and (c) follows.

Let W be the linear span of the vectors v_k, for $0 \leq k \leq n$. Then by definition of the vectors v_k, $Yv_k = v_{k+1}$. Therefore, Y leaves W invariant. By (c), H and X leave W invariant as well. It follows that W is a non-trivial invariant subspace of V, hence $V = W$ by irreducibility. The vectors v_k, for $0 \leq k \leq n$, must be linear independent since they are eigenvectors for H for distinct eigenvalues; hence (a).

Finally, we have established the second assertion of (c) for all $k \geq 0$, in particular for $k = n+1$. Now $v_{n+1} = 0$, hence $0 = (n+1)(\lambda - n)v_n$ and since $v_n \neq 0$ it follows that $\lambda = n$. This establishes (b).

It follows from (a) and (c) that the only primitive vectors in V are non-zero multiples of v_0.
□

Corollary 30.8 *Let V and V' be two irreducible finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules. Then $V \simeq V'$ if and only if $\dim V = \dim V'$. Moreover, if v and v' are primitive vectors of V and V', respectively, then there is a unique isomorphism $T : V \to V'$ mapping v onto v'.*

Proof: Clearly if $V \simeq V'$ then V and V' have equal dimension. Conversely, assume that $\dim V = \dim V' = n$ and that v and v' are primitive vectors of V and V' respectively. Then by the above lemma, the vectors $v_k = Y^k v$, $0 \leq k \leq n$ form a basis of V. Similarly the vectors $v'_k = Y^k v'$, $0 \leq k \leq n$ form a basis of V'. Any intertwining operator $T : V \to V'$ that maps v onto v' must map the basis v_k onto the basis v'_k, hence is uniquely determined. Let $T : V \to V'$ be the linear map determined by $Tv_k = v'_k$, for $0 \leq k \leq n$. Then T is a linear bijection. Moreover, by the above lemma we see that T intertwines the actions of H, X, Y on V and V'. It follows that T is equivariant, hence $V \simeq V'$. □

Completion of the proof of Theorem 30.3: The space $P_n(\mathbb{C}^2)$ is an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module, of dimension $n+1$. Hence if V is an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module of dimension $m \geq 1$, then $V \simeq P_n(\mathbb{C}^2)$, with $n = m - 1$. □

31 Roots and weights

Let $\mathfrak{t}$ be a finite dimensional commutative real Lie algebra, and let (ρ, V) be a representation of $\mathfrak{t}$ in a non-trivial complex linear space V (which we do not assume to be finite dimensional).

Let $\mathfrak{t}_\mathbb{C}^*$ denote the space of complex linear functionals on $\mathfrak{t}_\mathbb{C}$. Note that $\mathfrak{t}^*$, the space of real linear functionals on $\mathfrak{t}$ may be identified with the space of $\lambda \in \mathfrak{t}_\mathbb{C}^*$ that are real valued on $\mathfrak{t}$. Thus, $\mathfrak{t}^*$ is viewed as a real linear subspace of $\mathfrak{t}_\mathbb{C}^*$. Accordingly $i\mathfrak{t}^*$ equals the space of $\lambda \in \mathfrak{t}_\mathbb{C}^*$ such that $\lambda|_\mathfrak{t}$ has values in $i\mathbb{R}$.

If $\lambda \in \mathfrak{t}_\mathbb{C}^*$, then we define the following subspace of V :

$$V_\lambda = \bigcap_{H \in \mathfrak{t}} \ker(\rho(H) - \lambda(H)I). \tag{43}$$

In other words, V_λ equals the space of $v \in V$ such that

$$\rho(H)v = \lambda(H)v \qquad \text{for all} \qquad H \in \mathfrak{t}.$$

If $V_\lambda \neq 0$, then λ is called a *weight* of $\mathfrak{t}$ in V, and V_λ is called the associated *weight space.* The set of weights of $\mathfrak{t}$ in V is denoted by $\Lambda(\rho)$.

Lemma 31.1 *Let $T \in \mathrm{End}(V)$ be a ρ-intertwining linear endomorphism, then T leaves V_λ invariant, for every $\lambda \in \Lambda(\rho)$.*

Proof: Let $\lambda \in \Lambda(\rho)$. The endomorphism T commutes with $\rho(H)$ hence leaves the eigenspace $\ker(\rho(H) - \lambda(H))$ invariant, for every $H \in \mathfrak{t}$. Hence T leaves the intersection V_λ of all these spaces invariant. □

Lemma 31.2 *Let*

$$V' := \sum_{\lambda \in \Lambda(\rho)} V_\lambda. \tag{44}$$

Then for every $\mathfrak{t}$-invariant subspace $W \subset V'$,

$$W = \bigoplus_{\lambda \in \Lambda(\rho)} (W \cap V_\lambda). \tag{45}$$

In particular, the sum (44) is direct.

