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Introduction

Let $X = G/H$ be a homogeneous space of a Lie group $G$, and let $D: C^\infty(X) \to C^\infty(X)$ be a non-trivial $G$-invariant differential operator. One of the natural questions one can ask for the operator $D$ is whether it is solvable, in the sense that $DC^\infty(X) = C^\infty(X)$. If $G$ is the group of translations of $X = \mathbb{R}^n$ and $H$ is trivial, then $D$ has constant coefficients, and it is a well known result of Ehrenpreis and Malgrange that hence $D$ is solvable.

Assume for simplicity that $G/H$ carries an invariant measure. This measure induces a bilinear pairing of $C^\infty_c(X)$, the space of compactly supported smooth functions on $X$, with itself. Let $D^*$ denote the adjoint of $D$ with respect to this pairing. The strategy employed by Ehrenpreis and Malgrange was essentially to use the following properties of $D$:

(i) There exists a fundamental solution for $D$, that is, $\delta \in D'\mathcal{D}(X)^H$, where $\delta$ is the Dirac measure at the origin, and $\mathcal{D}(X)^H$ is the space of left-$H$-invariant distributions on $X$.
(ii) For each compact set $\Omega \subset X$ there exists a compact set $\Omega' \subset X$ such that

$$\text{supp } D^*f \subset \Omega \Rightarrow \text{supp } f \subset \Omega'$$

for all $f \in C^\infty_c(X)$.

In fact, for $X = \mathbb{R}^n$ one can take as $\Omega'$ the convex hull of $\Omega$. For this reason the support property (ii) has become known as the $D$-convexity of $X$. It follows from (i)–(ii) that $D$ is solvable.

The strategy has been applied in other cases as well, for example by Helgason in [14], where surjectivity is established for all non-trivial invariant differential operators on a Riemannian symmetric space. In a variant of the strategy (i) is replaced by the following weaker property (semi-global solvability):
(i') For each compact set \( \Omega \subset X \) and each function \( g \in C^\infty(X) \) there exists a function \( f \in C^\infty(X) \) such that \( Df = g \) on \( \Omega \).

The conjunction of (i') and (ii) is equivalent with the solvability of \( D \) (see Theorem 1). This is used by Rauch and Wigner in [19] where it is proved that the Casimir operator on a semisimple Lie group is solvable, and more generally by Chang in [6] where the Laplace-Beltrami operator on a semisimple symmetric space is shown to be solvable.

The purpose of the present paper is to give, also for a semisimple symmetric space \( X = G/H \), a sufficient condition on an invariant differential operator \( D \) to imply (ii), the \( D \)-convexity of \( X \). When \( G/H \) has rank one, our result follows from the above mentioned result of Chang, since the algebra \( \mathcal{D}(G/H) \) of all invariant differential operators in this case is generated by the Laplace-Beltrami operator. In general this is not so, and our result shows the \( D \)-convexity for a significantly larger class of operators \( D \). In particular, when \( G/H \) is split (that is, it has a vectorial Cartan subspace), all non-trivial elements of \( \mathcal{D}(G/H) \) satisfy our condition.

Though we do not consider the properties (i) or (i') in this paper, we notice that in the above-mentioned references, an important step towards obtaining (i') is to prove that \( D^* \) acts injectively on, say \( C_c^\infty(X) \) (see for example [6]). In fact the injectivity of \( D^* \) is an immediate consequence of (i'). In the present case of a semisimple symmetric space, the sufficient condition that we give for (ii) is also sufficient for \( D^* \) to be injective.

We also give a condition on \( D \), which is necessary for both the \( D \)-convexity and the injectivity. When \( G/H \) is not split, there exists a non-trivial operator in \( \mathcal{D}(G/H) \), which does not satisfy this condition. In particular, we conclude that \( D \)-convexity holds for all non-trivial elements of \( \mathcal{D}(G/H) \) if and only if \( G/H \) is split. This provides a large class of spaces \( G/H \) for which there exist non-solvable non-trivial invariant differential operators. Unfortunately, our necessary condition is weaker than the sufficient condition, and the complete classification of all \( D \in \mathcal{D}(G/H) \), for which \( D \)-convexity holds, remains open (for non-split \( G/H \)).

