Extra exercises 2019

Exercise 1. We assume that G is a Lie group, and (π, V) a finite dimensional continuous represenation of G. Show that for all $x \in G$ and $X, Y \in \mathfrak{g}$ the following identities are valid:

(a) $\pi(x)\pi_*(Y) = \pi_*(\mathrm{Ad}\,(x)Y)\pi(x)$;

(b) $\pi_*(X)\pi_*(Y) = \pi_*(Y)\pi_*(X) + \pi_*([X, Y])$.

Exercise 2. Let G and H be Lie groups. If (π, V) and (ρ, W) are finite dimensional continuous representations of G and H, respectively, then the exterior tensor product of π and ρ is defined to be the representation $\pi\widehat{\otimes}\rho$ of $G \times H$ in $V \otimes W$ given by

$$\pi\widehat{\otimes}\rho(g, h) = \pi(g) \otimes \rho(h), \qquad ((g, h) \in G \times H).$$

(a) Show that the representation $\pi\widehat{\otimes}\rho$ is continuous finite dimensional.

(b) Show that the character of $\pi\widehat{\otimes}\rho$ is the function $\chi_{\pi\widehat{\otimes}\rho} : G \times H \to \mathbb{C}$ given by

$$\chi_{\pi\widehat{\otimes}\rho}(g, h) = \chi_\pi(g)\chi_\rho(h).$$

(c) If G and H are compact show that the following assertions are equivalent.

(1) π is irreducible as a representation of G and ρ is irreducible as a representation of H;

(2) $\pi\widehat{\otimes}\rho$ is irreducible as a representation of $G \times H$.

Exercise 3. We assume that G and H are compact Lie groups, and that (π, V) is a finite dimensional continuous representation of $G \times H$. We identify G and H with the subgroups $G \times \{e_H\}$ and $\{e_G\} \times H$ of $G \times H$.

(a) For δ a finite dimensional irreducible representation of G, show that the projection operator $P_\delta : V \to V$ associated with $\pi|_G$ as in Exercise 33 commutes with $\pi(H)$.

We now assume that π is irreducible.

(b) Show that $P_\delta(V) = V$.

(c) Show that the canonical map $V_\delta \otimes \mathrm{Hom}_G(V_\delta, V) \to V$ is a linear isomorphism.

(d) Show that the natural representation ρ of H in $\mathrm{Hom}_G(V_\delta, V)$ is irreducible and that

$$\pi \simeq \delta\widehat{\otimes}\rho.$$

Exercise 4. The purpose of this exercise is to understand the group of fractional linear transformations.

We define $\mathbb{P}^1(\mathbb{C})$ to be the space of lines $\mathbb{C}z \subset \mathbb{C}^2$ with $z = (z_0, z_1) \in \mathbb{C} \setminus \{0\}$. The line $\mathbb{C}z$ is denoted by $[z_0 : z_1]$. Accordingly, $[z_0 : z_1] = [w_0 : w_1]$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $z = \lambda w$ (homogeneous coordinates).

(a) Show that the action of $\mathbb{C}^*$ on $\mathbb{C}^2 \setminus \{0\}$ defined by scalar multiplication is smooth and of principal fiber bundle type.

(b) Show that the map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1(\mathbb{C})$ factors through a bijection $\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^* \simeq \mathbb{P}^1(\mathbb{C})$. We equip $\mathbb{P}^1(\mathbb{C})$ with the structure of (complex) manifold which turns this map into a diffeomorphism.

(c) Show that the natural action of $\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ given by $(g, \mathbb{C}z) \mapsto g(\mathbb{C})$ is smooth.

(d) Show that $\mathbb{P}^1(\mathbb{C}) \simeq \mathrm{GL}(2, \mathbb{C})/B$, where B is the group of two by two invertible matrices with $zero$ in the lower left entry.

(e) Show that the map $j : \mathbb{C} \to \mathbb{P}(\mathbb{C})$ given by $z \mapsto [z : 1]$ is an open embedding whose image is the complement of a point, which we denote by ∞. Accordingly, we view $\mathbb{P}(\mathbb{C})$ a the Riemann sphere $\widehat{\mathbb{C}}$.

(f) For $z \in \widehat{\mathbb{C}}$ and $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Exercise 5. The purpose of this exercise is to understand covering homomorphisms of Lie groups.