Proof: We will first show that the sum (44) is direct. Let $\lambda_1, \ldots, \lambda_n$ be a collection of distinct weights in $\Lambda(\rho)$ and assume that $v_j \in V_{\lambda_j}$ are given such that $\sum_{j=1}^n v_j = 0$. Then it suffices to show that $v_j = 0$ for all $1 \leq j \leq n$. Since the weights are distinct, the sets $K_{ij} := \ker(\lambda_i - \lambda_j)$, for $i \neq j$ are hyperplanes in $\mathfrak{t}_\mathbb{C}$. The union $\cup_{i \neq j} K_{ij}$ is strictly contained in $\mathfrak{t}_\mathbb{C}$, hence we may select $H \in \mathfrak{t}_\mathbb{C}$ in the complement of this union. It follows that $s_j := \lambda_j(H)$, for $1 \leq j \leq n$, is a sequence of distinct complex numbers. Applying H repeatedly to the sum $v_1 + \cdots + v_n$ we find that

$$\sum_{j=1}^n s_j^l v_j = 0, \qquad (l \geq 0).$$

Let $T : \mathbb{C}^n \to V$ be the unique linear map sending the j-th standard basis vector e_j to v_j. Then it follows from the above that

$$T(\sum_{j=1}^n s_j^l e_j) = 0, \qquad (l \geq 0).$$

Let A be the linear map $\mathbb{C}^n \to \mathbb{C}^n$ which sends e_k to $\sum_{j=1}^n s_j^{k-1} e_j$ for $1 \leq k \leq n$. Then it follows that $TA = 0$. By the Vandermone determinant formula, $\det A = \prod_{i<j}(s_j - s_i) =\neq 0$, hence A is invertible. Therefore, $T = 0$ and we conclude that indeed $v_j = 0$ for all $1 \leq j \leq n$.

To complete the proof we note that W is a $\mathfrak{t}$-module, hence so is the quotient space V'/W. Let $w \in W$. Then $w = v_1 + \cdots + v_n$ for certain $v_j \in V_{\lambda_j}$ with $\lambda_1, \ldots, \lambda_n$ a collection of distinct weights in $\Lambda(\rho)$. Each canonical image $\bar{v}_j$ in V'/W is a weight vector of weights λ_j in V'/W. Furthermore, $\sum_{j=1}^n \bar{v}_j = \bar{w} = 0$. By the first result, applied to V'/W in place of V, it follows that $\bar{v}_j = 0$ hence $v_j \in W$ for all $1 \leq j \leq n$.

It follows from the above that $W = \sum_{\lambda \in \Lambda(\rho)} W \cap V_\lambda$. By the first result, applied to W in place of V, it follows that the sum is direct. □

The action of $\mathfrak{t}$ on V (or the representation ρ) is said to be *semisimple* if for every $X \in \mathfrak{t}$ the action of $\rho(X)$ is diagonalizable. The latter means that V decomposes as a direct sum of eigen spaces for $\rho(X)$.

Lemma 31.3 *If ρ is semisimple, then*

$$V = \bigoplus_{\lambda \in \Lambda(\rho)} V_\lambda. \tag{46}$$

Proof: We will prove the lemma by induction on the dimension of $\mathfrak{t}$. First assume $\dim \mathfrak{t} < 1$. Fix a non-zero element $X \in \mathfrak{t}$. Let S denote the set of eigenvalues of $\rho(X)$. Then by the assumed semisimplicity, V is the direct sum of the eigen spaces $V_s = \ker(\rho(X) - sI)$, for $s \in S$. For $s \in S$ we define $\lambda_s \in \mathfrak{t}_\mathbb{C}^*$ by $\lambda_s(X) = s$. Then for each $s \in \Sigma$ we have $V_{\lambda_s} = V_s$ and we see that $\Lambda(\rho) = \{\lambda_s \mid s \in S\}$ and (46) follows.

Let now $d > 1$ and assume the result has been established for $\mathfrak{t}$ of dimension smaller than d. We will then prove the result for $\mathfrak{t}$ of dimension d. We fix an element $X \in \mathfrak{t}$ and a complementary subspace $\mathfrak{t}_0$ such that $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathbb{R}X$. By the induction hypothesis, the space V decomposes as a direct sum of weight spaces V_μ for $\rho_0 := \rho|_{\mathfrak{t}_0}$, with $\mu \in \Lambda(\rho_0) \subset \mathfrak{t}_{0\mathbb{C}}^*$. Furthermore, V decomposes as the direct sum of the weight spaces V_s, for $s \in S$, defined as in first part of the proof. By commutativity of $\mathfrak{t}$, the operator $\rho(X) \in \mathrm{End}(V)$ is intertwining. By Lemma 31.1 each weight space V_μ is $\rho(X)$-invariant hence by Lemma 31.2 it decomposes as the direct sum of the spaces $V_\mu \cap V_s$, for $s \in S$. It follows that

$$V = \bigoplus_{\mu \in \Lambda(\rho_0), s \in S} V_\mu \cap V_s.$$

Let $\lambda_{\mu,s} \in \mathfrak{t}_\mathbb{C}^*$ be defined by $\lambda_{\mu,s}|_{\mathfrak{t}_0} = \mu$ and $\lambda_{\mu,s}(X) = s$, then

$$V_{\lambda_{\mu,s}} = V_\mu \cap V_s,$$

and we see that $\Lambda(V, \mathfrak{t})$ equals the set of $\lambda_{\mu,s} \in \mathfrak{t}_\mathbb{C}^*$, $(\mu \in \Lambda(\rho_0), s \in S)$, for which the above intersection is non-zero. Furthermore, V is the direct sum of the corresponding weight spaces.