In the special case where the semisimple symmetric space is Riemannian (that is, when \( H \) is compact), we have that \( G/H \) is split and thus our condition reduces to the requirement that \( D \) is non-trivial. In this case our result is part of the above-mentioned proof by Helgason that \( D \) is surjective (see [14, p. 473]). Helgason’s proof is based on his inversion formula and Paley-Wiener theorem for the Fourier transform on the Riemannian symmetric space \( X \). These results in turn rely heavily on the work of Harish-Chandra. Simplifications avoiding these strong tools were given by Chang [7] and Dadok [8]. In another special case, that of a semisimple Lie group considered as a symmetric space, our result was obtained by Duflo and Wigner [9].
All of the references mentioned above, except [14], use the uniqueness theorem of Holmgren to derive the D-convexity of $X$, and so do we. The main difficulty in the present generalization lies in the handling of the more complicated geometry of $X$. Our main tool to overcome this difficulty is the convexity theorem of [1].

In [3] (see also [4]) the result of the present paper will be applied to obtain injectivity of the Fourier transform on $C_0^\infty(X)$. Our reasoning will thus be the opposite of the original reasoning of Helgason in the Riemannian case: we shall deduce properties of the Fourier transform from the D-convexity.

Motivation

As mentioned in the introduction the main motivation for studying D-convexity is the following theorem. Here $G$ is a Lie group (with at most countably many connected components) and $H$ is a closed subgroup, of which we only assume that $G/H$ carries an invariant measure (this assumption is only used for defining $D^\ast$).

**THEOREM 1.** Let $D \in \mathcal{D}(G/H)$ be an invariant differential operator. Then $D$ is solvable if and only if (i') and (ii) hold.

**Proof.** This follows from [22, Ch. I, Thm. 3.3], using regularization by $C_0^\infty(G)$ to prove the equivalence of our definition of D-convexity with that of [22, Ch. I, Def. 3.1]. Note also the final remark of that section in loc. cit.

Notation

From now on, let $G$ be a real reductive Lie group of Harish-Chandra’s class, $\tau$ an involution of $G$, and $H$ an open subgroup of the fixed point group $G^\tau$. Then $X = G/H$ is a reductive symmetric space of Harish-Chandra’s class (see [2]). Let $K$ be a $\tau$-stable maximal compact subgroup of $G$, and let $\theta$ be the associated Cartan involution. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} + \mathfrak{f} + \mathfrak{p}$ be the eigen-decompositions of the Lie algebra $\mathfrak{g}$ induced by $\tau$ and $\theta$, then $\mathfrak{h}$ and $\mathfrak{f}$ are the Lie algebras of $H$ and $K$, respectively. Let $B$ be a non-degenerate, $G$- and $\tau$-invariant bilinear form on $\mathfrak{g}$ which extends the Killing form on $[\mathfrak{g}, \mathfrak{g}]$, and which is negative definite on $\mathfrak{f}$ and positive definite on $\mathfrak{p}$. Then the above-mentioned eigen-decompositions are orthogonal with respect to $B$.

Fix a maximal abelian subspace $a$ of $\mathfrak{p} \cap \mathfrak{q}$, and a maximal abelian subspace (a *Cartan subspace*) $a_1$ of $\mathfrak{q}$, containing $a$. Then $a = a_1 \cap \mathfrak{p}$. Let $m$ be the orthocomplement (with respect to $B$) of $a$ in its centralizer $\mathfrak{g}^a$, and let $a_m = a_1 \cap m$. Via the orthogonal decomposition $a_1 = a_m + a$ we view $a_1$ and
as subspaces of \( \mathfrak{a}_c^* \). Let \( \Sigma \) and \( \Sigma_1 \) denote the root systems of \( a \) and \( a_1 \) in \( \mathfrak{g}_c \), respectively, then \( \Sigma \) consists of the non-trivial restrictions to \( a \) of the elements of \( \Sigma_1 \). Denote by \( W \) and \( W_1 \) the Weyl groups of these two root systems, then \( W \) is naturally isomorphic to \( N_{W_1}(a)/Z_{W_1}(a) \), the normalizer modulo the centralizer of \( a \) in \( W_1 \), and to \( N_K(a)/Z_K(a) \), the normalizer modulo the centralizer of \( a \) in \( K \). Let \( W_{K \cap H} \) be the canonical image of \( N_{K \cap H}(a) \) in \( W \).

Recall that \( G = KA_{AH} \), and that if \( g = kah \) according to this decomposition, then the orbit \( W_{K \cap H} \log a \) is uniquely determined by \( g \). For a \( W_{K \cap H} \)-invariant set \( S \subset a \), we denote the subset \( K \exp(S)H \) of \( X \) by \( X_S \). Then \( S = \{ \log a | aH \in X_S \} \), and every \( K \)-invariant subset of \( X \) is of the form \( X_S \).

