omega = |\eta|^\alpha$ is a positive α-density on V and the space of all α-densities on V is given by $\mathbb{C}\omega$. Let now M be a smooth manifold of dimension n. Applying the construction of the space of α-densities fiberwise to the exterior bundle $\wedge^n TM$ we obtain the smooth complex line bundle $\mathcal{D}^\alpha = \mathcal{D}^\alpha_M$ of α-densities on M. A smooth section of $\mathcal{D}^\alpha$ is called a smooth α-density on M. A smooth map $\varphi : M \to N$ between two manifolds of equal dimension induces a pull-back map $\varphi^* : \Gamma(N, \mathcal{D}^\alpha_M) \to \Gamma(M, \mathcal{D}^\alpha_M)$. The density bundle with superscript $\alpha = 1$ will be denoted by $\mathcal{D}$; its sections are called smooth densities. A compactly supported continuous density ω on $\mathbb{R}^n$ is of the form $f\ dx$ with $dx = |dx_1 \wedge \ldots \wedge dx_n|$ and $f \in C_c(\mathbb{R}^n)$, and may be integrated in the usual way. On a manifold M, the integral of a compactly supported continuous density ω with support contained in a chart (U, χ) is defined by $\int_M \omega = \int_{\mathbb{R}^n} \chi^{-1*}\omega$. With the use of partitions of unity, the integral is extended to a continuous linear map $\omega \mapsto \int_M \omega$, $\Gamma^0_c(M, \mathcal{D}) \to \mathbb{C}$. Thus defined, integration is coordinate invariant. More precisely, if $\varphi : M \to N$ is a diffeomorphism of manifolds and $\omega \in \Gamma^0_c(N, \mathcal{D}_N)$, then

$$\int_N \omega = \int_M \varphi^*\omega. \tag{9.7}$$

In coordinate charts, this result reduces to the usual substitution of variables theorem for Riemann integration of compactly supported continuous functions.

We now return to the homogeneous manifold G/H and fix a positive density ω on $T_{eH}(G/H) = \mathfrak{g}/\mathfrak{h}$. Thus, $\mathbb{C}\omega$ is the space of all complex valued densities on $\mathfrak{g}/\mathfrak{h}$. The square root $\omega^{1/2}$ is a half density on $\mathfrak{g}/\mathfrak{h}$ and $\mathbb{C}\omega^{1/2}$ is the space of half densities on $\mathfrak{g}/\mathfrak{h}$.

The natural action of G on G/H induces a representation of G in $T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h}$ which coincides with the representation induced by the adjoint action of G on $\mathfrak{g}$. The mentioned representation of H in $\mathfrak{g}/\mathfrak{h}$ in turn induces a representation of H in $\mathbb{C}\omega^{1/2}$. In fact, the action of $h \in H$ on the latter space is given by scalar multiplication with

$$\delta(h) := |\det(\mathrm{Ad}(h)|_{\mathfrak{g}/\mathfrak{h}})|^{1/2}. \tag{9.8}$$

The bundle of half densities is now given by $\mathcal{D}^{1/2} = G \times_H \mathbb{C}\omega^{1/2}$. Its space of continuous sections may be identified with the space of continuous functions $\varphi : G \to \mathbb{C}\omega^{1/2}$ transforming according to the rule $\varphi(gh) = \delta(h)^{-1}\varphi(g)$. The action L of G on $\Gamma^0(G/H, \mathcal{D}^{1/2})$ coincides with the natural action by inverse pull-back. More precisely, if $g \in G$, let $l_g : G/H \to G/H$ be the natural action by left multiplication. By pull-back, the diffeomorphism l_g induces a continuous linear endomorphism l_g^* of $\Gamma^0(\mathcal{D}^{1/2})$. We now have $L_g\varphi = l_g^{*-1}\varphi$, for all $\varphi \in \Gamma^0(\mathcal{D}^{1/2})$ and $g \in G$. Similar remarks can be made for the bundle of densities.

It now follows from the invariance of integration of densities, mentioned in (9.7), that for all $\mu \in \Gamma^0_c(G/H, \mathcal{D})$ and all $g \in G$,

$$\int_{G/H} L_g\mu = \int_{G/H} \mu. \tag{9.9}$$

We now consider the continuous vector bundle $\mathcal{V} \otimes \mathcal{D}^{1/2}$ on G/H. Note that this bundle may be viewed as the homogeneous Banach bundle associated with the non-unitary continuous representation $\xi \otimes \delta$ of H in the Hilbert space $V \otimes \mathbb{C} \simeq \mathbb{C}$. The group G has a natural representation in the space

$$\Gamma^0_c(\mathcal{V} \otimes \mathcal{D}^{1/2})$$

of compactly supported continuous sections. This space may be equipped with an invariant inner product as follows.

The inner product $\langle \cdot\ ,\ \cdot \rangle$ on the fibers of $\mathcal{V}$ induces a sesquilinear map $(\varphi, \psi) \mapsto \langle \varphi\ ,\ \psi \rangle_{\mathcal{V}}$,

$$\Gamma^0_c(\mathcal{V} \otimes \mathcal{D}^{1/2}) \times \Gamma^0_c(\mathcal{V} \otimes \mathcal{D}^{1/2}) \to \Gamma^0_c(\mathcal{D}). \tag{9.10}$$

This map is G-equivariant, i.e., $\langle L_g\varphi\,,\, L_g\psi\rangle_\xi = L_g\langle\varphi\,,\,\psi\rangle_{\mathcal{V}}$. We define a positive definite inner product on $\Gamma_c^0(\mathcal{V}\otimes\mathcal{D}^{1/2})$ by

$$\langle\varphi\,,\,\psi\rangle = \int_{G/H}\langle\varphi\,,\,\psi\rangle_{\mathcal{V}}.$$

It follows from the equivariance of the pairing (9.10) combined with (9.9) that the inner product thus defined is invariant for the natural action L by G. In other words, for each $g\in G$ the endomorphism L_g of $\Gamma_c^0(\mathcal{V}\otimes\mathcal{D}^{1/2})$ is unitary.

The completion of $\Gamma_c^0(\mathcal{V}\otimes\mathcal{D}^{1/2})$ with respect to the given pre-Hilbert structure is denoted by

$$\mathcal{H}^\xi := \Gamma_2(\mathcal{V}\otimes\mathcal{D}^{1/2}).$$

For each $g\in G$ the map L_g extends to a unitary automorphism $\pi^\xi(g)$ of of $\mathcal{H}^\xi$. Moreover, $(g,\varphi)\mapsto\pi^\xi(g)\varphi$ defines a representation of G in $\mathcal{H}^\xi$. This representation is denoted by

$$\pi^\xi = \mathrm{Ind}_H^G(\xi)$$

and is called the representation of G unitarily induced by the unitary representation ξ of H.

Proposition 9.4 *The representation π^ξ is unitary.*

Proof. It remains to establish the continuity. This is done by invoking Lemma 2.4 as in Example 2.6, as follows. As a dense subspace we use $W=\Gamma_c^0(\mathcal{V}\otimes\mathcal{D}^{1/2})$. Let $\varphi\in W$. Then by an argument involving compactness and uniform continuity, it follows that $L_g\varphi(x)-\varphi(x)\to 0$ as $g\to e$, uniformly with respect to x. This in turn implies that $L_g\varphi-\varphi\to 0$ in $\mathcal{H}^\xi$. □

Finally, we observe that as in the beginning of this section, the space $\Gamma^0(\mathcal{V}\otimes\mathcal{D})$ may be identified with the space of continuous functions

$$\varphi: G\to V$$

transforming according to the rule

$$\varphi(gh)=\delta(h^{-1})\xi(h^{-1})\varphi(g),\qquad (g\in G,\ h\in H).$$

Accordingly the space $\Gamma_c^0(\mathcal{V}\otimes\mathcal{D})$ consists of continuous functions of the above type whose right H-invariant support has compact image in G/H. Moreover, the space $\Gamma_2(G/H,\mathcal{V}\otimes\mathcal{D}^{1/2})$ may be identified with the space of Borel measurable functions $\varphi: G\to V$ satisfying the above transformation rule and

$$\int_{G/H}\|\varphi\|_\xi^2<\infty,$$

where $\|\varphi\|_\xi^2$ defines the obvious measurable section of the density bundle $\mathcal{D}$. Moreover, two such functions are identified if they differ on a set of Lebesgue measure zero (by which we mean the set to have Lebesgue measure zero in every local coordinate patch).

We end this section with a brief discussion of *quasi-invariant measures.* We may select any nowhere vanishing density ω on $M:=G/H$. Then $l_g^{*-1}\omega=c(g,\,\cdot\,)\omega$, with $c: G\times M\to M$ a strictly positive continuous function. A measure μ on M with the similar property is called quasi-invariant for the G-action. We may now use the map

$$\varphi\mapsto\varphi\omega^{1/2}$$

to embed $\Gamma^0(\mathcal{V})$ into $\Gamma^0(\mathcal{V}\otimes\mathcal{D}^{1/2})$.

Accordingly, the Hilbert structure defined in this fashion is given by

$$(\varphi,\psi)\mapsto\int_{G/H}\langle\varphi\,,\,\psi\rangle\omega.$$

The natural unitary representation $\pi_{\mathcal{V}}$ of G in $\Gamma^0(\mathcal{V})$ is now given by

$$[\pi_{\mathcal{V}}(g)\varphi](m)=\omega^{-1/2}(l_g^{*-1}[\omega^{1/2}\varphi])(m)=c(g,m)^{1/2}[l_g^{*-1}\varphi](m).$$

This is the approach most found in the literature. It works in the more general context of locally compact groups, where the notion of a continuous density is not available.

10 The imprimitivity theorem

In the present section we will describe how the process of unitary induction, described in the previous section, can be inverted. We retain the notation of that section.

The central notion of the present section is that of a system of imprimitivity. We assume that G is a Lie group and that H is a closed subgroup of G.

Definition 10.1 Let M be a locally compact space, equipped with a continuous G-action. A *system of imprimitivity* for G, based on M, is a pair (π,P) with π a continuous unitary representation of G in a Hilbert space $\mathcal{H}$ and P a regular projection valued Borel measure in $\mathcal{H}$, based on M, such that the following compatibility condition holds. For every (Borel) measurable subset $E\subset M$ and every $x\in G$,

$$\pi(x)P(E)\pi(x)^{-1}=P(xE). \tag{10.1}$$

A more appropriate name for a system of imprimitivity would be a G-equivariant projection valued measure. Indeed, the continuous action of G on M induces an action of G on the Borel σ-algebra $\mathcal{B}_M$ through automorphisms. Moreover, the set $P(\mathcal{H})$ of orthogonal projections in $\mathcal{H}$ is equipped with the conjugation action induced by the unitary representation π. The above requirement on the projection valued measure $P:\mathcal{B}_M\to P(\mathcal{H})$ is precisely that the map P intertwines the G-actions. In spite of what has just been explained, we will continue to use the name imprimitivity theorem, since this has become the generally accepted terminology.

A particular example of a system of imprimitivity arises as follows. Let M be a smooth manifold, equipped with a continuous G-action and let $\pi:\mathcal{V}\to M$ be a continuous G-equivariant Hilbert bundle on M. By this we mean that $\mathcal{V}$ is equipped with a continuous G-action such that for each $m\in M$ the action by $g\in G$ maps the fiber $\mathcal{V}_m$ linearly and unitarily onto the fiber $\mathcal{V}_{gm}$. Let $\mathcal{D}^{1/2}$ denote the line bundle of half densities on M. We denote by π the natural representation of G on the space $\Gamma_c^0(\mathcal{V}\otimes\mathcal{D}^{1/2})$ of compactly supported continuous sections of the tensor product bundle $\mathcal{V}\otimes\mathcal{D}^{1/2}$.

We now have a sesquilinear pairing $\Gamma_c^0(M,\mathcal{V}\otimes\mathcal{D}^{1/2})\times\Gamma_c^0(M,\mathcal{V}\otimes\mathcal{D}^{1/2})\to\Gamma_c^0(M,\mathcal{D})$, given by $\langle\varphi\,,\,\psi\rangle_{\mathcal{V}}(m)=\langle\varphi(m)\,,\,\psi(m)\rangle$. It transforms according to the rule

$$\langle\pi(g)\varphi\,,\,\pi(g)\psi\rangle_{\mathcal{V}}=\alpha_g^{*-1}\langle\varphi\,,\,\psi\rangle_{\mathcal{V}}.$$

From this it follows that the pairing $\Gamma_c^0(M, \mathcal{V} \otimes \mathcal{D}^{1/2}) \times \Gamma_c^0(M, \mathcal{V} \otimes \mathcal{D}^{1/2}) \to \mathbb{C}$ given by

$$\langle \varphi \,,\, \psi \rangle = \int_M \langle \varphi \,,\, \psi \rangle_{\mathcal{V}}$$

is G-equivariant. Also, it is clearly skew symmetric and positive definite, so that it defines a G-equivariant pre-Hilbert structure. As in the proof of Proposition 9.4 it follows that π extends to a (continuous) unitary representation of G in the Hilbert completion $\mathcal{H} := \Gamma_2(\mathcal{V} \otimes \mathcal{D}^{1/2})$ of $\Gamma_c^0(\mathcal{V} \otimes \mathcal{D}^{1/2})$. We can now define a natural system of imprimitivity as follows. If $E \subset M$ is measurable, then $P(E) : \mathcal{H} \to \mathcal{H}$ is defined to be multiplication by the characteristic function 1_E of E. We refer to this system as the *natural system of imprimitivity* associated with the G-equivariant Hilbert bundle $\mathcal{V}$.