Lemma 31.4 *Let (ρ, V) be finite dimensional representation of $\mathfrak{t}$. Then $\Lambda(\rho)$ is a finite non-empty subset of $\mathfrak{t}_\mathbb{C}^*$.*

Proof: In view of Lemma 31.2 it follows that $\Lambda(\rho)$ has at most $\dim V$ elements.

Thus it remains to be shown that $\Lambda(\rho)$ is non-empty. For this we proceed by induction on the dimension of $\mathfrak{t}$.

First, assume $\dim \mathfrak{t} = 1$. Then $\mathfrak{t} = \mathbb{R}X$ for $X \in \mathfrak{t} \setminus \{0\}$. The map $\rho(X)$ has at least one eigenvalue s. Let $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ be defined by $\lambda(X) = s$. Then $V_\lambda \neq 0$ hence $\lambda \in \Lambda(\rho)$.

Next, assume that $\dim \mathfrak{t} > 1$. Then we fix a decomposition $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathbb{R}X$ with $\mathfrak{t}_0$ is a subspace of codimension 1 and $X \in \mathfrak{t} \setminus \{0\}$. By the induction hypothesis, $\rho_0 := \rho|_{\mathfrak{t}_0}$ has a weight $\lambda_0 \in \mathfrak{t}_{0\mathbb{C}}^*$. The associated weight space V_{λ_0} is $\rho(X)$-invariant and finite dimensional, hence contains an eigenvector $v \neq 0$. It follows that $\rho(\mathfrak{t})v \subset \mathbb{C}v$, from which we infer that v is contained in a weight space for ρ. □

Assumption: In the rest of this section we assume that G is a compact Lie group, with Lie algebra $\mathfrak{g}$.

Definition 31.5 A *torus* in $\mathfrak{g}$ is by definition a commutative subalgebra of $\mathfrak{g}$. A torus $\mathfrak{t} \subset \mathfrak{g}$ is called *maximal* if there exists no torus of $\mathfrak{g}$ that properly contains $\mathfrak{t}$.

From now on we assume that $\mathfrak{t}$ is a fixed maximal torus in $\mathfrak{g}$.

Lemma 31.6 *The centralizer of* $\mathfrak{t}$ *in* $\mathfrak{g}$ *equals* $\mathfrak{t}$.

Proof: Since $\mathfrak{t}$ is abelian, it is contained in its centralizer. Conversely, assume that $X \in \mathfrak{g}$ centralizes $\mathfrak{t}$. Then $\mathfrak{t}' = \mathfrak{t} + \mathbb{R}X$ is a torus which contains $\mathfrak{t}$. Hence $\mathfrak{t}' = \mathfrak{t}$ by maximality, and we see that $X \in \mathfrak{t}$. □

Let (π, V) be a finite dimensional representation of $\mathfrak{g}_{\mathbb{C}}$, the complexification of the Lie algebra $\mathfrak{g}$; i.e., π is a complex Lie algebra homomorphism from $\mathfrak{g}_{\mathbb{C}}$ into $\mathrm{End}(V)$ (the latter is the space of complex linear endomorphisms equipped with the commutator Lie bracket). Alternatively we will also say that V is a finite dimensional $\mathfrak{g}_{\mathbb{C}}$-module. We denote by $\Lambda(\pi) = \Lambda(\pi, \mathfrak{t})$ the set of weights of the representation $\rho = \pi|_{\mathfrak{t}}$ of $\mathfrak{t}$ in V. If $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, then as before, V_λ is defined as in (43), with $\pi|_{\mathfrak{t}}$ in place of ρ. Thus

$$V_\lambda = \{v \in V \mid \pi(H)v = \lambda(H)v \quad \text{for all} \quad H \in \mathfrak{t}\}.$$

From Lemma 31.4 we see that $\Lambda(\pi)$ is a finite non-empty subset of $\mathfrak{t}_{\mathbb{C}}^*$.

Let (π, V) be a finite dimensional continuous representation of G. Then the map $\pi : G \to \mathrm{GL}(V)$ is a homomorphism of Lie groups. Let $\pi_* = T_e\pi$. Then $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism, or, differently said, a representation of $\mathfrak{g}$ in V. The homomorphism π_* has a unique extension to a complex Lie algebra homomorphism from $\mathfrak{g}_{\mathbb{C}}$ into $\mathrm{End}(V)$ (we recall that V is a complex linear space by assumption). This extension is called the *induced infinitesimal representation* of $\mathfrak{g}_{\mathbb{C}}$ in V.

Lemma 31.7 *Let* π *be a finite dimensional continuous representation of* G. *Then* $\Lambda(\pi_*)$ *is a finite subset of* $i\mathfrak{t}^*$. *Moreover,*

$$V = \bigoplus_{\lambda \in \Lambda(\pi_*)} V_\lambda.$$

If V is equipped with a G-invariant inner product, then for all $\lambda, \mu \in \Lambda(\pi_*)$ with $\lambda \neq \mu$ we have $V_\lambda \perp V_\mu$.