**Invariant differential operators**

Let \( \mathbb{D}(G/H) \) be the algebra of invariant differential operators on \( G/H \). Let \( U(g) \) be the enveloping algebra of \( \mathfrak{g}_c \) and \( U(g)^H \) the subalgebra of \( H \)-invariant elements, then there is a natural isomorphism of the quotient \( U(g)^H/(U(g)^H \cap U(g)_H) \) with \( \mathbb{D}(G/H) \), induced by the right action \( R \) of \( U(g) \) on \( C^\infty(G) \) (see [15, p. 285]).

Let \( \Sigma_1^+ \) be a positive system for \( \Sigma_1 \), and let \( n_1 \) be the sum of the corresponding positive root spaces \( \mathfrak{g}_c^\alpha (\alpha \in \Sigma_1^+) \). We have the following direct sum decomposition

\[
\mathfrak{g}_c = n_1 + a_1 + \mathfrak{h}_c.
\]  

(1)

Using this decomposition and Poincare-Birkhoff-Witt, a map \( \gamma : U(g) \to U(a_1) \) is defined by \( u \equiv \gamma(u) \) modulo \( n_1 U(g) + U(g)_H \). From this map an algebra isomorphism \( \gamma \) of \( \mathbb{D}(G/H) \simeq U(g)^H/(U(g)^H \cap U(g)_H) \) onto \( S(a_1)^W \), the set of \( W_1 \)-invariant elements in the symmetric algebra of \( a_1 \) (which is isomorphic to \( U(a_1) \) because \( a_1 \) is abelian), is obtained by letting \( \gamma(u)(\lambda) = \gamma(u)(\lambda + \rho_1) \) for \( u \in U(g)^H \), \( \lambda \in a_1^* \) (see [11, p. 15, Thm. 3]). Here \( \rho_1 \in a_1^* \) is given by half the trace of the adjoint action on \( n_1 \). Thus \( \mathbb{D}(G/H) \) is identified as a polynomial algebra with \( \dim a_1 \) independent generators.

Assume that \( \Sigma_1^+ \) is chosen to be compatible with \( a \), that is, the set of nonzero restrictions to \( a \) of elements from \( \Sigma_1^+ \) is a positive system \( \Sigma^+ \) for \( \Sigma \). Let \( n \) be the sum of the corresponding positive root spaces \( \mathfrak{g}_c^\alpha (\alpha \in \Sigma^+) \), then we also have the following direct sum decomposition

\[
\mathfrak{g} = n + m + a + \mathfrak{h}.
\]  

(2)

Let \( \rho \in \mathfrak{a}^* \) and \( \rho_m \in \mathfrak{a}_{mc}^* \) be given by half the trace of the adjoint actions on \( n \), and on \( n_1 \cap m_c \), respectively.
Using the decomposition (2) a map $\eta: U(g) \to U(a)$ is defined by $u \equiv \eta(u)$ modulo $(n_c + m_c)U(g) + U(g)h_c$, and we obtain by restriction to $U(g)^H$ a homomorphism, also denoted $\eta$, from $\mathbb{D}(G/H) \simeq U(g)^H/(U(g)^H \cap U(g)h_c)$ into $S(a)$. Let $\eta(D) \in S(a)$ be defined by $\eta(D)(\lambda) = \eta(D)(\lambda + p)$.

**Lemma 1.** We have

$$\eta(D)(\lambda) = \gamma(D)(\lambda - \rho_m)$$

(3)

for all $D \in (G/H)$, $\lambda \in \mathfrak{a}^*$. Moreover $\eta(D) \in S(a)^W$, and $\eta(D)$ is independent of the choice of $\Sigma^+$. 

**Proof.** We first prove the following equation:

$$\rho_1 = \rho + \rho_m.$$  

(4)

We have

$$\rho_1 = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+} (\dim g_\alpha)\alpha$$  

and

$$\rho_m = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+, \alpha|_a = 0} (\dim g_\alpha)\alpha.$$  

Let

$$\tilde{\rho} = \rho_1 - \rho_m = \frac{1}{2} \sum_{\alpha \in \Sigma_1^+, \alpha|_a \neq 0} (\dim g_\alpha)\alpha,$$

then it is clear that $\tilde{\rho} = \rho$ on $a$. On the other hand, since the set of $\alpha \in \Sigma_1^+$ with $\alpha|_a \neq 0$ is $\sigma\theta$-invariant, we get that $\sigma\theta\tilde{\rho} = \tilde{\rho}$, and hence $\tilde{\rho} = 0$ on $a_m$, so that in fact $\tilde{\rho} = \rho$.