Of particular interest is the case that the action of G on M is *transitive,* so that $M \simeq G/H$ for a closed subgroup H of G. Assume that $\mathcal{V} = G \times_H V$, where (ξ, V) is a unitary representation of H. Then we are in the setting of Section 9 and the unitary representation π constructed above equals the unitarily induced representation $\pi^\xi = \mathrm{Ind}_H^G(\xi)$. The system of imprimitivity defined above is now denoted by P^ξ. It is given by

$$P^\xi(E)\varphi = 1_E \varphi, \qquad (\varphi \in \Gamma_2(G/H, \mathcal{V} \otimes \mathcal{D}^{1/2})),$$

for E a measurable subset of G/H. The system (π^ξ, P^ξ) is called the *system of imprimitivity induced by* ξ.

Let us return to the general setting of Definition 10.1. Thus, M is a locally compact Hausdorff space equipped with a continuous G-action. In the following, all systems of imprimitivity for G will be assumed to be based on M.

We shall now introduce the familiar notions of representation theory in the context of systems of imprimitivity. If (π_j, P_j), for $j = 1, 2$, are systems of imprimitivity for G, then by an *intertwining operator* or *G-equivariant map* from (π_1, P_1) to (π_2, P_2) we mean a continuous linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that

(a) $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$;

(b) $T \circ P_1(E) = P_2(E) \circ T$ for every measurable subset $E \subset M$.

The two systems (π_j, P_j) are said to be (unitarily) *equivalent* if there exists an isometric isomorphism T from $\mathcal{H}_1$ onto $\mathcal{H}_2$ that is G-equivariant.

An *invariant subspace* for a system (π, P) is a linear subspace $W \subset \mathcal{H}$ such that

(a) W is invariant for π, i.e., $\pi(g)W \subset W$ for all $g \in G$;

(b) W is invariant for P, i.e., $P(E) \subset W$ for every measurable subset $E \subset M$.

We leave it to the reader to verify that if W is an invariant subspace, then its orthocomplement $W^\perp$ in $\mathcal{H}$ is invariant as well. Moreover, if W is a closed invariant subspace, then W is a Hilbert space of its own right and there is an obvious way to define the restriction $(\pi, P)|_W = (\pi|_W, P|_W)$ of the system (π, P) to W.

A system (π, P) is called *irreducible* if 0 and $\mathcal{H}$ are the only closed invariant subspaces for (π, P) in $\mathcal{H}$.

If W is a closed invariant subspace for (π, P), then $W^\perp$ is closed and invariant as well, so that $\mathcal{H} = W \oplus W^\perp$. Obviously, (π, P) is the direct sum of its restrictions to W and $W^\perp$. Moreover, the orthogonal projections onto W and $W^\perp$ belong to the space $\mathrm{End}_G(\pi, P)$ of intertwining endomorphisms for the system (π, P). We leave it to the reader to give the proof of the following lemma, see also the proofs of Lemmas 6.1 and 6.2.

Lemma 10.2 *Let (π, P) be a system of imprimitivity for G. Then the following conditions are equivalent.*

(a) *The system (π, P) is irreducible.*

(b) *The system (π, P) cannot be expressed as the direct sum of two non-trivial systems of imprimitivity.*

(c) $\mathrm{End}_G(\pi, P) = \{0, I\}$.

Theorem 10.3 (Imprimitivity theorem) *Let G be a Lie group and H a closed subgroup. Let (π, P) be a system of imprimitivity for G, based on G/H. Then there exists a unitary representation $\xi = \xi_P$ of H such that (π, P) is unitarily equivalent to (π^ξ, P^ξ). The representation ξ is uniquely determined up to unitary equivalence. Finally, ξ is irreducible if and only if (π, P) is irreducible.*

Before proving the theorem, we shall give another characterization of a system of imprimitivity, based on the characterization of a regular projection valued measure given in Section 4. Let $(\pi, \mathcal{H})$ be a unitary representation of G and let P be a projection valued measure in $\mathcal{H}$, based on G/H. According to Lemma 4.12, P induces a continuous star homomorphism $C_b(G/H) \to B(\mathcal{H})$, which we denote by $f \mapsto P(f) := \int_M f dP$.

Lemma 10.4 *The projection valued measure P given above is a system of imprimitivity for π if and only if*

$$\pi(x)P(f)\pi(x)^{-1} = P(L_x f), \tag{10.2}$$

for all $f \in C_b(G/H)$ and $x \in G$.

Proof. If P is a system of imprimitivity for π, then the formula is obviously true for a function f from the space $\Sigma(G/H)$ of simple functions on G/H. By density and continuity it extends to the space $\mathcal{M}_b(G/H)$ of bounded measurable functions on G/H, so that in particular it holds on the subspace $C_b(G/H)$. Conversely, assume that the formula holds for all $f \in C_b(G/H)$. Since this space is dense in $\mathcal{M}_b(G/H)$, the formula extends to the latter space by continuity. In particular, if $E \subset G/H$ is measurable, the formula holds for the characteristic function $f = 1_E$. We now use that $L_x f = 1_{xE}$, so that (10.1) follows. □

We leave it to the reader to verify that in the same fashion the notions of intertwining operator, equivalence, invariant subspace, irreducibility, restriction, direct sum, etc. can be reformulated in terms of the map $f \mapsto P(f)$ instead of the map $E \mapsto P(E)$.

Let now (η, V) be an irreducible representation of H and let P^η be the associated induced system of imprimitivitiy. Then for $\psi \in C_b(G/H)$ the map $P^\eta(\psi) : \mathcal{H}^\eta \to \mathcal{H}^\eta$ is given by

$$P^\eta(\psi) : \; f \mapsto \psi f.$$

Indeed, it is a simple matter to check the formula for functions $\psi \in \Sigma(G/H)$. By continuity and density it extends to functions $\psi \in \mathcal{M}_b(G/H)$, hence in particular the formula is valid for bounded continuous functions.

Before starting with the proof of Theorem 10.3, we shall explain how to obtain the representation $\xi \simeq \eta$ in case $\pi = \pi^\eta$. Let ω be a positive density on $T_{eH}(G/H)$. Then the map

$$\mathrm{ev} : \Gamma^0(G/H, \mathcal{V}_\eta \otimes \mathcal{D}^{1/2}) \to V_\eta \otimes \mathbb{C}\omega^{1/2}$$

defined by $v \mapsto v(e)$, factors to a linear isomorphism from $\Gamma^0(G/H, \mathcal{V}_\eta \otimes \mathcal{D}^{1/2})/\ker(\mathrm{ev})$ onto $V_\eta \otimes \mathbb{C}\omega^{1/2}$. The idea is to use $\Gamma^0(G/H, \mathcal{V}_\eta \otimes \mathcal{D}^{1/2})/\ker(\mathrm{ev})$ as a model for $V_\xi \otimes \mathbb{C}\omega^{1/2}$. Moreover, the inner product on V_ξ may be obtained by factorization of the sesquilinear pairing $\Gamma^0(G/H, \mathcal{V}_\eta \otimes \mathcal{D}^{1/2}) \times \Gamma^0(G/H, \mathcal{V}_\eta \otimes \mathcal{D}^{1/2}) \to \mathbb{C}\omega$, given by

$$(v, w) = \mathrm{ev} \circ \langle v\,,\, w \rangle_\eta.$$

With suitable adaptations this procedure can be carried over to a representation $(\pi, \mathcal{H})$ that is not given in induced form. One may view (v, w) as a regular bounded Borel measure $B(v, w)$ on G/H, by testing with $f \in C_b(G/H)$. In terms of a general representation π this means

$$B(v, w)(f) := \langle P(f)v\,,\, w \rangle.$$

The problem is that $B(v, w)$ need not come from a continuous density on G/H, so that evaluation of $B(v, w)$ at the origin makes no sense. However, if v and w belong to the space $\mathcal{H}_0$ of continuous Gårding vectors for π, then $B(v, w)$ can be shown to be a continuous density. (In case $\pi = \pi^\eta$, the space $\mathcal{H}_0$ is contained in $\Gamma^0(G/H, \mathcal{V}_\eta \otimes \mathcal{D}^{1/2})$. For $v, w \in \mathcal{H}_0$ one then has that $B(v, w)$ coincides with the continuous density (v, w).)

The continuity of $B(v, w)$ allows the definition of a sesquilinear pairing $\beta : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{C}$, by the formula

$$\mathrm{ev}\ B(v, w) = \beta(v, w)\omega.$$

The pairing β factors to a pairing $\bar{\beta}$ on $\mathcal{H}_0/\ker(\beta)$ which can be shown to be positive definite. Moreover, the representation $\pi|_H$ factors to a representation π_0 of H in $\mathcal{H}_0/\ker(\beta)$. It can be shown that $\delta^{-1/2} \otimes \pi_0$ leaves $\bar{\beta}$ invariant. One may now take V_ξ to be the completion of the pre-Hilbert space $\mathcal{H}_0/\ker(\beta)$. For ξ one may take the unique extension $\delta^{-1/2} \otimes \pi_0$ to a unitary representation of H in V_ξ.

After this motivation, we prepare for the proof of Theorem 10.3 by introducing a suitable convolution product as follows.

We define the left regular representation of G on $C_b(G/H)'$ to be the contragredient of the left regular representation of G in $C_b(G/H)$. Thus, for $\mu \in C_b(G/H)'$ and $x \in G$ we put

$$L_x\mu = \mu \circ L_x^{-1}.$$

Moreover, for $\varphi \in C_c(G)$ and $\mu \in C_b(G/H)'$ we define the convolution product $\varphi * \mu \in C_b(G/H)'$ by

$$\varphi * \mu = \int_G \varphi(x) L_x\mu\, dx,$$

where dx is a choice of left Haar measure on G. We equip the space $\Gamma^0(G/H, \mathcal{D})$ of continuous densities on G/H with the usual Fréchet topology of uniform convergence on compact subsets. Via integration, the space $\Gamma^0(G/H, \mathcal{D})$ may be embedded into $C_c(G/H)'$. Accordingly, an element $\mu \in C_b(G/H)'$ is said to be continuous if its restriction to $C_c(G/H)$ coincides with a continuous density.

Lemma 10.5 *The map $(\varphi, \mu) \mapsto \varphi * \mu$ is a continuous bilinear map from $C_c(G) \times C_b(G/H)'$ to the space $\Gamma^0(G/H, \mathcal{D})$ of continuous densities on G/H.*

Proof. Let $p : G \to G/H$ denote the natural projection. We fix a left-invariant density dh on H and define $p_* : C_c(G) \to C_c(G/H)$ by $p_*(F)(xH) = \int_H F(xh)\, dh$. Then

$$\begin{aligned}\int_G \varphi(x) L_{x^{-1}} p_* F(y) &= \int_G \int_H \varphi(x) F(xyh)\; dh\, dx \\ &= \int_G \int_H \varphi(xh^{-1}y^{-1})\, dh\; F(x)\, dx \\ &= \int_G \mathcal{A}(L_{x^{-1}}\varphi)^\vee(y)\; F(x)\, dx,\end{aligned}$$

where we have used the notation

$$\mathcal{A}\psi(y) := \int_H \psi(yh)\; dh, \quad \text{and} \quad \psi^\vee(y) = \psi(y^{-1}).$$

Testing the expressions on the extreme sides of the above array with μ we obtain that

$$\langle \varphi * \mu\,,\, p_*(F)\rangle = \langle \Phi dx\,,\, F\rangle, \tag{10.3}$$

where $\Phi : G \to \mathbb{C}$ is defined by

$$\Phi(x) = \langle \mu\,,\, \mathcal{A}(L_{x^{-1}}\varphi)^\vee\rangle.$$

Since $x \mapsto \mathcal{A}(L_{x^{-1}}\varphi)^\vee$ is a continuous function on G with values in $C_b(G/H)$, it follows that Φ is a continuous function.

We now claim that there exists a continuous function $\chi : G \to \mathbb{C}$ with $p|_{\operatorname{supp}\chi} : \operatorname{supp}\chi \to G/H$ a proper map, and such that $f = p_*(\chi p^* f)$ for every $f \in C_c(G/H)$. Indeed, for every $aH \in G/H$ there exists an open neighborhood U of a in G/H such that the H-principal bundle $p : G \to G/H$ trivializes over U. From this trivialization a continuous function $\chi_U : p^{-1}(U) \to \mathbb{C}$ may be constructed with the required properties in $p{-}1(U)$. The functions χ_U may be patched together by using a partition of unity on G/H.

Substituting $F = \chi p^* f$ in (10.3) we see that, for all $f \in C_c(G/H)$,

$$\langle \varphi * \mu\,,\, f\rangle = \langle \Phi\, dx\,,\, \chi p^* f\rangle.$$

By application of a local analysis on G/H the desired continuity can now be deduced. □

Let us start with the proof of Theorem 10.3. We consider the continuous star homomorphism $f \mapsto P(f)$ from $C_b(G/H)$ to $B(\mathcal{H})$. For $v, w \in \mathcal{H}$ we define the element $B(v, w)$ in the continuous linear dual $C_b(G/H)'$ of $C_b(G/H)$ by

$$\langle B(v, w)\,,\, f\rangle = \langle P(f)v\,,\, w\rangle.$$

Then B is a continuous sesquilinear map $\mathcal{H} \times \mathcal{H} \to C_b(G/H)'$. In the following lemma we use the notation $\mathcal{H}_0$ for the space of continuous Gårding vectors defined above Lemma 6.3. According to the mentioned lemma, this space is dense in $\mathcal{H}$.