Proof: There exists a G-invariant inner product on V; assume such an inner product $\langle \cdot\,,\,\cdot \rangle$ to be fixed. Then π maps G into $\mathrm{U}(V)$, the associated group of unitary transformations. It follows that π_* maps $\mathfrak{g}$ into the Lie algebra $\mathfrak{u}(V)$ of $\mathrm{U}(V)$, which is the subalgebra of anti-Hermitian endomorphisms in $\mathrm{End}(V)$. It follows that for $X \in \mathfrak{g}$ the endomorphism $\pi_*(X)$ is anti-Hermitian, hence diagonalizable with imaginary eigenvalues. The direct sum decomposition now follows from Lemma 31.3. It remains to establish orthogonality of the summands. Let λ, μ be distinct weights in $\Lambda(\pi_*)$. Then there exists $H \in \mathfrak{t}$ such that $\lambda(H) \neq \mu(H)$. For $v \in V_\lambda$ and $w \in V_\mu$ we have

$$\lambda(H)\langle v\,,\,w\rangle = \langle \pi_*(H)v\,,\,w\rangle = -\langle v\,,\,\pi_*(H)w\rangle = -\overline{\mu(H)}\langle v\,,\,w\rangle = \mu(H)\langle v\,,\,w\rangle.$$

It follows that $\langle v\,,\,w\rangle = 0$. □

If $A \in \mathrm{End}(\mathfrak{g})$, then we denote by $A_\mathbb{C}$ the complex linear extension of A to $\mathfrak{g}_\mathbb{C}$. Obviously the map $A \mapsto A_\mathbb{C}$ induces a real linear embedding of $\mathrm{End}(\mathfrak{g})$ into $\mathrm{End}(\mathfrak{g}_\mathbb{C}) := \mathrm{End}_\mathbb{C}(\mathfrak{g}_\mathbb{C})$. Accordingly we shall view $\mathrm{End}(\mathfrak{g})$ as a real linear subspace of the complex linear space $\mathrm{End}(\mathfrak{g}_\mathbb{C})$ from now on. Thus, we may view Ad as a representation of G in the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. The associated infinitesimal representation is the adjoint representation ad of $\mathfrak{g}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. The associated collection $\Lambda(\mathrm{ad})$ of weights contains the weight 0. Indeed the associated weight space $\mathfrak{g}_{\mathbb{C}0}$ equals the centralizer of $\mathfrak{t}$ in $\mathfrak{g}_\mathbb{C}$, which in turn equals $\mathfrak{t}_\mathbb{C}$, by Lemma 31.6. Hence:

$$\mathfrak{g}_{\mathbb{C}0} = \mathfrak{t}_\mathbb{C}.$$

Definition 31.8 The weights of ad in $\mathfrak{g}_\mathbb{C}$ different from 0 are called the *roots* of $\mathfrak{t}$ in $\mathfrak{g}_\mathbb{C}$; the set of these is denoted by $R = R(\mathfrak{g}_\mathbb{C}, \mathfrak{t})$. Given $\alpha \in R$, the associated weight space $\mathfrak{g}_{\mathbb{C}\alpha}$ is called a *root space.*

It follows from the definitions that

$$\mathfrak{g}_{\mathbb{C}\alpha} = \{X \in \mathfrak{g}_\mathbb{C} \mid [H, X] = \alpha(H)X \quad \text{for all} \quad H \in \mathfrak{t}\}.$$

From Lemma 31.7 we now obtain the so called *root space decomposition* of $\mathfrak{g}_\mathbb{C}$, relative to the torus $\mathfrak{t}$.

Corollary 31.9 *The collection $R = R(\mathfrak{g}_\mathbb{C}, \mathfrak{t})$ of roots is a finite subset of $i\mathfrak{t}^*$. Moreover, we have the following direct sum of vector spaces:*

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\mathbb{C}\alpha}. \qquad (47)$$

Example 31.10 The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ has complexification $\mathfrak{sl}(2, \mathbb{C})$, consisting of all complex 2×2 matrices with trace zero. Let H, X, Y be the standard basis of $\mathfrak{sl}(2, \mathbb{C})$; i.e.

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now $\mathfrak{t} = i\mathbb{R}H$ is a maximal torus in $\mathfrak{su}(2)$. We recall that $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. Thus, if we define $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ by $\alpha(H) = 2$, then $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$ equals $\{\alpha, -\alpha\}$. Moreover, $\mathfrak{g}_{\mathbb{C}\alpha} = \mathbb{C}X$ and $\mathfrak{g}_{\mathbb{C}(-\alpha)} = \mathbb{C}Y$.

We recall that, by definition, the center $\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}}$ of $\mathfrak{g}$ is the ideal $\ker \mathrm{ad}$; i.e., it is the space of $X \in \mathfrak{g}$ that commute with all $Y \in \mathfrak{g}$.

Lemma 31.11 *The center of $\mathfrak{g}$ is contained in $\mathfrak{t}$ and equals the intersection of the root hyperplanes:*

$$\mathfrak{z}_{\mathfrak{g}} = \bigcap_{\alpha \in R} \ker \alpha.$$

In particular, if $\mathfrak{z}_{\mathfrak{g}} = 0$, then R spans the real linear space $i\mathfrak{t}^*$.