Since $m_c = m_c \cap n_1 + a_m + m_c \cap h_c$ it follows from (1) and (2) that $\eta(D)(\lambda) = \gamma(D)(\lambda)$. From this and (4) we get (3).

The proof will be completed by using the following observation: Every element $w \in W$ can be represented by an element $w \in N_{W, \Sigma}(n)$; this element also normalizes $a_m$, and can be chosen so that $\tilde{w} \rho_m = \rho_m$.

The $W$-invariance of $\eta(D)$ now follows from (3) and the $W$-invariance of $\gamma(D)$, in view of the above observation. By using this observation once more, it follows from (3) and the fact that $\gamma$ is independent of the choice of the positive system $\Sigma_1^+$, that $\eta$ is independent of the choice of $\Sigma^+$. \hfill \Box

Let $s: S(g) \to U(g)$ be the symmetrization map, then the restriction of $s$ to the set $S(q)^H$ of $H$-invariants in $S(q)$ gives rise to a linear bijection (also denoted by $s$) of $S(q)^H$ with $\mathbb{D}(G/H)$ (see [15, p. 287, Thm. 4.9]). A differential operator $D \in \mathbb{D}(G/H)$ is called *homogeneous* if it is the image of a homogeneous element.
LEMMA 2. Let $D \in \mathbb{D}(G/H)$ be non-constant and let $D = s(P)$, $P \in S(q)^H$. Then

$$\deg(\eta(D) - r(P)) < \deg P = \text{order } D. \quad (5)$$

In particular, if $D$ is homogeneous then $\deg(D) = \text{order } D$ if and only if $r(P) \neq 0$.

Proof: That $\text{order } D = \deg P$ follows from the explicit expression for $s(P)$ in [15, p. 287, Thm. 4.9]. Let $r_1(P)$ denote the restriction of $P$ to $a_1$, then it follows from [15, p. 305, Eq. (38)] that

$$\deg(\gamma(D) - r_1(P)) < \deg P. \quad (6)$$

It follows from (3) that $\eta(D) - r(P)$ and the restriction of $\gamma(D) - r_1(P)$ to $a$ have the same degree, and hence (5) follows from (6). If $P$ is homogeneous, then either $\deg r(P) = \deg P$ or $r(P) = 0$, and the final statement follows from (5). \bbox

Notice that $r_1(P)$ has the same degree as $P$ (to see this, let $P$ be homogeneous, then $\deg r_1(P) = \deg P$ unless $r_1(P) = 0$. But $r_1(P) = 0$ implies $P = 0$ by the $H$-invariance, because $\text{Ad}(H)(a_1)$ contains an open subset of $q$). Hence it follows from (6) that also $\gamma(D)$ has this degree (which equals the order of $D$). Thus $\gamma$ is a degree preserving isomorphism of $\mathbb{D}(G/H)$ onto $S(a_1)^W$.

However, a similar statement is not valid for $\eta(D)$; its degree can be strictly smaller than that of $D$. In fact $\eta$ is not injective in general: Since $\mathbb{D}(G/H)$ and $S(a)^W$ are polynomial algebras in $\dim a_1$ and $\dim a$ algebraically independent generators, respectively, $\eta$ is not injective if $a \neq a_1$ (otherwise it would cause the existence of an injection of the quotient field of $\mathbb{D}(G/H)$ into the quotient field of $S(a)^W$, which is impossible, since their transcendence degrees over $\mathbb{C}$ are $\dim a_1$ and $\dim a$, respectively (see [23, Ch. II, §12])). On the other hand, if $a_1 = a$, in which case the symmetric space $G/H$ is called split, then $\eta$ is injective since it equals $\gamma$. Examples of split symmetric spaces are the Riemannian symmetric spaces and the symmetric spaces of $K$-type (see [18]). In the special case (the 'group case') of a semisimple Lie group $G'$ considered as a symmetric space, where $G$ is $G' \times G'$ and $H$ is the diagonal, the notion of split for the space $G/H$ coincides with the notion of split (also called a normal real form) for $G'$.

Notice also that $\eta$ in general is not surjective. This can be seen already in the group case mentioned above, where $\mathbb{D}(G/H)$ is naturally isomorphic with $Z(g')$, the center of $U(g')$, and where $\eta$ by transference under a suitable isomorphism can be identified with the natural homomorphism of $Z(g')$ into $\mathbb{D}(G'/K')$. It is known from [13, 16] that this homomorphism is surjective when $G'$ is classical, but not surjective for certain exceptional groups $G'$.