Lemma 10.6 *If $v, w \in \mathcal{H}$, then the map $T : C_c(G) \times C_c(G) \to C_b(G/H)'$ given by*

$$T(f, g) := B(\pi(f)v, \pi(\bar{g})w)$$

is a continuous bilinear map from $C_c(G) \times C_c(G)$ to the space $\Gamma^0(G/H, \mathcal{D})$ of continuous densities on G/H. In particular, B maps $\mathcal{H}_0 \times \mathcal{H}_0$ into $\Gamma^0(G/H, \mathcal{D})$.

Proof. Let $v, w \in \mathcal{H}$. Then from (10.2) it follows that

$$L_x B(v,w) = B(\pi(x)v, \pi(x)w),$$

where we recall that $L_x B(v,w) := B(v,w) \circ L_x^{-1}$. It follows that for $f, g \in C_c(G)$,

$$\begin{aligned} B(\pi(f)v, \pi(\bar{g})w) &= \int_G \int_G f(x)g(y) L_x B(v, \pi(x^{-1}y)w)\, dy\, dx \\ &= \int_G \int_G f(x)g(xy) L_x B(v, \pi(y)w)\, dy\, dx \\ &= \int_G \int_G f(x)g(xy) L_x B(v, \pi(y)w)\, dx\, dy \\ &= \int_G h_{f,g,y} * B(v, \pi(y)w)\, dy, \end{aligned}$$

where $h_{f,g,y}(x) := f(x)g(xy)$. Now $(f,g,y) \mapsto h_{f,g,y}$ is a compactly supported continuous $C_c(G)$-valued function and $(v,w,y) \mapsto B(v,\pi(y)w)$ is continuous as well. Combining this with Lemma 10.5, we deduce that the map $(f,g,v,w) \mapsto B(\pi(f)v, \pi(\bar{g})w)$ is continuous from $C_c(G) \times C_c(G) \times \mathcal{H} \times \mathcal{H}$ to $\Gamma^0(G/H, \mathcal{D})$. □

Lemma 10.7 *Let $v \in \mathcal{H}_0$. Then $B(v,v)$, viewed as a continuous density, is positive on G/H.*

Proof. Denote by μ the element $B(v,v)$, viewed as a continuous density on G/H. Let $g \in C_c(G/H)$, $g \geq 0$. We define $f := \sqrt{g}$. Then $f \in C_c(G/H)$ and $f \geq 0$, hence $P(f) = P(f)^*$. It follows that

$$\int_{G/H} g\, \mu = \langle B(v,v)\, ,\, f^2 \rangle = \langle P(f)v\, ,\, P(f)v \rangle \geq 0.$$

□

We now fix a positive density ω on $\mathfrak{g}/\mathfrak{h}$, so that the full space of densities on $\mathfrak{g}/\mathfrak{h}$ equals $\mathbb{C}\omega$. On the space of continuous densities on G/H we have a well defined evaluation map $\mathrm{ev}_e : \Gamma^0(\mathcal{D}) \to \mathbb{C}\omega$. We define the sesquilinear pairing $\beta : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{C}$ by

$$\beta(v,w)\, \omega = \mathrm{ev}_e B(v,w). \tag{10.4}$$

Then it is readily verified that β is skew symmetric; moreover, it follows from Lemma 10.7 that β is positive semi-definite.

Lemma 10.8 *Let $v, w \in H$. Then*

$$\beta(\pi(h)v, \pi(h)w) = \delta(h)^2 \beta(v,w), \qquad (h \in H),$$

where the function δ is defined as in (9.8).

Proof. It follows from (10.4) that

$$\begin{aligned} \beta(\pi(h)v, \pi(h)w)\omega &= \mathrm{ev}_e L_h B(v,w)) \\ &= [\mathrm{Ad}(h)|_{\mathfrak{g}/\mathfrak{h}}]^* \mathrm{ev}_e(B(v,w)) \\ &= \delta(h)^2 \beta(v,w)\omega. \end{aligned}$$

The proof is complete. □

It follows from the above lemma that $\ker\beta$ is a H-invariant subspace of $\mathcal{H}_0$. We equip the quotient space $\mathcal{H}_0/\ker\beta$ with the induced positive definite inner product $\bar{\beta}$ and with the representation representation π_0 of H obtained by factorization. Let ξ be the representation of H in $\mathcal{H}_0/\ker\beta$ defined by

$$\xi(h) = \delta(h)^{-1/2}\pi_0(h).$$

Then it follows from Lemma 10.8 that $\xi(h)$ is a unitary isomorphism for every $h \in H$. Let V be the Hilbert space completion of $\mathcal{H}_0/\ker\beta$. Then for every $h \in H$ the map $\xi(h)$ has a unique extension to a unitary isomorphism of V.

Lemma 10.9 *The unitary representation ξ of H in V is continuous.*

Proof. Let $v \in \mathcal{H}_0$. Then by Lemma 2.4 it suffices to show that the map $h \mapsto \xi(h)[v]$ is continuous at e. For this it suffices to show that the function $h \mapsto \bar{\beta}(\xi(h)[v], [v])$ is continuous at e. For this in turn it suffices to show that the map $h \mapsto \beta(\pi(h)v, v)$ is continuous at e.

The vector v is a linear combination of vectors of the form $\pi(f)w$ with $f \in C_c(G)$ and $w \in \mathcal{H}$. Thus, it suffices to establish the continuity for v of the form $\pi(f)w$. For such a v we have

$$\beta(\pi(h)v, v)\omega = \beta(\pi(L_h f)w, \pi(f)w)\omega = \mathrm{ev}_e B(\pi(L_h f)w, \pi(f)w).$$

The result now follows by application of Lemma 10.6. $\square$

We now come to the end of the proof of the imprimitivity theorem. Let $\pi^\xi := \mathrm{Ind}_H^G(\xi)$ and let P^ξ be the system of imprimitivity for π^ξ induced from ξ.

Lemma 10.10 *The pair (π^ξ, P^ξ) is unitarily equivalent to (π, P).*

Proof. We define the map $T : \mathcal{H}_0 \to C(G, V)$ as follows. If $v \in \mathcal{H}_0$, then

$$T(v)(g) = [\pi(g)^{-1}v], \qquad (g \in G).$$

It is now readily seen that

$$T(v)(gh) = \delta(h)^{-1/2}\xi(h)^{-1}T(v)(g),$$

and that T intertwines π with the left-regular representation L of G in $C(G, V)$. Thus, via the choice of the positive density ω on $\mathfrak{g}/\mathfrak{h}$ made above, T may be identified with a map from $\mathcal{H}_0$ into the space $\Gamma^0(G/H, \mathcal{V} \otimes \mathcal{D}^{1/2})$ of continuous sections of the bundle $\mathcal{V} \otimes \mathcal{D}^{1/2}$, see Section 9. Moreover, if $v, w \in \mathcal{D}$, then the continuous density $\langle Tv\, ,\, Tw\rangle_\xi$ on G/H is given by

$$\begin{aligned}
\langle Tv\, ,\, Tw\rangle_\xi(g) &= \bar{\beta}([\pi(g^{-1})v], [\pi(g^{-1})w])\omega \\
&= \beta(\pi(g^{-1})v, \pi(g^{-1})w)\omega \\
&= \mathrm{ev}_e L_g^{-1} B(v, w) = B(v, w)(g),
\end{aligned}$$

so that

$$\langle Tv\, ,\, Tw\rangle_\xi = B(v, w).$$

From this we infer that

$$\begin{aligned}
\langle Tv\, ,\, Tw\rangle &= \int_{G/H} \langle Tv\, ,\, Tw\rangle_\xi \\
&= \langle B(v, w)\, ,\, 1_{G/H}\rangle \\
&= \langle P(1)v\, ,\, w\rangle = \langle v\, ,\, w\rangle.
\end{aligned}$$

It follows that the map T is an isometry. It therefore has a unique extension to an isometry $\mathcal{H} \to \mathcal{H}^\xi := \Gamma_2(G/H, \mathcal{V} \otimes \mathcal{D}^{1/2})$, which intertwines π with π^ξ. It remains to be shown that the map T is surjective and interwines P with P^ξ. This requires some effort.

Lemma 10.11 *The map $\Phi : C_c(G/H) \otimes \mathcal{H} \to \mathcal{H}^\xi$ given by $\Phi(f \otimes v) = fTv$ has a dense image.*

Proof. It is easily verified that the map Φ intertwines the representations $L \otimes \pi$ and π^ξ of G. Assume its image not to be dense. Then the orthocomplement $\mathcal{W} := \text{image}(\Phi)^\perp$ is a closed G-invariant subspace of $\mathcal{H}^\xi$. Its space $\mathcal{W}_0$ of Gårding vectors is a non-trivial and G-invariant subspace of $\Gamma^0(G/H, \mathcal{V} \otimes \mathcal{D}^{1/2})$. By G-invariance there exists an non-trivial element $\varphi \in \mathcal{W}_0$ with $\varphi(e) \neq 0$. Now $\varphi(e) \in V$, hence there exists a $v \in \mathcal{H}_0$ such that $\bar{\beta}(\varphi(e), [v]) \neq 0$. It follows that the function $x \mapsto \langle \varphi(x) \, , \, Tv(x) \rangle_\xi = \bar{\beta}(\varphi(x), [\pi(x)v])$ is a continuous section of the density bundle on G/H, which is non-zero in a neighborhood of eH. This implies the existence of a continuous function $f \in C_c(G/H)$ with $\langle \varphi \, , \, fTv \rangle = 0$, contradiction. □

Lemma 10.12 *The isometry $T : \mathcal{H} \to \mathcal{H}^\xi$ is surjective and intertwines P with P^ξ.*

Proof. We consider the map $\Psi : C_c(G/H) \otimes \mathcal{H} \to \mathcal{H}^\xi$ defined by

$$\Phi(f, v) = T(P(f)v) - fTv.$$

By an easy calculation it follows that this map intertwines the representations $L \otimes \pi$ and π^ξ. We will show that the image of Φ is perpendicular to the image of the map Ψ defined above.

Let $f, g \in C_c(G/H)$, and let $v, w \in \mathcal{H}_0$. Then

$$\langle fTv \, , \, gTw \rangle_\xi = f\bar{g}B(v, w).$$

From this it follows that

$$\begin{aligned} \langle fTv \, , \, gTw \rangle &= \langle f\bar{g}B(v, w) \, , \, 1 \rangle \\ &= \langle B(v, w) \, , \, f\bar{g} \rangle \\ &= \langle P(f)v \, , \, P(g)w \rangle \\ &= \langle TP(f)v \, , \, TP(g)w \rangle. \end{aligned}$$

Applying this two times, we see that

$$\langle fTv \, , \, gTw \rangle = \langle P(f)v \, , \, P(g)w \rangle = \langle TP(f)v \, , \, gTw \rangle,$$

whence $fTv - T(P(f)v) \perp \text{im}(\Phi)$.

As we have seen, the image of Φ is dense in $\mathcal{H}^\xi$. It follows that $\Psi = 0$. This in turn implies that the image of Φ is contained in the image of T. Hence the image of T is dense. Since T is an isometry, this completes the proof. □

Completion of the proof of Thm. 10.3. We will now show that ξ is uniquely determined up to equivalence.

For this we assume that (η, W) is a unitary representation of H, π^η the associated induced representation and P^η the induced system of imprimitivity. Let $S : \mathcal{H} \to \mathcal{H}^\eta$ be an isometric isomorphism onto, intertwining the pairs (π, P) and (π^η, P^η). Then we must show that η is unitarily equivalent to the representation ξ constructed above.

Via the equivalence S we will identify the two systems (π, P) and (π^η, P^η), and describe the above construction of ξ in terms of π^η.

The space $\mathcal{H}_0$ of Gårding vectors for the latter representation is a dense subspace of the space $\Gamma^0(G/H, \mathcal{W} \otimes \mathcal{D}^{1/2})$ of continuous sections. Moreover, it is readily seen that for $v, w \in \mathcal{H}_0$, the continuous density $B(v, w)$ on G/H equals $\langle v \,,\, w \rangle_\eta$. In particular, $\beta(v, w)\omega = \mathrm{ev}_e B(v, w) = \langle v(e) \,,\, w(e) \rangle_\eta$, showing that the map $\mathrm{ev} : \mathcal{H}_0 \to W$, factors to an isometry $\varphi : \mathcal{H}/\ker\beta \to W$, whose image is dense by the above mentioned density of $\mathcal{H}_0$. Moreover, the map φ is readily seen to intertwine ξ with η. It follows that φ extends to a unitary equivalence of ξ with η.