Proof: The center of $\mathfrak{g}$ centralizes $\mathfrak{t}$ in particular, hence is contained in $\mathfrak{t}$, by Lemma 31.6. Let $H \in \mathfrak{t}$ and assume that H centralizes $\mathfrak{g}$; then H centralizes $\mathfrak{g}_{\mathbb{C}}$, hence every root space of $\mathfrak{g}_{\mathbb{C}}$. This implies that $\alpha(H) = 0$ for all $\alpha \in R$. Conversely, if $H \in \mathfrak{t}$ is in the intersection of all the root hyperplanes, then H centralizes $\mathfrak{t}_{\mathbb{C}}$ and every root space $\mathfrak{g}_{\mathbb{C}\alpha}$. By the root space decomposition it then follows that $H \in \mathfrak{z}$. This establishes the characterization of the center.

If $\mathfrak{z} = 0$, then the root hyperplanes $\ker \alpha$ $(\alpha \in R)$ have a zero intersection in $\mathfrak{t}$. This implies that the set $R \subset i\mathfrak{t}^*$ spans the real linear space $i\mathfrak{t}^*$. □

Lemma 31.12 *Let (π, V) be a finite dimensional representation of $\mathfrak{g}_{\mathbb{C}}$. Then for all $\lambda \in \Lambda(\pi)$ and all $\alpha \in R \cup \{0\}$ we have:*

$$\pi(\mathfrak{g}_{\mathbb{C}\alpha})V_\lambda \subset V_{\lambda+\alpha}.$$

In particular, if $\lambda + \alpha \notin \Lambda(\pi)$, then $\pi(\mathfrak{g}_{\mathbb{C}\alpha})$ anihilates V_λ.

Proof: Let $X \in \mathfrak{g}_{\mathbb{C}\alpha}$ and $v \in V_\lambda$. Then, for $H \in \mathfrak{t}$,

$$\begin{aligned}
\pi(H)\pi(X)v &= \pi(X)\pi(H)v + [\pi(H), \pi(X)]v \\
&= \lambda(H)\pi(X)v + \pi([H, X])v = [\lambda(H) + \alpha(H)]\pi(X)v.
\end{aligned}$$

Hence $\pi(X)v \in V_{\lambda+\alpha}$. If $\lambda+\alpha$ is not a weight of π, then $V_{\lambda+\alpha} = 0$ and it follows that $\pi(X)v = 0$. □

Corollary 31.13 *If $\alpha, \beta \in R \cup \{0\}$, then*

$$[\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\beta}] \subset \mathfrak{g}_{\mathbb{C}(\alpha+\beta)}.$$

In particular, if $\alpha + \beta \notin R \cup \{0\}$, then $\mathfrak{g}_{\mathbb{C}\alpha}$ and $\mathfrak{g}_{\mathbb{C}\beta}$ commute.

Proof: This follows from the previous lemma applied to the adjoint representation. □

We shall write $\mathbb{Z}R$ for the $\mathbb{Z}$-linear span of R, i.e., the $\mathbb{Z}$-module of elements of the form $\sum_{\alpha\in R} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}$.

In the following corollary we do not assume that π comes from a representation of G.

Corollary 31.14 *Let (π, V) be a finite dimensional representation of $\mathfrak{g}_\mathbb{C}$. Then*

$$W := \bigoplus_{\lambda\in\Lambda(\pi)} V_\lambda \tag{48}$$

is a non-trivial $\mathfrak{g}_\mathbb{C}$-submodule. If π is irreducible, then $W = V$. Moreover, if $\lambda, \mu \in \Lambda(\pi)$, then $\lambda - \mu \in \mathbb{Z}R$.

Proof: By Lemma 31.4 the set $\Lambda(\pi)$ is non-empty and finite, and therefore W is a non-trivial subspace of V. From Lemma 31.12 we see that W is $\mathfrak{g}_\mathbb{C}$-invariant. If π is irreducible, then $W = V$. To establish the last assertion we define an equivalence relation on $\Lambda(\pi)$ by $\lambda \sim \mu \iff \lambda - \mu \in \mathbb{Z}R$. If S is a class for $\sim$, then $V_S = \oplus_{\lambda\in S} V_\lambda$ is a non-trivial $\mathfrak{g}_\mathbb{C}$-invariant subspace of V, by Lemma 31.12. Hence $V_S = V$ and it follows that $S = \Lambda(\pi)$. □

Remark 31.15 If $\mathfrak{g}$ has trivial center, then the above result actually holds for every finite dimensional V-module. To see that a condition like this is necessary, consider $\mathfrak{g} = \mathbb{R}$, the Lie algebra of the circle. Define a representation of $\mathfrak{g}$ in $V = \mathbb{C}^2$ by

$$\pi(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

Then $\Lambda(\pi) = \{0\}$, but $V_0 = \mathbb{C} \times \{0\}$ is not all of V.

Note that this does not contradict the conclusion of Lemma 31.7, since π is not associated with a continuous representation of the circle group in $\mathbb{C}^2$.