For $v \in S(a_1)$ or $v \in S(a)$ we define $v^*$ by $v^*(v) = v(-v)$, where $v \in a_1^*$ or $v \in a^*$. 

For $\gamma(D) = \gamma(D) - r(P)$ and the restriction of $\gamma(D) - r_1(P)$ to $a$ have the same degree, and hence (5) follows from (6). If $P$ is homogeneous, then either $\deg r(P) = \deg P$ or $r(P) = 0$, and the final statement follows from (5). \bbox

Notice that $r_1(P)$ has the same degree as $P$ (to see this, let $P$ be homogeneous, then $\deg r_1(P) = \deg P$ unless $r_1(P) = 0$. But $r_1(P) = 0$ implies $P = 0$ by the $H$-invariance, because $\text{Ad}(H)(a_1)$ contains an open subset of $q$). Hence it follows from (6) that also $\gamma(D)$ has this degree (which equals the order of $D$). Thus $\gamma$ is a degree preserving isomorphism of $\mathbb{D}(G/H)$ onto $S(a_1)^W$.

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For $v \in S(a_1)$ or $v \in S(a)$ we define $v^*$ by $v^*(v) = v(-v)$, where $v \in a_1^*$ or $v \in a^*$.
LEMMA 3. Let $D \in \mathcal{D}(G/H)$. Then $\gamma(D^*) = \gamma(D)^*$ and $\eta(D^*) = \eta(D)^*$.

Proof. Choose $u \in U(g)^H$ such that $D = R_u$, and let $v \mapsto \tilde{v}$ be the antiautomorphism of $U(g)$ determined by $\tilde{v} = -v$ for $v \in g$. Using [15, Ch. I, Thm. 1.9 and Lemma 1.10] it is easily seen that $D^* = R_u$. The equality for $\gamma$ will follow if we prove that $\gamma(u) = \gamma(u)^*$ for $u \in U(g)^H$. Using [11, p. 16, Cor. 4] it is now seen that it suffices to consider the case of a Riemannian symmetric space, that is, we may assume that $H$ is compact. In this special case, the statement is proved in [15, p. 307]. This proves that $\gamma(D^*) = \gamma(D)^*$.

From (3) we now get that

$$\eta(D^*)(\lambda) = \gamma(D^*)(\lambda - \rho_m) = \gamma(D)(-\lambda + \rho_m).$$

Using the fact that there exists an element $w$ in the Weyl group of the root system of $\alpha_m$ in $m$ such that $w\rho_m = -\rho_m$, and that this Weyl group is a subgroup of $W_1$, we get that

$$\gamma(D)(-\lambda + \rho_m) = \gamma(D)(-\lambda - \rho_m) = \eta(D)(-\lambda),$$

proving the equality for $\eta$. \qed

In the final section of this paper we relate $\eta(D)$ to the radial part of $D$ with respect to the $KAH$ decomposition. In particular we shall prove that the condition $\eta(D) = 0$ has the following strong consequence:

LEMMA 4. Let $D \in \mathcal{D}(G/H)$ and assume that $\eta(D) = 0$. Then $Df = 0$ for all $K$-invariant smooth functions $f$ on $G/H$.

Convexity

We are now ready to state our main theorem:

THEOREM 2. Let $D \in \mathcal{D}(G/H)$ be non-zero.

(i) If $\deg \eta(D) = \text{order } D$ then

$$\text{supp } f \subset X_S \iff \text{supp } Df \subset X_S \iff \text{supp } D^*f \subset X_S$$

for all $f \in C_c^\infty(X)$ and all convex, compact $W_{K \cap H}$-invariant sets $S \subset \alpha$. In particular, $X$ is $D$-convex, and $D^*$ is injective on $C_c^\infty(X)$.

(ii) If $\eta(D) = 0$ there exists for each closed ball $S \subset \alpha$, centered at the origin, a function $f \in C_c^\infty(X)$ such that $D^*f = 0$ and $\text{supp } f = X_S$. In particular, $X$ is not $D$-convex, and $D^*$ is not injective on $C_c^\infty(X)$.
Proof. We first prove (i). The implication of \( \text{supp} \, Df \subset X_S \) from \( \text{supp} \, f \subset X_S \) is obvious. Assume \( \text{supp} \, Df \subset X_S \). Expanding \( f \) as a sum of \( K \)-finite functions, we have, since \( X_S \) is \( K \)-invariant, that \( f \) is supported in \( X_S \) if and only if all the summands are supported in \( X_S \). Moreover, \( D \) can be applied termwise to the sum, and hence we see that we may assume \( f \) to be \( K \)-finite. Then the support of \( f \) is \( K \)-invariant, and it suffices to prove that \( \text{supp} \, f \cap AH \subset \exp(S)H \).