We will finish the proof by showing that ξ is irreducible if and only if (π^ξ, P^ξ) is irreducible. Clearly, if ξ is reducible, then V admits an orthogonal direct sum decomposition $V = V_1 \oplus V_2$ into closed non-trivial subspaces. Let ξ_1, ξ_2 be the restrictions of ξ to V_1 and V_2, respectively. Let (π^1, P^1) and (π^2, P^2) be the induced systems of imprimitivity. Then it readily follows that (π^ξ, P^ξ) is equivalent to the orthogonal direct sum of (π^j, P^j), for $j = 1, 2$. It follows that (π^ξ, P^ξ) is reducible. Conversely, assume that (π^ξ, P^ξ) is reducible. Then this system admits an orthogonal direct sum decomposition in terms of non-trivial systems (π^1, P^1) and (π^2, P^2), both based on G/H. Each of the summands (π^j, P^j) is equivalent to a system induced by an irreducible representation ξ_j of H. Let $\eta = \xi_1 \oplus \xi_2$, then by what we just said, (π^η, P^η) is equivalent to the direct sum of the (π^j, P^j), hence equivalent to (π^ξ, P^ξ). By the already established uniqueness of ξ it follows that $\xi \simeq \eta \simeq \xi_1 \oplus \xi_2$. □

The imprimitivity theorem is due to G. Mackey [15], in the more general context of locally compact groups. In the same context, a simpler proof was given by Ørsted [18]. It is essentially this simpler proof that we have presented above. However, by working with the density bundle we avoided the use of quasi-invariant measures.

11 Representations of semi-direct products

In this section we assume that G is a Lie group and that N is an abelian closed normal subgroup. In addition, we assume that H is a closed subgroup such that the natural map $H \to G/N$ is an isomorphism of Lie groups. Equivalently, the latter means that the multiplication map $H \times N \to G$ is a diffeomorphism, exhibiting G as the semi-direct product

$$G \simeq H \ltimes N.$$

We will investigate the sructure of the irreducible representations of G with the help of the imprimitivity theorem. For simplicity of exposition we assume here that N is connected, so that the results of Section 8 become available. We will use the notation of that section, with N in place of A. As in the mentioned section we define Γ to be the kernel of $\exp : \mathfrak{n} \to N$ and define $\Gamma^\vee$ to be the subspace of $i\mathfrak{n}^*$ consisting of ν with $\nu(\Gamma) \subset 2\mathbb{Z}\pi i$. Then $\chi \mapsto \chi_*$ is a homeomorphism from $\widehat{N}$ onto $\Gamma^\vee$.

The group H acts on N by conjugation. This naturally induces an action of H on the space of characters $\widehat{N}$, given by the formula

$$h \cdot \chi(n) = \chi(h^{-1} n h).$$

This action in turn induces a representation $\hat{\alpha}$ of H in the space of functions on $\widehat{N}$ given by

$$\hat{\alpha}(h)\psi(\chi) = \psi(h^{-1} \cdot \chi).$$

Let $\Delta : H \to \,]\,0, \infty\,[$ be the function determined by the formula

$$\mathbb{A}\mathrm{d}(h)^* dn = \Delta(h)^{-1} dn.$$

We define a representation α of H in the space $L^1(N)$ by the formula

$$\alpha(h)\varphi(n) = \Delta(h)\varphi(h^{-1}nh).$$

We leave it to the reader to check that with these definitions,

$$(\alpha(h)\varphi)\hat{} = \hat{\alpha}(h)\hat{\varphi}, \tag{11.1}$$

for all $\varphi \in L^1(N)$ and $h \in H$. On the other hand, if $(\rho, \mathcal{H})$ is a unitary representation of N, then, for all $\varphi \in L^1(N)$,

$$\rho(\alpha(h)\varphi) = \rho(n)\rho(\varphi)\rho(n)^{-1}, \tag{11.2}$$

for all $h \in H$.

If $(\pi, \mathcal{H})$ is a unitary representation of the semi-direct product $G = H \ltimes N$ we let P_π denote the projection-valued measure on $\widehat{N}$ associated with $\pi|_N$ as in Theorem 8.2.

The following results express the fundamental importance of the notion of system of imprimitivity for the representation theory of the semidirect product $G = H \ltimes N$.

Proposition 11.1 *If $(\pi, \mathcal{H})$ is a unitary representation of $G = H \ltimes N$, then $(\pi|_H, P_\pi)$ is a system of imprimitivity for H in $\mathcal{H}$, based on $\widehat{N}$.*

Proof. We fix $h \in H$. Then using (11.1) and (11.2) we see that for all $f \in L^1(N)$,

$$\pi(h)P(\hat{f})\pi(h)^{-1} = \pi(h)\pi(f)\pi(h)^{-1} = \pi(\alpha(h)f) = P((\alpha(h)f)\hat{}\,) = P(\hat{\alpha}(h)\hat{f}).$$

Since the space of functions $\hat{f}$, $f \in L^1(N)$ is dense in the space of continuous functions on $\widehat{N}$ tending to zero at infinity, it follows that $\pi(h)P(\varphi)\pi(h^{-1}) = P(\hat{\alpha}(h)\varphi)$ for all such functions φ. In particular, the identity is valid for all $\varphi \in C_c(\widehat{N})$. With notation as in Section 4 it follows that for all $v, w \in \mathcal{H}$ the measures $\hat{\alpha}(h)^*\mu_{v,w}$ and $\mu_{\pi(h)v,\pi(h)w}$ are equal. Hence, P is a system of imprimitivity for $\pi|_H$. □

Proposition 11.2 *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then $(\pi|_H, P_\pi)$ is irreducible if and only if π is irreducible. The map $\pi \mapsto (\pi|_H, P_\pi)$ induces a bijection between the equivalence classes of irreducible unitary representations of G and the equivalence classes of systems of imprimitivity for H based on $\widehat{N}$.*

Proof. This result follows by a straightforward application of Theorem 8.2. □

By a section for the H-action on $\widehat{N}$ we mean a subset S of $\widehat{N}$ which intersects every H-orbit in precisely one point. We shall call a section *σ-compact* if it is a countable union of compact subsets of $\widehat{N}$. Such a section is Borel measurable, which is the weaker assumption under which G. Mackey proved the following result.

Theorem 11.3 (Mackey) *Let π be an irreducible unitary representation of $G = H \ltimes N$. Assume that the H-action on $\widehat{N}$ admits a σ-compact section S. Then there exists a unique single H-orbit O in $\widehat{N}$ such that $P = P_\pi$ is zero on $\widehat{N} \setminus O$.*

Proof. We consider the projection map $p : \widehat{N} \to S$ determined by $\{p(\xi)\} = H\xi \cap S$ for all $\xi \in \widehat{N}$. The group H admits a covering by countably many compact subsets $H_n, n \in \mathbb{N}$. If C is compact subset of S then the preimage $p^{-1}(C)$ equals $H \cdot C$, which is the union of the countable collection of compact sets $H_n C$, hence measurable. Since every closed subset of S is a countable union of compact sets, it follows that the preimage of every closed subset of S is measurable. The closed sets generate the Borel σ-algebra and p^{-1} is a morphism of σ-algebras. It follows that the projection map p is Borel measurable. One now readily checks that

$$Q := p_* P : E \mapsto P(p^{-1}E)$$

defines a projection-valued Borel measure on S. Let E be a measurable subset of S; from the H-invariance of $p^{-1}(E)$ it readily follows that $Q(E) = P(p^{-1}(E))$ is a $\pi|_H$-intertwining projection operator of $\mathcal{H}$. Since N is abelian, it follows that $Q(E)$ intertwines $\pi|_N$ as well. Therefore, $Q(E)$ is a an intertwining projection for $(\pi, \mathcal{H})$. By irreducibility of π we find that $Q(E)$ has only 0 and I as its possible values.

Let $\mathcal{B}_0$ denote the collection of Borel measurable subsets E of S with $Q(E) = 0$. If $E \subset F$ are measurable subsets of S and $F \in \mathcal{B}_0$ then $E \in \mathcal{B}_0$ as well. This is readily seen by using non-negativity of the measures $\mu_{Q,v,v} = \langle Q(\,\cdot\,)v\,,\, v\rangle$, for all $v \in \mathcal{H}$.

The space S has a countable basis $\{U_n\}_{n\in\mathbb{N}}$ of open sets. Let $\mathbb{N}_0$ be the collection of $n \in \mathbb{N}$ with $Q(U_n) = 0$. Let U be the union of the sets U_n, for $n \in \mathbb{N}_0$. Then U may be written as a countable disjoint union of measurable sets from $\mathcal{B}_0$. Hence, by countable additivity it follows that $Q(U) = 0$. By additivity it follows that $Q(S \setminus U) = I$.

In particular, $S \setminus U$ contains a point a. We claim that $S \setminus U = \{a\}$. Indeed, let $b \in S$, $b \neq a$. Then there exist $k, l \in \mathbb{N}$ such that $a \in U_k$, $b \in U_l$ and $U_k \cap U_l = \emptyset$. It follows that $Q(U_k)Q(U_l) = 0$ hence $k \in \mathbb{N}_0$ or $l \in \mathbb{N}_0$. Since $a \notin U$, we see that $l \in \mathbb{N}_0$ hence $b \in U$. This establishes the claim that $S \setminus U = \{a\}$. Let O be the H-orbit containing a. Then $P(O) = Q(\{a\}) = I$ and it follows that $P(\widehat{N} \setminus O) = 0$, by additivity. □

In the following we assume that π is an irreducible unitary representation of $G = H \ltimes N$ and that the associated projection valued measure $P = P_\pi$ is supported on a single H-orbit O in $\widehat{N}$. Let $\xi \in O$, then the map $G \to \widehat{N}, h \mapsto h \cdot \xi$ induces an injective immersion $\iota : G/G_\xi \hookrightarrow \widehat{N}$, with image O. The immersion is continuous, hence measurable. On the other hand, each closed subset of G/G_ξ is a countable union of compact sets. Therefore, the image of every closed subset of G/G_ξ is a measurable subset of O. It follows that ι is a bijection from the Borel measurable subsets of G/G_ξ onto the measurable subsets of O. This implies that $\iota^*(P) : S \mapsto P(\iota(S))$ defines a projection valued Borel measure on G/G_ξ.

Lemma 11.4 *The projection valued measure $\iota^* P$ on G/G_ξ is a system of imprimitivity for π, based on G/G_ξ.*

Proof. We put $Q = \iota^*(P)$. Let $g \in G$ and let $E \subset G/G_\xi$ be a measurable subset, then

$$\pi(g)Q(E)\pi(g)^{-1} = \pi(g)P(\iota(E))\pi(g)^{-1} = P(g\iota(E)) = P(\iota(gE)) = Q(gE).$$

Thus, (10.1) follows. It remains to be shown that the measure Q is regular. We may write G/G_ξ as a disjoint union of measurable subsets S_n, $(n \geq 1)$, with compact closure. On a neighborhood U_n of each S_n the map ι is an embedding of manifolds. This implies that the restriction of Q to U_n is regular. Let now $E \subset G/G_\xi$ be measurable and let $x \in \mathcal{H}$. Let $\varepsilon > 0$. Then for each $n \geq 1$ there exists a closed subset $C_n \subset E \cap S_n$ and an open subset $V_n \subset U_n$ containing $E \cap S_n$ such that $\|Q(V_n \setminus C_n)x\|^2 = \langle Q(V_n \setminus C_n)x\,,\, x\rangle < 2^{-n}\varepsilon$. The set E is the disjoint union of the sets $E_n := E \cap S_n$. Therefore, $Q(E)x = \sum_n Q(E_n)x$ as an orthogonal direct sum, and it follows that for some N we have $\sum_{n>N} \|Q(E_n)x\|^2 < \varepsilon$. Let $C = \cup_{n \leq N} C_n$ and let V be the union of the open sets V_n. Then V is open, C is closed and $C \subset E \subset V$. Moreover,

$$V \setminus C \subset \cup_{n\geq 1}(V_n \setminus C_n) \;\cup \cup_{n>N} E_n.$$

By countable additivity and positivity of the measure $\langle Q(\,\cdot\,)x\,,\, x\rangle = \|Q(\,\cdot\,)x\|^2$, it follows that

$$\|Q(V \setminus C)x\|^2 \leq \sum_{n\geq 1} \|Q(V_n \setminus C_n)x\|^2 + \sum_{n>N} \|Q(E_n)x\|^2 < \sum_n 2^{-n}\varepsilon + \varepsilon = 2\varepsilon.$$

We conclude that Q is regular. □

We keep the assumption that π is an irreducible unitary representation of $G = H \ltimes N$. Then $(\pi|_H, \iota^* P)$ is an irreducible system of imprimitivity for the group H, based on H/H_χ. It follows that there exists a unique irreducible representation ξ of H_χ such that $(\pi_H, \iota^* P)$ is unitarily equivalent to the induced system (π^ξ, P^ξ), where

$$\pi^\xi := \mathrm{Ind}_{H_\chi}^H \xi.$$

Moreover, $\mathcal{H}^\xi$ denotes the Hilbert space $\Gamma_2(H/H_\chi, \mathcal{V}_\xi \otimes \mathcal{D}^{1/2})$ in which ξ is realized. Let $T : \mathcal{H} \to \mathcal{H}^\xi$ be the unitary equivalence. Then T intertwines $\iota^* P$ with P^ξ. Via T we transfer the unitary representation π to a unitary representation ρ of G in $\mathcal{H}^\xi$. Then $\rho|_H = \pi^\xi$. Moreover, $\rho|_N$ can be retrieved from the system of imprimitivity P^χ as follows. An element $n \in N$ defines the function $\mathrm{ev}_n : \widehat{N} \to \mathbb{C}$ given by $\mathrm{ev}_n(\eta) = \eta(n)$. The representation $\rho|_N$ is connected with P by the formula

$$\rho(n) = \int_{\widehat{N}} \mathrm{i}(n)\, dP,$$

see (8.3). By transference under T we obtain that

$$\rho(n) = \int_{H/H_\chi} \iota^*(\mathrm{i}(n))\; d(\iota^* P) = \int_{H/H_\chi} \iota^*(\mathrm{i}(n))\; dP^\xi.$$