Lemma 31.16 *Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{g}$, and R the associated collection of roots. If $\alpha \in R$ then $-\alpha \in R$.*

Proof: Let τ be the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to the real form $\mathfrak{g}$. That is: $\tau(X + iY) = X - iY$ for all $X, Y \in \mathfrak{g}$. One readily checks that τ is an automorphism of $\mathfrak{g}_\mathbb{C}$, considered as a real Lie algebra (by forgetting the complex linear structure). Let $\alpha \in R$, and let $X \in \mathfrak{g}_{\mathbb{C}\alpha}$. Then for every $H \in \mathfrak{t}$,

$$[H, \tau(X)] = \tau[H, X] = \tau(\alpha(H)X) = \overline{\alpha(H)}\tau(X) = -\alpha(H)\tau(X).$$

For the latter equation we used that α has imaginary values on $\mathfrak{t}$. It follows that $-\alpha \in R$ and that τ maps $\mathfrak{g}_{\mathbb{C}\alpha}$ into $\mathfrak{g}_{\mathbb{C}-\alpha}$ (in fact is a bijection between these root spaces; why?). □

We recall that we identify $i\mathfrak{t}^*$ with the real linear subspace of $\mathfrak{t}_{\mathbb{C}}^*$ consisting of λ such that $\lambda|\mathfrak{t}$ has values in $i\mathbb{R}$; the latter condition is equivalent to saying that $\lambda|_{i\mathfrak{t}}$ is real valued. One readily verifies that the restriction map $\lambda \mapsto \lambda|_{i\mathfrak{t}}$ defines a real linear isomorphism from $i\mathfrak{t}^*$ onto the real linear dual $(i\mathfrak{t})^*$. In the following we shall use this isomorphism to identify $i\mathfrak{t}^*$ with $(i\mathfrak{t})^*$. Now R is a finite subset of $(i\mathfrak{t})^* \setminus \{0\}$. Hence the complement of the hyperplanes $\ker \alpha \subset i\mathfrak{t}$, for $\alpha \in R$ is a finite union of connected components, which are all convex. These components are called the *Weyl chambers* associated with R. Let $\mathcal{C}$ be a fixed chamber. By definition every root is either positive or negative on $\mathcal{C}$. We define the *system of positive roots* $R^+ := R^+(\mathcal{C})$ associated with $\mathcal{C}$ by

$$R^+ = \{\alpha \in R \mid \alpha > 0 \quad \text{on} \quad \mathcal{C}\}.$$

By what we said above, for every $\alpha \in R$, we have that either α or $-\alpha$ belongs to R^+, but not both. It follows that

$$R = R^+ \cup (-R^+) \quad \text{(disjoint union)}. \tag{49}$$

We write $\mathbb{N}R^+$ for the subset of $\mathbb{Z}R$ consisting of the elements that can be written as a sum of the form $\sum_{\alpha \in R^+} n_\alpha \alpha$, with $n_\alpha \in \mathbb{N}$.

Lemma 31.17 $\mathbb{N}R^+ \cap (-\mathbb{N}R^+) = 0$.

Proof: Let $\mu \in \mathbb{N}R^+$. Then $\mu \geq 0$ on $\mathcal{C}$, the chamber corresponding to R^+. If also $-\mu \in \mathbb{N}R^+$, then $\mu \leq 0$ on $\mathcal{C}$ as well. Hence $\mu = 0$ on $\mathcal{C}$. Since $\mathcal{C}$ is a non-empty open subset of $i\mathfrak{t}^*$, this implies that $\mu = 0$. $\square$

Lemma 31.18 *The spaces*

$$\mathfrak{g}_{\mathbb{C}}^+ := \sum_{\alpha \in R^+} \mathfrak{g}_{\mathbb{C}\alpha}, \qquad \mathfrak{g}_{\mathbb{C}}^- := \sum_{\beta \in -R^+} \mathfrak{g}_{\mathbb{C}\beta}$$

are $\mathrm{ad}(\mathfrak{t})$*-stable subalgebras of* $\mathfrak{g}_{\mathbb{C}}$*. Moreover,*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^+ \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^-.$$

Proof: Let $\alpha, \beta \in R^+$ and assume that $[\mathfrak{g}_{\mathbb{C}\alpha}, \mathfrak{g}_{\mathbb{C}\beta}] \neq 0$. Then $\alpha + \beta \in R \cup \{0\}$, and $\alpha + \beta > 0$ on $\mathcal{C}$. This implies that $\alpha + \beta \in R^+$, hence $\mathfrak{g}_{\mathbb{C}(\alpha+\beta)} \subset \mathfrak{g}_{\mathbb{C}}^+$. It follows that $\mathfrak{g}_{\mathbb{C}}^+$ is a subalgebra. For similar reasons $\mathfrak{g}_{\mathbb{C}}^-$ is a subalgebra. Both subalgebras are $\mathrm{ad}(\mathfrak{t})$ stable, since root spaces are. The direct sum decomposition is an immediate consequence of (47) and (49). $\square$

We are now able to define the notion of a highest weight vector for a finite dimensional $\mathfrak{g}_{\mathbb{C}}$-module, relative to the system of positive roots R^+. This is the appropriate generalization of the notion of a primitive vector for $\mathfrak{sl}(2, \mathbb{C})$.

Definition 31.19 Let V be a (not necessarily finite dimensional) $\mathfrak{g}_{\mathbb{C}}$-module. Then a *highest weight vector* of V is by definition a non-trivial vector $v \in V$ such that

(a) $\mathfrak{t}_{\mathbb{C}} v \subset \mathbb{C}v$;

(b) $Xv = 0$ for all $X \in \mathfrak{g}_{\mathbb{C}}^{+}$.