Let \( m = \text{order} \, D \), then \( m = \deg \eta(D) \) by the assumption on \( D \). Let \( u_0 \) denote the homogeneous part of \( \eta(D) \) of degree \( m \), then \( u_0 \neq 0 \). Notice that \( u_0 \) is also the homogeneous part of \( \eta(D) \) of degree \( m = \deg \eta(D) \) for any choice of \( \Sigma^+ \).

Assume that \( \text{supp} \, f \cap AH \neq \exp(S)H \), and write

\[
\text{supp}_a f = \{ Y \in a | \exp(Y)H \in \text{supp} \, f \}.
\]

Then \( \text{supp}_a f \) is compact and not contained in \( S \). By the convexity of \( S \) there exists a non-empty open set of linear forms \( \lambda \in a^* \) with the property that

\[
0 < \max_{Y \in S} \lambda(Y) < \max_{Y \in \text{supp}_a f} \lambda(Y).
\]

(7)

Since \( u_0 \neq 0 \) there exists a \( \lambda \in a^* \) with \( u_0(\lambda) \neq 0 \), and satisfying (7). Let \( Y_0 \in \text{supp}_a f \) be a point where the value on the right side of (7) is attained. Then \( Y_0 \notin S \) and we have that

\[
\lambda(Y) \leq \lambda(Y_0), \quad (Y \in \text{supp}_a f).
\]

(8)

Let \( a_0 = \exp Y_0 \), then

\[
a_0H \notin \text{supp} \, Df
\]

(9)

by the assumption on \( \text{supp} \, Df \), and

\[
a_0H \in \text{supp} \, f.
\]

(10)

Choose a positive system \( \Sigma^+ \) such that \( \lambda \) is antidominant, and let \( n \) and \( N \) be given correspondingly. Let \( \Omega \) denote the open (see [21, Prop. 7.1.8]) subset \( \Omega = NMAH \) of \( X = G/H \), and define \( g \in C^\infty(\Omega) \) by \( g(nmaH) = \lambda(\log a) \) for \( n \in N, \ m \in M, \ a \in A \). We claim that

\[
f = 0 \quad \text{on} \quad \{ x \in \Omega | g(x) > g(a_0) \}.
\]

(11)

To prove (11) let \( x = nmaH \in \Omega \cap \text{supp} \, f \). Then we must show that \( g(x) \leq g(a_0) \).
or equivalently, that $\lambda(\log a) \leq \lambda(Y_0)$. To see that this holds, write

$$nma = k \exp(Z)h, \quad (k \in K, \ Z \in a, \ h \in H_e)$$

according to the $G = KAH_e$ decomposition; here $H_e$ denotes the identity component of $H$. Then

$$\exp(Z)h \in KNa = KNaN,$$

and by the convexity theorem of [1, Thm. 3.8] it follows that $\log a = U + V$, where $U$ is contained in the convex hull of $W_{K \cap H}Z$, and $V$ belongs to a certain subcone of the closed convex cone $\{ V \in a | \langle V, Y \rangle \geq 0, \ Y \in a^+ \}$, which is dual to the positive Weyl chamber $a^+$. In particular, $\lambda(V) \leq 0$ by the antidominance of $\lambda$, and hence

$$\lambda(\log a) \leq \lambda(U) \leq \max_{w \in W_{K \cap H}} \lambda(wZ).$$

Now $\exp(wZ)H = w \exp(Z)H = wk^{-1}xH$ for $w \in W_{K \cap H}$, and from $x \in \text{supp } f$ and the $K$-invariance of the support we then see that $\exp(wZ)H \in \text{supp } f$. Hence $wZ \in \text{supp } f$, and we conclude by (8) that

$$\lambda(\log a) \leq \lambda(Y_0).$$

This implies (11).