If E is a measurable subset of H/H_χ then $P^\xi(E)$ equals multiplication with 1_E on $\mathcal{H}^\xi = \Gamma_2(H/H_\chi, \mathcal{V}_\xi \otimes \mathcal{D}^{1/2})$. Thus, if ψ is a simple function on H/H_χ, then $\int_{H/H_\chi} \psi dP^\xi$ is multiplication by ψ, and by density and continuity the same holds for all bounded measurable functions ψ on H/H_χ. In particular, it follows that $\rho(n)$ is given by multiplication by the function $c(n,\,\cdot\,) := \iota^*(\mathrm{i}(n))$. The latter function is given by

$$c(n, gH_\chi) = (g\chi)(n) = \chi(g^{-1} n g). \tag{11.3}$$

We define the representation $\xi \otimes \chi$ of $G_\chi = H_\chi N$ in $\mathcal{H}_\xi$ by

$$\xi \otimes \chi(hn) = \chi(n)\xi(h).$$

Theorem 11.5 *Assume that π is an irreducible unitary representation of $G = H \ltimes N$ such that the associated system of imprimitivity P_π is supported by a single H-orbit O in $\widehat{N}$. Let $\chi \in O$. Then the representation π is unitarily equivalent to $\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi)$.*

Proof. The inclusion $j : H \to G$ induces a diffeomorphism $H/H_\chi \to G/G_\chi$. We leave it to the reader to verify that the Hilbert bundle on G/G_χ associated with the representation $\xi \otimes \chi$ of G_χ equals the tensor product of the bundles $\mathcal{V}_\xi$ and $\mathcal{V}_\chi$ associated with ξ and χ respectively. Moreover, the pull-back of $\mathcal{V}_\chi$ to H/H_χ is readily seen to be the trivial bundle, and the pull-back of $\mathcal{V}_\xi$ is the associated bundle on H/H_χ. Therefore, the pull-back by j induces a unitary equivalence of the H-representations

$$\mathcal{H}^\xi \simeq \mathcal{H}^{\xi\otimes\chi}.$$

We shall now calculate how the action of N on the second of these spaces transfers to an action on the first. Let $f \in \Gamma^0(G/G_\chi, \mathcal{V}_\xi \otimes \mathcal{D}^{1/2})$. Then f may be viewed as a continuous function on G with values in V_ξ, that transforms according to the rule

$$f(ghn) = \chi(n)^{-1}\xi(h)^{-1}\delta(nh)^{-1/2}f(g),$$

for $g \in G, h \in H_\chi, n \in N$. Under this identification and the similar one for the space $\mathcal{H}^\xi$ pull-back becomes restriction to H. We have

$$\pi^{\xi\otimes\chi}(n)f(g) = f(n^{-1}g) = f(g(g^{-1}ng)^{-1}) = \chi(g^{-1}ng)f(g).$$

This shows that in the space $\mathcal{H}^\xi$, the action of $\pi^{\xi\otimes\chi}(n)$ is given by multiplication by the function $c(\,\cdot\,, n)$ given in (11.3). We conclude that j^* induces a unitary equivalence of $\pi^{\xi\otimes\chi} \simeq \rho$ both as representations of H and of N. Whence the result. □

We can finally formulate the following result, due to Mackey. Again he proved it under the weaker assumption that the action of H on N allows a measurable section.

Theorem 11.6 (Mackey) *Let $G = H \ltimes N$ as above and assume that the action of H on N allows a σ-compact section. If $\chi \in \widehat{N}$ and $\xi \in \widehat{H}_\chi$, then*

$$\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi)$$

is an irreducible unitary representation of H. Every irreducible unitary representation of G is of this form. Finally, if $\eta \in \widehat{N}$, $\omega \in \widehat{H}_\eta$ then

$$\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi) \simeq \mathrm{Ind}_{G_\eta}^G(\omega \otimes \eta)$$

if and only if the data (χ, ξ) and (η, ω) are conjugate by an element of G.

Proof. The above calculations show that the induced representation $\pi^{\xi\otimes\chi}$ is completely determined by the induced system of imprimitivity P^ξ, which is irreducible by Theorem 10.3. It follows that the induced representation is irreducible.

That every irreducible representation of G is of this induced form has been proved in the beginning of this section.

For the latter assertion, we first assume the induced representations to be equivalent. The equivalence yields an equivalence of the systems of imprimitivity based on $\widehat{N}$, which are located on the orbits $G\chi$ and $G\eta$. These orbits must be equal, by the uniqueness statement of Theorem 10.3. It follows that $\eta = g\chi$ for some $g \in G$. This implies that $G_\eta = gG_\chi g^{-1}$. Let $g\xi$ be the representation of G_η defined by $g\xi(x) = \xi(g^{-1}xg)$. Then we must show that $g\xi \simeq \omega$. For this we observe that right multiplication by g induces a diffeomorphism

$$r_g : G/G_\eta \to G/G_\chi.$$

Moreover, pull-back by r_g induces a unitary equivalence $\pi^{\xi\otimes\chi} \to \pi^{g\xi\otimes\eta}$. Let $j_\eta : G/G_\eta \to G\eta$ be the natural map, and let j_χ be defined similarly. Then it is readily seen that $j_\chi \circ r_g = j_\eta$. It follows from this and the statement of Lemma 11.4 that pull-back by r_g induces an equivalence between the natural systems of imprimitivity. We have thus reduced to the situation that $\chi = \eta$ and it remains to be shown that in this situation, $\xi \simeq \omega$. Now this follows from the fact that the representations $\pi^{\xi,\chi}$ and $\pi^{\omega,\chi}$ are equivalent, hence have the same systems of imprimitivity on $\widehat{N}$ hence on G/G_χ. It follows from the imprimitivity theorem that $\xi \simeq \omega$.

In the course of the above arguments, we have also proved the following result.

Proposition 11.7 *Let $\chi \in \widehat{N}$ and let ξ be an irreducible unitary representation of H_χ. Then the induced representation $\pi = \mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi)$ is an irreducible representation of G. The restriction $\pi|_H$ is naturally unitarily equivalent to $\pi^\xi := \mathrm{Ind}_{H_\chi}^H(\xi)$. Moreover, let $j_*(P^\xi)$ denote the push-forward of P^ξ by the natural map $j : H/H_\chi \to \widehat{N}$. Then the system of imprimitivity $(\pi|_H, P_\pi)$ for H, based on $\widehat{N}$, associated with the representation π, is equivalent to $(\pi^\xi, j_*(P^\xi))$*

Exercise 11.8 Let G, H be Lie groups, and let (π, V) and (ρ, W) be unitary representations of G and H. The algebraic tensor product $V \otimes W$ of V and W carries the tensor product inner product. The Hilbert completion with respect to this inner product is denoted by $V\widehat{\otimes}W$. Show that the algebraic exterior tensor product $\pi \otimes \rho$, a representation of $G \times H$ in $V \otimes W$, extends to a (continuous) unitary representation of $G \times H$.

We now assume that H is abelian. Show that the tensor product representation $\pi\widehat{\otimes}\rho$ is irreducible. Conversely, show that each irreducible unitary representation of $G \times H$ is of tensor product form.

12 The structure of the Lorentz group

In this section we will discuss the structure of the Lorentz group. Given an element $x = (x_1, \dots, x_4)$ of $\mathbb{R}^4$, we agree to write $x' = (x_1, \dots x_3)$, so that $x = (x', x_4)$. We define the Lorentz inner product β on $\mathbb{R}^4$ by

$$\beta(x, y) = \langle x' \, , \, y' \rangle - x_4 y_4, \qquad (x, y \in \mathbb{R}^4),$$

where $\langle \cdot \, , \, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^3$.

The *Lorentz group* is defined to be the group $H := \mathrm{O}(3,1)$ of transformations $g \in \mathrm{GL}(4, \mathbb{R})$ that leave the form β invariant, i.e., $\beta(gx, gy) = \beta(x, y)$ for all $x, y \in \mathbb{R}^4$. Being a closed subgroup of $\mathrm{GL}(4, \mathbb{R})$, H is is a Lie group. We shall now describe its Lie algebra.

Let J be the 4×4 diagonal matrix whose first three diagonal entries are 1 and whose bottom diagonal entry equals -1. Then, for $x, y \in \mathbb{R}^4$,

$$\beta(x, y) = \langle x \, , \, Jy \rangle,$$

where $\langle \cdot\ ,\ \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^4$. It follows that an element $g \in \mathrm{GL}(4,\mathbb{R})$ belongs to H if and only if $a^t J a = J$, or, equivalently,

$$Ja^t J = a^{-1},$$

where a^t denotes the usual transposed. From this we see that the Lie algebra $\mathfrak{h} = \mathfrak{o}(3,1)$ consists of all matrices $X \in \mathrm{M}(4,\mathbb{R})$ with

$$JX^t J = -X.$$

From this in turn we easily infer that $\mathfrak{h}$ consists of all 4×4 matrices of the form

$$X = \begin{pmatrix} A & b \\ b^t & 0 \end{pmatrix}, \tag{12.1}$$

with $A \in \mathfrak{so}(3)$, i.e., $A^t = -A$ and $b \in R^3$. It follows from this description that the algebra $\mathfrak{h}$ is invariant under the Lie algebra involution $\theta : X \mapsto -X^t$ of $\lambda(4,\mathbb{R})$ (the so called standard Cartan involution). Let $\mathfrak{k} = \mathfrak{h} \cap \ker(\theta - I)$ and let $\mathfrak{p} = \mathfrak{h} \cap (\theta + I)$. Then $\mathfrak{k}$ is the subalgebra of $\mathfrak{h}$ consisting of matrices of the above form with $\mathfrak{b} = 0$ and $\mathfrak{p}$ consists of the matrices with $A = 0$. In particular,

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}, \tag{12.2}$$

as a direct sum of linear spaces. This decomposition Identifying $\mathfrak{so}(3) \simeq \mathfrak{k}$ and $\mathbb{R}^3 \simeq \mathfrak{p}$ in the obvious way, we see that the decomposition (12.2) is concretely given by

$$\mathfrak{so}(3) \times \mathbb{R}^3 \ni (A, b) \mapsto \begin{pmatrix} A & b \\ b^t & 0 \end{pmatrix}.$$

It is readily checked that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}]$. It follows from the given description that $\mathfrak{h}_c := \mathfrak{k} \oplus i\,\mathfrak{p}$ is isomorphic to the Lie algebra $\mathfrak{so}(4)$ of $\mathrm{SO}(4)$. In fact, a Lie algebra isomorphism $\mathfrak{so}(4) \to \mathfrak{h}_c$ is given by conjugation by the matrix with column vectors e_1, e_2, e_3, ie_4. The algebra $\mathfrak{so}(4)$ is semisimple, as it is compact and without center. Its Killing form is negative definite. We thus see that $\mathfrak{h}$ and $\mathfrak{so}(4)$ have the same complexification. It follows that the Killing form of $\mathfrak{h}$ is non-degenerate; hence $\mathfrak{h}$ is a semisimple Lie algebra as well. The decomposition (12.2) is known as a Cartan decomposition for $\mathfrak{h}$.

The map $\theta : a \mapsto (a^t)^{-1}$ is the standard Cartan involution of $\mathrm{GL}(4,\mathbb{R})$. The associated Cartan decomposition of $\mathrm{GL}(4,\mathbb{R})$ is given by

$$\mathrm{GL}(4,\mathbb{R}) = \mathrm{O}(4) \exp \mathfrak{s},$$

where $\mathfrak{s}$ denotes the space of symmetric 4×4-matrices. We recall from standard theory that the map $(k, X) \mapsto k \exp X$ is a diffeomorphism from $\mathrm{O}(4) \times \mathfrak{s}$ onto $\mathrm{GL}(4,\mathbb{R})$. If $g \in \mathrm{GL}(4,\mathbb{R})$, then the elements $X \in \mathfrak{s}$ and $k \in \mathrm{O}(4)$ are found from $\exp 2X = x^t x = \theta(x)^{-1} x$ and $k = g \exp(-X)$.

The Lorentz group H is invariant under θ. This implies that the Cartan decomposition induces the following Cartan decomposition of the Lorentz group H,

$$H = K \exp \mathfrak{p},$$

where $K = H \cap \mathrm{O}(3,1) = \mathrm{O}(3) \times \mathrm{O}(1)$, and where $\mathfrak{p} = \mathfrak{h} \cap \mathfrak{h}$. Moreover, the map $K \times \mathfrak{p} \to H$, $(k, X) \mapsto k \exp X$ is the restriction of the smooth diffeomorphism $\mathrm{O}(4) \times \mathcal{S} \to \mathrm{GL}(4,\mathbb{R})$ to a smooth submanifold, hence a diffeomorphism of its own right. We leave it to the reader to verify that the adjoint action of K on $\mathfrak{h}$ leaves $\mathfrak{p}$ invariant and corresponds to the natural action of $\mathrm{O}(3)$ on $\mathbb{R}^3$.