Lemma 31.20 *Any finite dimensional $\mathfrak{g}_{\mathbb{C}}$-module has a highest weight vector.*

Proof: We define the $\mathfrak{g}_{\mathbb{C}}$-submodule W of V as the sum of the $\mathfrak{t}_{\mathbb{C}}$-weight spaces, see Corollary 31.14.

Let $\mathcal{C}$ be the positive chamber determining R^{+}. Fix $X \in \mathcal{C}$. Then $\alpha(X) > 0$ for all $\alpha \in R^{+}$. We may select $\lambda_0 \in \Lambda(\pi)$ such that the real part of $\lambda(X)$ is maximal. Then $\lambda_0 + \alpha \notin \Lambda(\pi)$ for all $\alpha \in R^{+}$. By Lemma 31.12 this implies that $\pi_{*}(\mathfrak{g}_{\mathbb{C}\alpha})V_{\lambda} \subset V_{\lambda_0+\alpha} = 0$ for all $\alpha \in R^{+}$. Hence $\mathfrak{g}_{\mathbb{C}}^{+}$ annihilates V_{λ_0}. Thus, every non-zero vector of V_{λ_0} is a highest weight vector. □

Definition 31.21 Let V be a (not necessarily finite dimensional) $\mathfrak{g}_{\mathbb{C}}$-module. A vector $v \in V$ is said to be *cyclic* if it generates the $\mathfrak{g}_{\mathbb{C}}$-module V, i.e., V is the smallest $\mathfrak{g}_{\mathbb{C}}$-submodule containing v.

Obviously, if V is irreducible, then every non-trivial vector is cyclic.

Proposition 31.22 *Let V be a $\mathfrak{g}_{\mathbb{C}}$-module and $v \in V$ a cyclic highest weight vector.*

(a) *There exists a (unique) $\lambda \in \Lambda(V)$ such that $v \in V_{\lambda}$. Moreover, $V_{\lambda} = \mathbb{C}v$.*

(b) *The space V is equal to the span of the vectors v and $\pi(X_1)\cdots\pi(X_n)v$, with $n \in \mathbb{N}$ and $X_j \in \mathfrak{g}_{\mathbb{C}}^{-}$, for $1 \leq j \leq n$.*

(c) *Every weight $\mu \in \Lambda(V)$ is of the form $\lambda - \nu$, with $\nu \in \mathbb{N}R^{+}$.*

(d) *The module V has a unique maximal proper submodule W.*

(e) *The module V has a unique non-trivial irreducible quotient.*

Proof: The first assertion of (a) follows from the definition of highest weight vector. We define an increasing sequence of linear subspaces of V inductively by $V_0 = \mathbb{C}v$ and $V_{n+1} = V_n + \pi(\mathfrak{g}_{\mathbb{C}}^{-})V_n$. Let W be the union of the spaces V_n. We claim that W is an invariant subspace of V. To establish the claim, we note that by definition we have $\pi(\mathfrak{g}_{\mathbb{C}}^{-})V_n \subset V_{n+1}$; hence W is $\mathfrak{g}_{\mathbb{C}}^{-}$ invariant. The space V_0 is $\mathfrak{t}$- and $\mathfrak{g}_{\mathbb{C}}^{+}$-invariant; by induction we will show that the same holds for V_n. Assume that V_n is $\mathfrak{t}$- and $\mathfrak{g}_{\mathbb{C}}^{+}$-invariant, and let $v \in V_n$, $Y \in \mathfrak{g}_{\mathbb{C}}^{-}$. Then for H in $\mathfrak{t}$ we have $HYv = YHv + [H, Y]v$. Now $v \in V_n$ and by the inductive hypothesis it follows that $Hv \in V_n$. Hence $YHv \in V_{n+1}$. Also $[H, Y] \in \mathfrak{g}_{\mathbb{C}}^{-}$ and it follows that $[H, Y]v \in V_{n+1}$. We conclude that $HYv \in V_{n+1}$. It follows from this that

$$\pi(\mathfrak{t})\pi(\mathfrak{g}_{\mathbb{C}}^{-})V_n \subset V_{n+1}.$$

Hence V_{n+1} is $\mathfrak{t}$-invariant.

Let now $v \in V_n$, $Y \in \mathfrak{g}_{\mathbb{C}}^{-}$ and $X \in \mathfrak{g}_{\mathbb{C}}^{+}$. Then $XYv = YXv + [X, Y]v$. Now $Xv \in V_n$ by the induction hypothesis and we see that $YXv \in V_{n+1}$. Also, $[X, Y] \in \mathfrak{g}_{\mathbb{C}}$. By the induction hypothesis it follows that $\mathfrak{g}_{\mathbb{C}} V_n \subset V_{n+1}$. Hence $[X, Y]v \in V_{n+1}$. We conclude that $XYv \in V_{n+1}$. It follows from this that

$$\pi(\mathfrak{g}_{\mathbb{C}}^+)\pi(\mathfrak{g}_{\mathbb{C}}^-)V_n \subset V_{n+1}.$$

Hence V_{n+1} is $\mathfrak{g}_{\mathbb{C}}^+$-invariant. This establishes the claim that W is a $\mathfrak{g}_{\mathbb{C}}$-invariant subspace of V.