Let $\sigma(D)$ be the principal symbol of $D$. We have

$$\sigma(D)(dg(a_0)) = \frac{1}{m!} D((g - g(a_0))^m)(a_0). \quad (12)$$

It follows immediately from the definition of $g$ that $R_u g = 0$ for $u \in U(g)h_e$. Moreover, since $g$ is left $NM$-invariant, and since $n$ and $m$ are normalized by $A$, we also have that $R_u g(a) = 0$ for $a \in A$, $u \in (n + m)U(g)$. Hence $Dg(a) = R_{\sigma(D)}g(a)$. Applying the same reasoning to the function $(g - g(a_0))^m$ we obtain that

$$D((g - g(a_0))^m)(a) = R_{\sigma(D)}(g - g(a_0))^m(a) = m! u_0(\lambda). \quad (13)$$

Combining (12) and (13) we obtain that $\sigma(D)(dg(a_0)) = u_0(\lambda)$ and hence

$$\sigma(D)(dg(a_0)) \neq 0 \quad (14)$$

by the assumption on $\lambda$. 
From (9), (11) and (14) it follows by Holmgren's uniqueness theorem ([17, Thm. 5.3.1]) that \( f = 0 \) on a neighbourhood of \( a_0 H \), contradicting (10). This completes the proof of the first biimplication in (i). From Lemma 3 we get that \( D^* \) also satisfies the assumption of (i), and hence the remaining statements in (i) follow.

We now prove (ii). Let \( S \) be the ball of radius \( R \) centered at the origin, and let \( \varphi \in C^\infty(\mathbb{R}) \) be positive on \([0; R^2[\) and zero on \([R^2; \infty[\). Define \( f(kaH) = \varphi(\|\log a\|^2) \) for \( k \in K, a \in A \). Then \( f \in C^\infty(X) \) by [10, Thm. 4.1], and we clearly have \( \text{supp } f = X_S \). Now (ii) follows from Lemma 4.

\[ \square \]

COROLLARY 1

(i) If \( X = G/H \) is split, then \( X \) is \( D \)-convex and \( D \) is injective on \( C^\infty_c(X) \) for all non-trivial invariant differential operators \( D \).

(ii) If \( X \) is not split there exists a non-trivial invariant differential operator \( D \), such that \( X \) is not \( D \)-convex and such that \( D \) is not injective on \( C^\infty_c(X) \).

REMARK 1. By regularization it follows that the statements of Theorem 2 and its corollary hold with \( C^\infty_c(X) \) replaced by the space of compactly supported distributions on \( X \).

REMARK 2. An explicit example of an operator \( D \) as in part (ii) of Theorem 2 and its corollary is given in [5] (see also [20]), where it is shown that the "imaginary part" \( C' \) of the Casimir operator on a complex semisimple Lie group \( G' \) is not solvable. Viewing \( G' \) as a symmetric space for \( G' \times G' \) it is easily seen that \( \eta(C') = 0 \) (see [5, p. X.8]).

The radial part

Let \( D \in \mathfrak{D}(G/H) \). Choose a positive system \( \Sigma^+ \) and let \( A^+ \subset A \) be the corresponding open chamber. Via the canonical map from \( G \) to \( G/H \) we identify \( A^+ \) with a submanifold of \( X \). According to [15, p. 259] there exists a unique differential operator \( \Pi(D) \) on \( A^+ \) such that \( (Df)|_{A^+} = \Pi(D)(f|_{A^+}) \) for all \( K \)-invariant smooth functions \( f \) on \( X \). \( \Pi(D) \) is called the radial part of \( D \). The following result establishes a connection between \( \Pi(D) \) and \( \eta(D) \). It is a generalization of [12, p. 267, Lemma 26] (see also [15, p. 308, Prop. 5.23]).

Let \( \mathfrak{M}^+ \) denote the ring of analytic functions \( \varphi \) on \( A^+ \) which can be expanded in an absolutely convergent series on \( A^+ \) with zero constant term:

\[ \varphi = \sum_{\nu \in \Lambda} c_\nu e^{-\nu}, \quad c_\nu \in \mathbb{C}, \quad c_0 = 0 \]

where the sum is over the set \( \Lambda = \mathbb{N}\Sigma^+ \) and where \( e^{-\nu} \) is defined by \( e^{-\nu}(a) = e^{-\nu(\log a)} \).
PROPOSITION 1. Let $D \in \mathbb{D}(G/H)$. There exist a finite number of elements $v_i \in S(a)$ and functions $g_i \in \mathfrak{R}^+$ such that

$$\Pi(D) = e^{-\rho} R_{\eta(D)} \circ e^\rho + \sum_i g_i R_{v_i}$$

(15)
on $A^+$. Moreover the order $m$ of $\Pi(D)$ equals the degree of $\eta(D)$, and we can select the $v_i$ such that

$$\deg v_i \leq m - 1$$

(16)

for all $i$ (where a negative degree of $v_i$ means that $v_i = 0$). In particular, $\Pi(D) = 0$ if and only if $\eta(D) = 0$.