From the Cartan decomposition $H = K \exp \mathfrak{p}$ it follows that the connected components of H are in one-to-one correspondence with the connected components of K. In particular, it follows that $H^\circ = K^\circ \exp \mathfrak{p}$ where $K^\circ = \mathrm{SO}(3)$. Moreover, the group $K = \mathrm{O}(3) \times \mathrm{O}(1)$ has four connected components. Representatives for these components are $I, \sigma, \tau, \sigma\tau$, where

$$\sigma = \begin{pmatrix} -I_3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} I_3 & 0 \\ 0 & -1 \end{pmatrix},$$

The first of these matrices may be interpreted as space inversion, the second as time inversion.

It follows from the Cartan decomposition that $K^\circ = \mathrm{SO}(3)$ is a deformation retract of the connected Lorentz group $H^\circ = \mathrm{O}(3,1)^\circ = \mathrm{SO}(3,1)^\circ$. From this we see that the fundamental group of H° equals that of $\mathrm{SO}(3)$, which in turn equals $\mathbb{Z}_2$. It follows that the universal covering of H° is two-fold.

We will show that the universal covering group equals $\mathrm{SL}(2,\mathbb{C})$. For this, we will first show that $\mathrm{SL}(2,\mathbb{C})$ is simply connected. Next, we will explicitly define a surjective group homomorphism $\psi : \mathrm{SL}(2,\mathbb{C}) \to H^\circ$ with kernel $\{\pm I\}$.

The map $\theta : x \mapsto x^{*-1}$, where $*$ denotes the Hermitian adjoint, is the standard Cartan involution of $\mathrm{GL}(2,\mathbb{C})$, viewed as a real Lie group. The associated Cartan decomposition is given by

$$\mathrm{GL}(2,\mathbb{C}) = \mathrm{U}(2) \exp \mathcal{S},$$

where $\mathcal{S}$ denotes the space of Hermitian 2×2 complex matrices. By standard theory, the map $\mathrm{U}(2) \times \mathcal{S} \to \mathrm{GL}(2,\mathbb{C})$, $(u, X) \mapsto u \exp X$ is a diffeomorphism. Let $\mathcal{S}_0$ be the linear space of elements in $\mathcal{S}$ of trace zero. Then $\mathcal{S}_0 = \mathcal{S} \cap \mathfrak{sl}(2,\mathbb{C})$. The above map restricts to a diffeomorphism $\mathrm{SU}(2) \times \mathcal{S}_0 \to \mathrm{SL}(2,\mathbb{C})$. Since both $\mathrm{SU}(2)$ and $\mathcal{S}_0$ are simply connected, it follows that $\mathrm{SL}(2,\mathbb{C})$ is simply connected.

We will now define an explicit map $\psi : \mathrm{SL}(2,\mathbb{C}) \to H^\circ$. For this we consider the representation ξ of $\mathrm{SL}(2,\mathbb{C})$ in $\mathcal{S}$ given by the formula

$$\xi(x)A = xAx^*,$$

for $A \in \mathcal{S}$ and $x \in \mathrm{SL}(2,\mathbb{C})$. Here x^* denotes the Hermitian conjugate of the matrix x. The restriction of ξ to $\mathrm{SU}(2)$ equals the natural action of $\mathrm{SU}(2)$ on $\mathcal{S}$ by conjugation.

The associated infinitesimal representation ξ_* of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathcal{S}$ is given by the formula

$$\xi_*(X)A = XA + AX^*,$$

for $X \in \mathfrak{sl}(2,\mathbb{C})$ and $A \in \mathcal{S}$. We note that $\mathfrak{sl}(2,\mathbb{C}) = su(2) + isu(2)$ as a real linear space. If $X \in su(2)$, then $\xi_*(X)A = [X, A]$. If $X \in isu(2)$, then $\xi_*(X)A = XA + AX$. We will now fix a basis of $\mathfrak{sl}(2,\mathbb{C})$ and compute the action of the basis elements on $\mathcal{S}$. For $isu(2)$ we choose the basis consisting of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$, see Section 3. For $su(2)$ we choose the basis consisting of the matrices $S_j = -i\sigma_j$, for $j = 1, 2, 3$. We have

$$\mathcal{S} = i\, su(2) \oplus \mathbb{R}I,$$

hence may choose the basis $f_j := \sigma_j$, $(j = 1, 2, 3)$, and $f_4 := I$ for $\mathcal{S}$. By a straightforward computation it follows that, for $j = 1, 2, 3$,

$$\text{mat} \circ \xi_*(S_j) = \begin{pmatrix} 2R_j & 0 \\ 0 & 0 \end{pmatrix},$$

where R_j denotes the infinitesimal rotation in $\mathfrak{so}(3)$ about the x_j-axis. We conclude that

$$\text{mat} \circ \xi_*(su(2)) = \begin{pmatrix} \mathfrak{so}(3) & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, $\xi_*(\sigma_j)\sigma_k = \sigma_j\sigma_k + \sigma_k\sigma_j$ equals 0 for $k \neq j$, and $2I = 2f_4$ for $k = j$. Finally, $\xi_*(\sigma_j)f_4 = \sigma_j I + I\sigma_j = 2\sigma_j = 2f_j$. From this we conclude that

$$\text{mat} \circ \xi_*(\sigma_j) = \begin{pmatrix} 0 & e_j \\ e_j^t & 0 \end{pmatrix},$$

for $j = 1, 2, 3$. This implies that mat $\circ \xi_*(isu(2))$ consists of the matrices of the form

$$\begin{pmatrix} 0 & b \\ b^t & 0 \end{pmatrix},$$

with $b \in \mathbb{R}^3$. It follows from these computations that mat $\xi_*(\mathfrak{sl}(2, \mathbb{C}))$ consists of all matrices of the form

$$\begin{pmatrix} A & b \\ b^t & 0 \end{pmatrix},$$

with $A \in \mathfrak{so}(3)$ and $b \in \mathbb{R}^3$. We thus see that the map mat $\circ \xi_*$ is a Lie algebra isomorphism from $\mathfrak{sl}(2, \mathbb{C})$ onto $\mathfrak{so}(3, 1)$. Using the commutative diagram

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{C}) & \stackrel{\text{mat} \circ \xi}{\longrightarrow} & \mathrm{GL}(4,\mathbb{R}) \\ \uparrow & & \uparrow \\ \mathfrak{sl}(2,\mathbb{C}) & \stackrel{\text{mat} \circ \xi_*}{\longrightarrow} & \mathfrak{so}(3,1), \end{array}$$

where the vertical arrows represent the exponential maps, we see that mat $\circ \xi$ maps $\mathrm{SL}(2, \mathbb{C})$ onto the identity component $\mathrm{SO}(3, 1)^\circ$.

Lemma 12.1 *The group* $\mathrm{SL}(2, \mathbb{C})$ *is simply connected. Moreover, the map* $\psi :=$ mat $\circ \xi$ *constructed above is a surjective group homomorphism* $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, 1)^\circ$ *with kernel* $\{\pm I\}$.

It follows from the lemma that the homomorphism ψ establishes a two-fold covering of $\mathrm{SO}(3, 1)^\circ$ by $\mathrm{SL}(2, \mathbb{C})$.

Proof. The simple connectedness of $\mathrm{SL}(2, \mathbb{C})$ and the surjectivity of ψ were established above. From this it follows by general It remains to determine the kernel of ψ. This is most easily done by using the Cartan decompositions. Let $x \in \mathrm{SL}(2, \mathbb{C})$ and assume that $\psi(x) = I$. Write $x = u \exp X$ with $u \in \mathrm{SU}(2)$ and $X \in \mathcal{S}_0$. Then $e = \xi(x) = \xi(u) \exp(\xi_*(X))$. Since $\xi(u) \in K^\circ = \mathrm{SO}(3)$ and $\xi_*(X) \in \mathfrak{p}$, it follows that the latter decomposition of $\xi(x)$ is compatible with the Cartan decomposition $H^\circ = K^\circ \exp \mathfrak{p}$. From this we conclude that $X = 0$ and $\xi(u) = e$. Now $\xi|_{\mathrm{SU}(2)} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the standard double cover, with kernel $\{\pm I\}$. Hence, $u = \pm I$. □

13 The Poincaré group

The Lorentz group $H = \mathrm{O}(3,1)$ naturally acts on $\mathbb{R}^4$ by transformations that preserve the linear, hence the additive group structure. We define the Poincaré group G as the associated semidirect product of Lie groups,

$$G = \mathrm{O}(3,1) \ltimes \mathbb{R}^4.$$

Let M be $\mathbb{R}^4$ viewed as a flat pseudo-Riemannian manifold for the pseudo-Riemannian structure induced by the Lorentzian form β at every point. We consider the group $\mathrm{Aut}\,(M)$ of smooth automorphisms of M. Via the natural action on $\mathbb{R}^4$, the Lorentz group becomes embedded in M. If $a \in \mathbb{R}^4$, we use the notation $T_a : x \mapsto x + a$ for the translation by a. The map $a \mapsto T_a$ defines an embedding of $\mathbb{R}^4$ into $\mathrm{Aut}\,(M)$. The two embeddings $\mathrm{O}(3,1) \to \mathrm{Aut}\,(M)$ and $\mathbb{R}^4 \to \mathrm{Aut}\,(M)$ induce a group homomorphism $\mathrm{O}(3,1) \ltimes \mathbb{R}^4 \to \mathrm{Aut}\,(M)$. The following lemma asserts that the Poincaré group should be considered as the symmetry group of special relativity.

Lemma 13.1 *The natural map* $\mathrm{O}(3,1) \ltimes \mathbb{R}^4 \to \mathrm{Aut}\,(M)$ *is an isomorphism.*

Proof. Injectivity of the map is obvious. It remains to show that the natural map is surjective. For this let $A \in \mathrm{Aut}\,(M)$ and let $a = A^{-1}(0)$. Then $A \circ T_a$ stabilizes the origin 0. It follows that the tangent map $h := T_0(A \circ T_a)$ preserves the Lorentzian form, hence belongs to $\mathrm{O}(3,1)$. Viewing h as an element of $\mathrm{Aut}\,(M)$, we obtain that $h^{-1} \circ A \circ T_a$ is an automorphism of M that fixes the origin 0 and has tangent map equal to the identity at 0. In local exponential coordinates it is seen that $h^{-1} \circ A \circ T_a = I$. By repeated use of local exponential coordinates it follows that this identity holds on an open and closed subset of M, hence everywhere on M. We conclude that $A = hT_{-a}$. □

Let $\psi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,1)^\circ$ be the universal covering of the connected Lorentz group of the previous section. Then via ψ, the group $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{R}^4$. We denote the corresponding semi direct product group by $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$. It follows that the map

$$\tau \times I : \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4 \to \mathrm{SO}(3,1)^\circ \ltimes \mathbb{R}^4$$

realizes the universal covering of the connected Poincaré group G°.

14 Projective representations of the Poincaré group

In this section all Lie algebras and linear spaces are assumed to be defined over the ground field $\mathbb{R}$, unless specified otherwise. If $\mathfrak{g}$ is Lie algebra, then by a $\mathfrak{g}$-module we mean a real linear space V equipped with a representation of $\mathfrak{g}$. We will denote the associated bilinear map $\mathfrak{g} \times V \to V$ by $(X, v) \mapsto Xv$. The fact that V is a $\mathfrak{g}$-module is reflected by the rule that $XYv - YXv = [X,Y]v$ for all $X, Y \in \mathfrak{g}$ and $v \in V$. We note that the real linear dual V^* is a $\mathfrak{g}$-module as well; the module structure is defined by $X\lambda(v) = \lambda(-Xv)$, for $\lambda \in V^*$, $X \in \mathfrak{g}$ and $v \in V$. If V and W are two $\mathfrak{g}$-modules, then $V \otimes V$ is a $\mathfrak{g}$-module and so is its quotient $\wedge^2 V$. Note that $X(v_1 \wedge v_2) = Xv_1 \wedge v_2 + v_1 \wedge Xv_2$. Accordingly, it makes sense to define the subspace

$$(\wedge^2 V)^{\mathfrak{g}} := \{\omega \in \wedge^2 V \mid \mathfrak{g}\,\omega = 0\}.$$

If V is $\mathfrak{g}$-module over $\mathbb{R}$, then V may be viewed as a real Lie algebra, on which $\mathfrak{g}$ acts by derivations. Accordingly, the Lie algebra semi-direct product $\mathfrak{g} \ltimes V$ is defined by the following requirements. As a linear space, it is the direct sum of $\mathfrak{g}$ and V. Moreover, $\mathfrak{g}$ is a subalgebra, $\mathfrak{g} \ltimes V$ is the direct sum of $\mathfrak{g}$-modules, and finally, V is required to be an abelian subalgebra. Thus, the commutator structure of $\mathfrak{g} \ltimes V \simeq \mathfrak{g} \times V$ is given by $[(X, v), (Y, w)] = ([X, Y], Xw - Yv)$.

The following result is of crucial importance for our discussion of projective representations of the Lorentz group.

Proposition 14.1 *Let $\mathfrak{g}$ be a semisimple Lie algebra, and V a finite dimensional $\mathfrak{g}$-module. If $(\wedge^2 V^*)^{\mathfrak{g}}$ is trivial, then $H^2(\mathfrak{g} \ltimes V) = 0$.*

Before we give a proof of this proposition, we recall the fact that a finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if every finite dimensional $\mathfrak{g}$-module is completely reducible. By this we mean that for every submodule $\mathfrak{a}$ of a finite dimensional $\mathfrak{g}$-module V there exists a $\mathfrak{g}$-invariant complementary space $\mathfrak{b}$. Equivalently, this means that every short exact sequence

$$0 \to \mathfrak{a} \to \mathfrak{b} \xrightarrow{p} \mathfrak{c} \to 0$$

of finite dimensional $\mathfrak{g}$-modules splits, i.e., there exists a $\mathfrak{g}$-module homomorphism $\varphi : \mathfrak{c} \to \mathfrak{b}$ such that $p \circ \varphi = 1_{\mathfrak{c}}$.