We recall that a smooth map $f : M \to N$ between connected smooth manifolds is a smooth covering if for every $q \in N$ there exists an open neighborhood $V \ni q$ in N such that $f^{-1}(V)$ can be written as $\cup_{j \in J} U_j$, with $\{U_j \mid j \in J\}$ a non-empty disjoint collection of open subsets in M such that for each $j \in J$ the map $f|_{U_j}$ is a diffeomorphism from U_j onto V. By using connectedness of N, it is easy to check that such a covering f is surjective. The covering f is said to be a covering with base points if it comes equipped with a pair of points $(m_0, n_0) \in M \times N$ such that $f(m_0) = n_0$.

(a) Let $f : G \to H$ be a homomorphism of connected Lie groups. Show that the following assertions are equivalent:

(1) $T_e f$ is bijective;

(2) f is a smooth covering (with base points e_G, e_H).

Suppose now that M is a smooth manifold and $m_0 \in M$. Then there exists a smooth covering $\pi : \widetilde{M} \to M$ with a pair of basepoints $(\tilde{m}_0, m_0) \in \widetilde{M} \times M$ such that $\widetilde{M}$ is connected and simply connected. Any such covering has the following universal property.

If $p : (Q, q_0) \to (N, n_0)$ is a covering with basepoints and if $f : M \to N$ is a smooth map with $f(m_0) = n_0$ then there exists a unique smooth map $\tilde{f} : \widetilde{M} \to Q$ such that $\tilde{f}(\tilde{m}_0) = \tilde{q}_0$ and such that the following diagam commutes:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\tilde{f}} & Q \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & N \end{array}$$

From the universal property it follows that $(\tilde{M}, \tilde{m}_0)$ is determined modulo isomorphism (of obvious type) of based coverings. For this reason, it is called the universal cover with base point of (M, m_0).

(b) Let G be a connected Lie group and $\pi : \widetilde{G} \to G$ its universal covering with base point $\tilde{e} \in \tilde{G}$. Show that $\widetilde{G}$ has a unique structure of Lie group with neutral element $\tilde{e}$ such that π is a Lie group homomorphism.

(c) If $f : G \to H$ is a Lie group homomorphism of connected Lie groups, $\pi : \widetilde{G} \to G$ the universal covering homomorphism and $p : K \to H$ a covering homomorphism, show that there exists a unique Lie group homomorphism $\tilde{f} : \tilde{G} \to K$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\tilde{f}} & K \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{f} & H \end{array}$$

(d) Let $\widetilde{\mathfrak{g}}$ denote the Lie algebra of $\widetilde{G}$. Show that $\pi_* : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism of Lie algebras.

Exercise 6. This exercise is a continuation of the previous one. Its purpose is establish the following result.

Lemma *Let G, H be two connected Lie groups with algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, then there exists a unique homomorphism $f : G \to H$ of Lie groups such that $T_e f = \varphi$.*

(a) Show that $\mathfrak{k} = \mathrm{graph}(\varphi)$ is a Lie subalgebra of the direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$.

(b) Let K be the connected Lie subgroup of $G \times H$ generated by $\exp(\mathfrak{k})$. Show that the projection map $p_1 : G \times H \to G$ restricts to a smooth covering homomorphism $p_1|_K : K \to G$.

(c) Show that $p_1|_K : K \to G$ is an isomorphism of Lie groups.

(d) Complete the proof of the above theorem.

(e) Let V be a finite dimensional complex vector space. Show that the map $\pi \mapsto \pi_*$ induces a bijection between the set of continuous representations of G in V and the set of Lie algebra representations of $\mathfrak{g}$ in V.

Conclusion: If G is connected and simply connected, every finite dimensional representation of $\mathfrak{g}$ can be 'lifted' to a continuous finite dimensional representation of G.

Without proof we mention the following beautiful result of Hermann Weyl: if G is a compact connected Lie group, and $\mathfrak{g}$ has trivial center, then the universal cover $\widetilde{G}$ is compact. Thus, the finite dimensional representations of $\mathfrak{g}$ can be lifted to continuous representations of $\tilde{G}$ and the global methods based on compactness can still be used.

Example: $\mathrm{SO}(3)$ is compact and its Lie algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ has trivial center. The group $\mathrm{SU}(2)$ is simply connected and the natural homomorphism $p : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a covering. Therefore, $p : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is the universal covering homomorphism. Every finite dimensional representation ρ of $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$ lifts to a finite dimensional continuous representation $\tilde{\rho}$ of $\mathrm{SU}(2)$.

(f) Show that ρ lifts to $\mathrm{SO}(3)$ if and only $\tilde{\rho}(x) = I$ for $x \in \ker p = \{-I, I\}$.