Proof. The existence of the $v_i$ and $g_i$ such that (15) holds follows from $[2, \text{Lemma 3.9}]$. It remains to prove (16) (from the lemma of loc. cit. we only get that $\deg v_i < \text{order}(D)$, which is not sharp enough to conclude (16), because the order of $\Pi(D)$ in general may be smaller than that of $D$).

Let

$$\Pi(D) = \sum_{v \in \Lambda} e^{-\gamma} R_{v}$$

(17)

be the expansion of $\Pi(D)$ derived from (15), where $v_0 \in S(a)$ and where $v_0$ is given by $v_0(\lambda) = \eta(D)(\lambda + \rho)$. We claim that

$$\deg v_v \leq \deg v_0 - 1$$

(18)

for all $v \neq 0$,

from which both the statement that $\text{order } \Pi(D) = \deg \eta(D)$ and (16) follow. We shall obtain (18) by means of a recursion formula for the $v_v$, derived from the relation $L_X D = D L_X$, where $L_X$ is the Laplace-Beltrami operator on $X$ given in terms of the Casimir operator $\Omega U(g) H$ by $L_X = R_\omega$.

The radial part of $L_X$ is easily computed (see $[10, \text{Eq. (4.12)}]$):

$$\Pi(L_X) = J^{-1/2}(L_A \circ J^{1/2} - L_A(J^{1/2}))$$

(19)

where $L_A$ is the Laplacian on $A$, and $J = \prod_{\gamma \in \Delta_+} (e^{\gamma} - e^{-\gamma})^{p_\gamma}(e^{\gamma} + e^{-\gamma})^{q_\gamma}$. Here $p_\gamma$ and $q_\gamma$ are certain integers given by root space dimensions, see $[21, \text{Thm. 8.1.1}]$.

Put $\tilde{\Pi}(D) = J^{1/2} \Pi(D) \circ J^{-1/2}$, then it follows from the commutation relation $[L_X, D] = 0$ and (19) that $\tilde{\Pi}(D)$ commutes with $L_A - d$, where $d$ is the function $J^{-1/2} L_A(J^{1/2})$. Expanding $d$ in a power series $d(a) = \sum_{\gamma \in \Lambda} d_\gamma a^{-\gamma}$ on $A^+$ and expanding $\tilde{\Pi}(D)$ in analogy with (17) as

$$\tilde{\Pi}(D) = \sum_{v \in \Lambda} e^{-\gamma} R_{\tilde{v}}$$
we obtain the following expression
\[
\sum_{v, \gamma \in \Lambda} ([L_{A^+}, e^{-v}] \mathcal{R}_{\hat{v}_v} - d_v e^{-v}[e^{-\gamma}, \mathcal{R}_{\hat{v}_{\hat{v}_{\gamma}}}] ) = 0.
\]
Comparing coefficients to $e^{-v}$ we get
\[
[L_{A^+}, e^{-v}] \mathcal{R}_{\hat{v}_v} = \sum_{\gamma \in \Lambda, v - \gamma \in \Lambda} d_v e^{-(v-\gamma)}[e^{-\gamma}, \mathcal{R}_{\hat{v}_{\hat{v}_{\gamma}}}],
\]
where the sum is finite. In this equation, if $v \neq 0$ and $\hat{v}_v \neq 0$, the left side is a differential operator on $A^+$ of order $1 + \deg \hat{v}_v$, whereas the order of the operator on the other side is less than the maximum of the degrees of all $\hat{v}_{v-\gamma}$, $\gamma \in \Lambda \setminus \{0\}$. In particular, it follows by an easy induction that $\deg \hat{v}_v \leq \deg \hat{v}_0 - 2$ for $v \neq 0$.

In the series
\[
\Pi(D) = J^{-1/2} \tilde{\Pi}(D) \circ J^{1/2} = J^{-1/2} \sum_{v \in \Lambda} e^{-v} \mathcal{R}_{\hat{v}_v} \circ J^{1/2}
\]
it is seen that the only contribution in degree $\deg \hat{v}_0$ is obtained in the $e^0$ term. Hence $\upsilon_0$ and $\hat{v}_0$ have the same degree (in fact it is easily seen that $\hat{v}_0 = \eta(D)$), and $\upsilon_v$ has a lower degree for all other $v$. From this the claimed property (18) of the $\upsilon_v$ follows.

The final statement of the proposition follows from the previous statements.

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