Since W contains the cyclic vector v, it follows that $W = V$. In view of the definition of the spaces V_k assertion (b) follows.

Let $w = \pi(Y_1) \cdots \pi(Y_n)v$, with $n \in \mathbb{N}$, $Y_j \in \mathfrak{g}_{\mathbb{C}(-\alpha_j)}$, $\alpha_j \in R^+$. Then w belongs to the weight space $V_{\lambda-\nu}$, where $\nu = \alpha_1 + \cdots + \alpha_n \in \mathbb{N}R^+$. Since v and such elements w span $W = V$, we conclude that every weight μ in $\Lambda(V)$ is of the form $\lambda - \nu$ with $\nu \in \mathbb{N}R^+$. This establishes (c).

It follows from the above description that V equals the vector sum of $\mathbb{C}v$ and V_-, where V_- denotes the sum of the weight spaces V_μ with $\mu \in \Lambda(V) \setminus \{\lambda\}$. This implies that $V_\lambda = \mathbb{C}v$, whence the second assertion of (a).

We now turn to assertion (d). Let U be a submodule of V. In particular, U is a $\mathfrak{t}_{\mathbb{C}}$-invariant subspace. Let $\Lambda(U)$ be the collection of $\mu \in \Lambda(V)$ for which $U_\mu := U \cap V_\mu \neq 0$. In view of Lemma 31.2, U is the direct sum of the spaces U_μ, for $\mu \in \Lambda(U)$. If U is a proper submodule, then $U_\lambda = 0$, hence $\Lambda(U) \subset \Lambda(V) \setminus \{\lambda\}$ and we see that $U \subset V_-$. It follows that the vector sum W of all proper submodules satisfies $W \subset V_-$ hence is still proper. Therefore, V has W as unique maximal proper submodule.

The final assertion (e) is equivalent to (d). To see this, let $p : V \to V'$ be a surjective $\mathfrak{g}_{\mathbb{C}}$-module homomorphism onto a non-trivial $\mathfrak{g}_{\mathbb{C}}$-module. Then $U \mapsto p^{-1}(U)$ defines a bijection from the collection of proper submodules of V' onto the collection of proper submodules of V containing $\ker p$. It follows that V' is irreducible if and only if $\ker p$ is a proper maximal submodule of V. The equivalence of (d) and (e) now readily follows. □

Corollary 31.23 *Let V be a finite dimensional irreducible $\mathfrak{g}_{\mathbb{C}}$-module. Then V has a highest weight vector v, which is unique up to a scalar factor. Let λ be the weight of v. Then all assertions of Proposition 31.22 are valid.*

Proof: It follows from Lemma 31.20 that V has a highest weight vector. Let v be any highest weight vector in V and let λ be its weight. By irreducibility of V, the vector v is cyclic. Hence all assertions of Proposition 31.22 are valid. Note that $W = \{0\}$ is the unique maximal proper submodule.

Let w be a second highest weight vector and let μ be its weight. Then all assertions of Proposition 31.22 are valid. Hence $\mu \in \lambda - \mathbb{N}R^+$ and $\lambda \in \mu - \mathbb{N}R^+$, from which $\mu - \lambda \in \mathbb{N}R^+ \cap (-\mathbb{N}R^+) = \{0\}$. It follows that $\mu = \lambda$; hence $w \in V_\lambda = \mathbb{C}v$. □

Remark 31.24 For obvious reasons the above weight λ is called the *highest weight* of the irreducible $\mathfrak{g}_{\mathbb{C}}$-module V, relative to the choice R^+ of positive roots.

The following theorem is the first step towards the classification of all finite dimensional irreducible representations of $\mathfrak{g}_{\mathbb{C}}$.

Theorem 31.25 *Let V and V' be irreducible $\mathfrak{g}_{\mathbb{C}}$-modules. If V and V' have the same highest weight (relative to R^+), then V and V' are isomorphic (i.e., the associated $\mathfrak{g}_{\mathbb{C}}$-representations are equivalent).*

Proof: We denote the highest weight by λ and fix associated highest weight vectors $v \in V_\lambda \setminus \{0\}$ and $v' \in V'_\lambda \setminus \{0\}$. We consider the direct sum $\mathfrak{g}_{\mathbb{C}}$-module $V \oplus V'$ and denote by W the smallest $\mathfrak{g}_{\mathbb{C}}$-submodule containing the vector $w := (v, v')$. Then w is a cyclic weight vector of W, of weight λ.

Let $p : V \oplus V' \to V$ be the projection onto the first component, and $p' : V \oplus V' \to V'$ the projection onto the second. Then p and p' are $\mathfrak{g}_{\mathbb{C}}$-module homomorphisms. Since $p(w) = v$, it follows that $p|_W$ is surjective onto V. Similarly, $p'|_W$ is surjective onto V'. It follows that V, V' are both irreducible quotients of W, hence isomorphic by Proposition 31.22 (e). $\square$

Remark 31.26 In the above proof it is easy to deduce that in fact W is irreducible, and $p|_W$ and $p'|_W$ are isomorphisms from W onto V and V', respectively.