Proof of Proposition 14.1. We write $\mathfrak{s} = \mathfrak{g} \ltimes V$. Let $\omega : \mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$ be a closed alternating two-form. We must show that ω is exact.

It is readily seen that the restriction $\omega_{\mathfrak{g}}$ of ω to $\mathfrak{g} \times \mathfrak{g}$ is a closed alternating two-form on $\mathfrak{g}$. Since $H^2(\mathfrak{g}) = 0$, there exists a linear functional $\lambda \in \mathfrak{g}^*$ such that $\omega_{\mathfrak{g}} = d\lambda$. We extend λ to a linear functional on $\mathfrak{s}$ by triviality on V. Then $\omega_1 := \omega - d\lambda$ represents the same class in $H^2(\mathfrak{s})$ as ω, and has trivial restriction to $\mathfrak{g} \times \mathfrak{g}$.

We now investigate the map $\mathfrak{g} \times V \to \mathbb{R}$ given by $(X, v) \mapsto \omega_1(X, v)$. For $X \in \mathfrak{g}$ we define the linear map $\rho(X) \in \mathrm{End}(\mathbb{R} \oplus V)$ by

$$\rho(X)(t, v) = (\omega_1(X), Xv).$$

It follows from the closedness of the form ω_1 that ρ defines a representation of $\mathfrak{g}$ in $\mathbb{R} \oplus V$. Moreover, the following sequence is a short exact sequence of $\mathfrak{g}$-modules,

$$0 \to \mathbb{R} \to \mathbb{R} \oplus V \to V \to 0,$$

where the first map is inclusion on the first component, and the second map projection onto the second. By what we said just before the beginning of the proof, the sequence splits. Let $\varphi : V \to \mathbb{R} \oplus V$ be a splitting map. Then $\varphi = (\mu, I_V)$, where $\mu \in V^*$. The fact that φ is a homomorphism implies that $\mu([X, v]) = \omega_1(X, v)$ for all $X \in \mathfrak{g}$ and $v \in V$. We extend μ to a linear functional on $\mathfrak{g} \ltimes V$ by requiring it to be trivial on $\mathfrak{g}$. Then $d\mu = 0$ on $\mathfrak{g} \times \mathfrak{g}$. Moreover, $\omega_1 - d\mu = 0$ on $\mathfrak{g} \times V$. It follows that

$$\omega_2 := \omega_1 - d\mu = \omega - d\lambda - d\mu$$

is a closed alternating two-form on $\mathfrak{s} = \mathfrak{g} \ltimes V$ that vanishes on $\mathfrak{g} \times \mathfrak{s}$. It follows that ω_2 is completely determined by its restriction $\bar{\omega}_2$ to $V \times V$. Let $X \in \mathfrak{g}$ and $v, w \in V$. Identifying these elements with their images in $\mathfrak{s}$, we have, by closedness of ω_2, that

$$\bar{\omega}_2(Xv, w) + \bar{\omega}_2(v, Xw) = \bar{\omega}_2([X, v], w) + \bar{\omega}_2(v, [X, w]) = \omega_2(X, [v, w]) = 0.$$

It follows that $\bar{\omega}_2$ belongs to $(\wedge^2 V^*)^{\mathfrak{g}}$ hence equals zero, so that $\omega_2 = 0$. We conclude that $\omega = d\lambda + d\mu$. □

We now turn our attention to the Poincaré group $\mathrm{O}(3,1) \ltimes \mathbb{R}^4$. Its Lie algebra equals the semidirect product $\mathfrak{so}(3,1) \ltimes \mathbb{R}^4$, where the representation of $\mathfrak{so}(3,1)$ in $\mathbb{R}^4$ is the natural one.

Lemma 14.2 $H^2(\mathfrak{so}(3,1) \ltimes \mathbb{R}^4) = 0$.

Proof. Since $\mathfrak{so}(3,1)$ is semisimple, it suffices to verify the condition of Proposition 14.1. Let ω be an alternating two form on $\mathbb{R}^4$ that is $\mathfrak{so}(3,1)$-invariant. Let β be the Lorentzian inner product on $\mathbb{R}^4$. Then by non-degenerateness of β, there exists a unique linear map $T \in \mathrm{End}(\mathbb{R}^4)$ such that

$$\omega(v,w) = \beta(Tv, w),$$

for all $v, w \in \mathbb{R}^4$. By invariance of ω and β it follows that T commutes with the action of $\mathfrak{so}(3,1)$. From this it follows that $T = cI$ for a scalar $c \in \mathbb{R}$. It follows that $\omega = c\beta$. Since ω is anti-symmetric and $\beta(e_1, e_1) > 0$, it follows that $c = 0$. Thus, $\omega = 0$. □

We recall that a projective representation of a Lie group G in a complex Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ such that the action map $G \times \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ is continuous. Moreover, The projective representation π is called irreducible if there is no non-zero proper closed subspace V of $\mathcal{H}$ such that $\mathbb{P}(V)$ is $\pi(G)$-invariant.

Theorem 14.3 *Let $\mathcal{H}$ be complex Hilbert space. Every projective representation $\pi : \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4 \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ lifts to a unique unitary representation $\widetilde{\pi}$ of $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$ in $\mathcal{H}$. Moreover, π is irreducible if and only if $\widetilde{\pi}$ is irreducible.*

Proof. The fact that a lifting $\tilde{\pi}$ exists follows from Corollary 3.12 combined with Lema 14.2 Assume that ρ is a second lifting. Then there is a unique map $\tau : \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4 \to \mathbb{T}$ such that $\tilde{\pi}(x) = \rho(x)\varphi(x)$, for all x. Since $\mathbb{T}$ is central in $\mathrm{U}^1(\mathcal{H})$, it follows that φ is a homomorphism of Lie groups. The restriction $\bar{\varphi}$ of φ to $\mathrm{SL}(2,\mathbb{C})$ is trivial, since $\mathrm{SL}(2,\mathbb{C})$ has no finite dimensional unitary representations, except for the trivial one (the reason is that $\mathfrak{sl}(2,\mathbb{C})$ has no compact ideals). Here we could also reason directly as follows. The derivative $\bar{\varphi}_*$ is a Lie algebra homomorphism $\mathfrak{sl}(2,\mathbb{C}) \to \mathbb{R}$. As a real Lie algebra, $\mathfrak{sl}(2,\mathbb{C})$ is simple. Hence, $\ker\bar{\varphi} = 0$. It follows that $\varphi = 1$ on $\mathrm{SL}(2,\mathbb{C})$. The subspace $\ker\varphi_* \cap \mathbb{R}^4$ is $\mathrm{SO}(3,1)$-invariant, hence must be all of $\mathbb{R}^4$. It follows that $\varphi_* = 0$ hence $\varphi = 1$ on $G := \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$.

Finally, if $V \subset \mathcal{H}$ is closed, then V is $\widetilde{\pi}(G)$-invariant if and only if $\mathbb{P}(V)$ is $\pi(G)$-invariant. Thus, π is irreducible if and only $\widetilde{\pi}$ is. □

Corollary 14.4 *Every irreducible unitary representation of $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$ in $\mathcal{H}$ naturally induces an irreducible projective representation of $\mathrm{SO}(3,1)^\circ \ltimes \mathbb{R}^4$ in $\mathcal{H}$. This sets up a bijective correspondence between the irreducible projective representations of the connected Poincaré group and the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$.*

Proof. We write $G = \mathrm{SO}(3,1)^\circ \ltimes \mathbb{R}^4$, and $\widetilde{G}$ for its universal cover $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$. We recall that the covering homomorphism $\psi : \widetilde{G} \to G$ has kernel $\{(\pm I, 0)\}$, which also equals the center of $\widetilde{G}$. It follows that for any irreducible unitary representation $\tilde{\pi}$, the group $\ker\psi$ acts by scalars. Therefore, the representation $\tilde{\pi}$ induces a projective representation $\pi : \widetilde{G} \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$ which is trivial on $\ker\psi$, hence factors to a projective representation $\bar{\pi}$ of G. Conversely, let $\bar{\pi}$ be an irreducible projective representation of G, then $\pi = \bar{\pi} \circ \psi$ is an irreducible projective representation of $\widetilde{G}$, which is induced by its unique unitary lift $\tilde{\pi}$. □

15 Orbits for the Lorentz group

We will now investigate the orbit structure for the natural action of the Lorentz group $H = \mathrm{O}(3,1)$ on $\mathbb{R}^4$. In particular, we will determine the H° and hence also the $\mathrm{SL}(2,\mathbb{C})$-orbits. As a preparation, we introduce the one dimensional linear space $\mathfrak{a} := \mathbb{R}Y$, with

$$Y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. By a simple computation, it follows that

$$\exp tY = \begin{pmatrix} \cosh t & & \sinh t \\ & I & \\ \sinh t & & \cosh t \end{pmatrix}.$$

This shows that $A := \exp \mathfrak{a}$ is a closed vectorial subgroup of H.

Exercise 15.1 Investigate the relation of the matrix $\exp tY$ with the Lorent boost.

We start by considering the action of the connected component H°. For $c \in \mathbb{R}$ we define the sets

$$X_c := \{x \in \mathbb{R}^4 \mid \beta(x,x) = c\}.$$

Since β is H-invariant, the sets X_c are H-invariant.

If $c < 0$ then all points $x = (x', x_4) \in X_c$ satisfy $x_4 \neq 0$. In this case X_c is the disjoint union of the two subsets

$$X_c^\pm := \{x \in X_c \mid \pm x_4 > 0\}.$$

Clearly, each of the sets $X_c^\pm$ is open and closed in X_c; therefore, H° leaves both of them invariant.

Lemma 15.2 *Let $c < 0$. Put $c := -m^2$ with $m > 0$, and $v^\pm := \pm m e_4$. Then for each choice of sign, the map $h \mapsto hv^\pm$ is a surjection from H° onto $X_c^\pm$, which induces a diffeomorphism $H^\circ/\mathrm{SO}(3) \simeq X_c^\pm$. Finally, $X_c = Hv^+$.*

Proof. Fix the sign $\varepsilon \in \{+,-\}$. We will first show that the associated map is a surjection. Since $\exp tY\, e_4 = \sinh t\, e_1 - \cosh t\, e_4$, we see that Av^ε equals the set of points $(y_1, 0, 0, y_4)$ with $\varepsilon y_4 > 0$ and $y_1^2 - y_4^2 = -c$.

Let $x \in X_c^\varepsilon$. Then $x = (x', x_4)$, with $\|x'\|^2 - x_4^2 = c$ and $\varepsilon x_4 > 0$. Let $y_1 := \|x'\|$ and $y_4 = x_4$. Then $x' = k'(y_1, 0, 0)$ for an element $k' \in K^\circ = \mathrm{SO}(3)$ and $x_4 = y_4$. It follows that $x \in K^\circ(y_1, 0, 0, y_4) \subset K^\circ A v^\varepsilon$. Thus, $K^\circ \times A \to X_c^\varepsilon$, $(k, a) \mapsto kav^\varepsilon$ is surjective. Hence, $h \mapsto hv^\varepsilon$ is surjective.

The stabilizer of v^ε in H° equals $\mathrm{SO}(3)$, naturally embedded in the upper left 3×3 corner. Thus, the map $h \mapsto hv^\varepsilon$ induces an immersion of $H/\mathrm{SO}(3)$ onto X_c^ε which is a diffeomorphism onto the closed submanifold X_c^ε. The final assertion follows from the fact that time inversion τ maps v^+ onto v^-. $\square$

It follows from the above result that

$$H^\circ = \mathrm{SO}(3) A\, \mathrm{SO}(3).$$

Indeed, if $h \in H^\circ$, then $he_4 = kae_4$ for some $k \in K^\circ$ and $a \in A$. It follows that $h = kal$ with l in the stabilizer of of e_4 in H°, which equals $\mathrm{SO}(3)$.

Lemma 15.3 *Let $c > 0$. Put $c := m^2$ with $m > 0$, and $v := me_1$. Then the map $h \mapsto hv$ is a surjection from H° onto X_c, which induces a diffeomorphism $H^\circ/\mathrm{SO}(2,1)^\circ \simeq X_c$. In particular, $Hv = X_c$.*

Proof. We first show that the map is a surjection. Since $\exp tY\, e_1 = \cosh t\, e_1 + \sinh t\, e_4$, we see that Av equals the set of points $(y_1, 0, 0, y_4)$ with $y_1 > 0$ and $y_1^2 - y_4^2 = c$. Let $x \in X_c$ and put $y_1 = \|x'\|$ and $y_4 = x_4$. Then $x' = k(y_1, 0, 0')$ for an element $k \in \mathrm{SO}(3)$ and we see that $x \in K^\circ(y_1, 0, 0, y_4)$. Since $y_1^2 - y_4^2 = \beta(x, x) = c$, the element $(y_1, 0, 0, y_4)$ equals $\exp tYv$ for a unique $t \in \mathbb{R}$. We conclude that $X_c = K^\circ Av$. In particular, $X_c = H^\circ v$. The stabilizer of e_1 in H equals $O(2,1)$, embedded in the 3×3 lower right corner. The intersection of this group with $K^\circ = \mathrm{SO}(3) \times \{1\}$ equals $\mathrm{SO}(2)$, which is also the intersection of $\mathrm{SO}(1,2)$ with K°. In view of the Cartan decomposition it follows that $\mathrm{SO}(1,2)^\circ$ equals the stabilizer of e_1 in H°. The result follows. □

Note that it follows from the above proof that

$$H^\circ = \mathrm{SO}(3) A\, \mathrm{SO}(2,1)^\circ.$$

In the following we shall use the notation

$$X_0^\pm := \{x \in X_0 \mid \pm x_4 > 0\}.$$

Thus, X_0^+ is the forward and X_0^- the backward light cone.

Lemma 15.4 *The set X_0 decomposes into the three H°-orbits $\{0\}$ and $X_0^\pm$. Let $v^\pm = e_1 \pm e_4$. Then the map $h \mapsto hv^\pm$ induces a diffeomorphism from H°/L onto $X_0^\pm$, with L isomorphic to $\mathrm{SO}(2) \ltimes \mathbb{R}^2$, the Euclidean motion group of $\mathbb{R}^2$. Finally, $X_0^+ \cup X_0^-$ is a single H-orbit.*

Proof. Clearly $\{0\}$ is an H° orbit. If $x \in X_0 \setminus \{0\}$ then $\|x'\|^2 = x_4^2$ and $x \neq 0$ hence $x_4 \neq 0$. It follows that x belongs to X_0^+ or X_0^-. We see that X_0 is the disjoint union of the sets $\{0\}$ and $X_0^\pm$. Clearly, $X_0^\pm$ are both open and closed in $X \setminus \{0\}$, hence invariant under H°.

Fix a sign $\varepsilon \in \{+, -\}$. We will show that $X_0^\varepsilon = H^\circ v^\varepsilon$. We start by observing that $\exp tYv^\varepsilon = e^t v^\varepsilon$, hence $Av^\varepsilon = \mathbb{R}_{>0} v^\varepsilon$. Let $x \in X_0^\varepsilon$. Then $\varepsilon x_4 > 0$ and $\|x'\|^2 = x_4^2 = e^{2t}$ for a unique $t = t(x) \in \mathbb{R}$ that depends smoothly on x. There exists a $k' \in K = \mathrm{SO}(3)$ such that $x = k' e^t e_1$. Clearly, k' is uniquely determined modulo the stabilizer $K_L \simeq \mathrm{SO}(2)$ of e_1 in K and the map $\bar{k}' : x \mapsto k' K_L$ is smooth from X_0^ε onto K/K_L. We see that $x = k' \exp tY(e_1 + \varepsilon e_4)$, whence $X_0^\varepsilon = KAv^\varepsilon$.

We will now determine the stabilizer L of v^ε in H°. Clearly, $K_L \subset L$, and K_L commutes with A. It follows from the above computations that the map $H^\circ \to K_L \backslash H^\circ$ given by $h \mapsto [\bar{k}'(hv^\varepsilon), \exp t(hv^\varepsilon)]^{-1} h$ is a smooth map onto the image of $L_K \backslash L$ in H°. From this we conclude that $L_K \backslash L$ hence L is connected.

The Lie algebra $\mathfrak{l}$ of L equals the set of $X \in \mathfrak{h}$ with $Xv^\varepsilon = 0$. It is straightforward to check that $\mathfrak{l}$ consists of the matrices of the form (12.1) with $Ae_1 = -b$ and $b_1 = 0$. Clearly,

$\mathfrak{l} \cap \mathfrak{k}$ consists of such matrices with $b = 0$ and $Ae_1 = 0$, hence equals $\{0\} \times \mathfrak{so}(2) \times \{0\}$. As a complementary subspace we may take the space $\mathfrak{l}_v$ consisting of $X \in \mathfrak{l}$ with $A_{ij} = 0$ for $i, j \geq 0$. This space is spanned by the matrix X_2 which occurs for $b = e_2$ and by the matrix X_3 which occurs for $b = e_3$. One readily checks that X_2 and X_3 commute. The conjugation action of $K = \mathrm{SO}(3)$ on $\mathfrak{p}$ corresponds to the natural action on $\mathbb{R}^3$. It follows that the conjugation action of K_L on $\mathfrak{l}_v = \mathbb{R}X_2 \oplus \mathbb{R}X_3 \simeq \mathbb{R}^2$ corresponds to the natural action of $\mathrm{SO}(2)$ on $\mathbb{R}^2$. Since L is connected, we deduce that $L = K_L \exp \mathfrak{l}_v \simeq \mathrm{SO}(2) \ltimes \mathbb{R}^2$. The final assertion follows from the observation that time inversion τ maps v^+ onto v^-. □

16 Wigner's classification

We now turn to the physical application we had in mind all along, and which motivated E. Wigner's milestone paper [21]. Consider a free elementary particle in flat space time (the context of special relativity). Its quantum mechanical state space is $\mathbb{P}(\mathcal{H})$ with $\mathcal{H}$ a complex Hilbert space. The symmetry group of space-time must be an open subgroup S of the Poincaré group, and since the laws of physics should be the same in every inertial frame, it is reasonable to expect that change of frame and change of state space should be S-equivariantly related. This leads to a group homomorphism $\pi : S \to \mathrm{Aut}\,(\mathbb{P}(\mathcal{H}))$, a projective representation of S in $\mathcal{H}$. The elementary nature of the particle leads to the assumption that the representation should be irreducible.

At this point we shall restrict ourselves to the classification of the irreducible projective representations of the connected Poincaré group. The classification of the irreducible representations of a non-connected open subgroup of the Poincaré group is an interesting problem, which has physical relevance in connection with parity (invariance under space inversion). For these matters, we refer to the literature.

We shall now give the description of the irreducible projective representations of the connected Poincaré group. By Corollary 14.4 they are the representations naturally induced by the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$. In view of Theorem 11.6 these are classified by two parameters. The first of these is a representative ξ of an orbit in $\widehat{\mathbb{R}}^4 \simeq i(\mathbb{R}^4)^*$. Via the Lorentz inner product we define a real linear isomorphism $v \mapsto \xi_v$, $\mathbb{R}^4 \to i(\mathbb{R}^4)^*$ by

$$\xi_v(x) = e^{i\beta vx}, \qquad (x \in \mathbb{R}^4).$$

By invariance of β under the Lorentz group, the linear isomorphisms $v \mapsto \xi_v$ intertwines the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{R}^4$ determined by $\psi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,1)^\circ$ with the induced action on the character space.

Accordingly, for the ξ we may take ξ_v where v runs through a collection of representatives of the $\mathrm{SO}(3,1)^\circ$-orbits in $\mathbb{R}^4$. According to the previous section, as a set $\mathcal{R}$ of such representatives we take $\mathcal{R} = \mathcal{R}_4^+ \cup \mathcal{R}_4^- \cup \mathcal{R}_L \cup \mathcal{R}_1 \cup \{0\}$, where

$$\mathcal{R}_4^\pm = \{\pm m e_4 \mid m > 0\}, \quad \mathcal{R}_L = \{e_1 + e_4, e_1 - e_4\}, \quad \mathcal{R}_1 = \{m e_1 \mid m > 0\}.$$

The set $\mathcal{R}$ clearly gives a σ-compact section for the $\mathrm{SL}(2,\mathbb{C})$-actin on $\widehat{\mathbb{R}}^4$, so that the hypothesis of Theorem 11.6 is fulfilled. For each representative v we denote the stabilizer in $\mathrm{SL}(2,\mathbb{C})$ by $\widetilde{H}_v$, In the physics literature, this group is called the *little group,* after Wigner.

The second classification parameter is an element of $\widehat{\widetilde{H}}_v$, the set of equivalence classes of irreducible unitary representations of $\widetilde{H}_v$. From the description of the orbits given in the previous section, it follows that the little group $\widetilde{H}_v$ equals

(a) $\mathrm{SU}(2)$ for $v \in \mathcal{R}_4^{\pm}$,

(b) $\mathrm{U}(1) \ltimes \mathbb{R}^2$ for $v = e_1 \pm e_4$,

(c) $\mathrm{SL}(2,\mathbb{R})$, for $v \in \mathcal{R}_1$,

(d) $\mathrm{SL}(2,\mathbb{C})$, for $v = 0$.

Theorem 16.1 (Wigner) *The (unitary equivalence classes of) irreducible projective representations of the connected Poincaré group are precisely those induced by the (equivalence classes of) irreducible unitary representations of its double covering* $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^4$. *The latter are given by*

$$\pi_{v,\rho} = \mathrm{Ind}_{\widetilde{H}_v \ltimes \mathbb{R}^4}^{\mathrm{SL}(2,\mathbb{C})\ltimes\mathbb{R}^4}(\rho \otimes \xi_v),$$

where $v \in \mathcal{R}$ *and* $\rho \in \widehat{\widetilde{H}}_v$. *The given list contains no double occurrences.*

In his paper [21], Wigner observed that the physically relevant representations occur for $v \in \mathcal{R}_4^+$ and $v = e_1 + e_4$. In the first case, $v = me_4$, with $m > 0$, and the little group equals $\widetilde{H}_v = \mathrm{SU}(2)$. The irreducible unitary representations of $\mathrm{SU}(2)$ are classified by their highest weight s times the positive root, where $s \in \frac{1}{2}\mathbb{N}$. The associated irreducible representation π_s has dimension $2s+1$. The parameter m has the interpretation of rest mass of the particle, whereas s has the interpretation of internal spin. The representation with $s = 1/2$ represents the electron, those with $s = 0$ the mesons.

The second family of physically relevant representations is associated with the representative $v = e_1 + e_4$. This case corresponds to rest mass zero. The little group equals the double cover of the Euclidean motion group $\mathrm{SO}(2) \ltimes \mathbb{R}^2$. The unitary representations of this group are described by the lemma below. The representations that occur in physics are the representations $\nu_{0,s}$ with $s \in \frac{1}{2}\mathbb{Z}$. The representations with $s = \pm 1/2$ represent the neutrinos, those with $s = \pm 1$ the photons.

In the following lemma we denote by χ_m the character of $\mathrm{U}(1)$ given by $(e^{it}) \mapsto e^{imt}$.

Lemma 16.2 *The equivalence classes of irreducible unitary representations of the double cover* $E = \mathrm{U}(1) \ltimes \mathbb{R}^2$ *of the Euclidean motion group* $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ *are* ν_ρ *for* $\rho > 0$ *and* $\nu_{0,s}$ *for* $s \in \frac{1}{2}\mathbb{Z}$, *where*

(a) $\nu_\rho = \mathrm{Ind}_{\mathbb{R}^2}^{E}(e^{i\langle \rho e_1 \,,\, \cdot\,\rangle})$;

(b) $\nu_{0,s} = \chi_{2s} \otimes 1$.

Proof. We consider the group homomorphism $p : U(1) \to \mathrm{SO}(2), e^{it} \mapsto R_{2t}$, where R_φ denotes the rotation by angle φ. Then $\mathrm{U}(1)$ acts on $\mathbb{R}^2$ via p. The associated semidirect product $E = \mathrm{U}(1) \ltimes \mathbb{R}^2$ is the double cover of the Euclidean motion group $\mathrm{SO}(2) \ltimes \mathbb{R}^2$, which may be realized as the stabilizer of $e_1 + e_4$ in $\mathrm{SL}(2,\mathbb{C})$. Since $\mathbb{R}^2$ is an abelian normal subgroup, we can use Theorem 11.6 to classify the irreducible unitary representations of E.

We identify an element $a \in \mathbb{R}^2$ with the unitary character $x \mapsto e^{i\langle a\,,\,x\rangle}$ in $\widehat{\mathbb{R}}$. Via this identification, the natural action of $\mathrm{SO}(2)$ on $\widehat{\mathbb{R}}$ corresponds with the natural representation of $\mathrm{SO}(2)$ on $\mathbb{R}^2$. The orbits for this action are the circles $C_\rho = \mathrm{SO}(2)\rho e_1$ with center 0 and radius $\rho \geq 0$. For $\rho > 0$ the stabilizer of ρe_1 in $\mathrm{SO}(2)$ is trivial, and we obtain the series of representations in (a). The stabilizer of 0 in $\mathrm{SO}(2)$ is the full group $\mathrm{SO}(2)$. The irreducible representations of $\mathrm{SO}(2)$ correspond to the characters χ_{2s}, for $s \in \frac{1}{2}\mathbb{Z}$. The origin $0 \in \mathbb{R}^2$ corresponds to the trivial character of $\mathbb{R}^2$. This gives the series of representations in (b). $\square$

It follows from Wigner's theorem that in order to give the full classification of the irreducible projective representations of the connected Poincaré group it is necessary to classify the full unitary duals of the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, which are the double covers of the Lorentz groups $\mathrm{SO}(2,1)^\circ$ and $\mathrm{SO}(3,1)^\circ$. Except for the trivial one, all such representations are necessarily infinite dimensional. This follows from the fact that their Lie algebras are simple with non-definite Killing forms, so that the groups do not admit compact factors. The classification of the irreducible unitary representations of the two mentioned groups was achieved in the important paper of V. Bargmann, [3]. It was the starting point of the development of the theory of representations of real semisimple Lie groups of the non-compact type, dominated by the work of Harish-Chandra. See [13] for an overview.

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