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The Plancherel decomposition for a reductive symmetric space II. Representation theory

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Abstract. We obtain the Plancherel decomposition for a reductive symmetric space in the sense of representation theory. Our starting point is the Plancherel formula for spherical Schwartz functions, obtained in part I. The formula for Schwartz functions involves Eisenstein integrals obtained by a residual calculus. In the present paper we identify these integrals as matrix coefficients of the generalized principal series.

Contents

1 Introduction

In this paper we establish the Plancherel decomposition for a reductive symmetric space $\mathrm{X} = G/H$, in the sense of representation theory. Here G is a real reductive group of Harish-Chandra's class and H is an open subgroup of the group G^σ of fixed points for an involution σ of G. This paper is a continuation of the paper [12] in the sense that we derive the Plancherel decomposition from its main result [12], Thm. 23.1, the Plancherel formula for the space $\mathcal{C}(\mathrm{X} : \tau)$ of τ-spherical Schwartz functions on X. Here (τ, V_τ) is a finite dimensional unitary representation of K, a σ-stable maximal compact subgroup of G. At the end of the paper, we make a detailed comparison of our results with those of P. Delorme [21].

The results of this paper were found and announced in the fall of 1995, when both authors were visitors of the Mittag–Leffler Institute in Djursholm, Sweden. At the same time Delorme announced a proof of the Plancherel theorem. For more historical comments, we refer the reader to the introduction of [12].

Before giving a detailed outline of the results of this paper, we shall first give some background and describe the main result of [12], which serves as the basis for this paper. The space X carries an invariant measure dx; accordingly, the regular representation L of G in $L^2(\mathrm{X})$ is unitary. The Plancherel decomposition amounts to an explicit decomposition of L as a direct integral of irreducible unitary representations of G. These representations will turn out to be discrete series representations of X and generalized principal series representations of the form

$$\pi_{Q,\xi,\nu} = \mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1), \tag{1.1}$$

with $Q = M_Q A_Q N_Q$ a $\sigma\theta$-stable parabolic subgroup of G with the indicated Langlands decomposition, ξ a discrete series representation of the symmetric space $\mathrm{X}_Q := M_Q / M_Q \cap H$, and ν a unitary character of $A_Q / A_Q \cap H$. To keep the exposition simple, we assume here, and in the rest of the introduction, that the number of open H-orbits on $Q\backslash G$ is one. In general, there are finitely many open orbits, parametrized by a set ${}^Q\mathcal{W}$ of representatives, and then ξ should be taken from the discrete series of the spaces $\mathrm{X}_{Q,v} := M_Q / M_Q \cap vHv^{-1}$, for $v \in {}^Q\mathcal{W}$.

Let θ be the Cartan involution associated with K; it commutes with σ. Let $\mathfrak{a}_\mathrm{q}$ be a maximal abelian subspace of the intersection of the -1 eigenspaces for θ and σ in $\mathfrak{g}$, the Lie algebra of G. We denote by $\mathcal{P}_\sigma$ the collection of $\theta\sigma$-stable parabolic subgroups of G containing $A_\mathrm{q} := \exp \mathfrak{a}_\mathrm{q}$. For $Q \in \mathcal{P}_\sigma$ we put $\mathfrak{a}_{Q\mathrm{q}} := \mathfrak{a}_Q \cap \mathfrak{a}_\mathrm{q}$. In [12] we defined a spherical Fourier transform $\mathcal{F}_Q$ in terms of a so called normalized Eisenstein integral

$$E^\circ(Q : \nu) = E^\circ_\tau(Q : \nu).$$

The Eisenstein integral is a $\mathbb{D}(\mathrm{X})$-finite and $1 \otimes \tau$-spherical function in $C^\infty(\mathrm{X}) \otimes \mathrm{Hom}(\mathcal{A}_{2,Q}, V_\tau)$, depending meromorphically on the parameter $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Here $\mathcal{A}_{2,Q} = \mathcal{A}_{2,Q}(\tau)$ is defined as the space of Schwartz functions $\mathrm{X}_Q \to V_\tau$ that are $\tau_Q := \tau|_{K\cap M_Q}$-spherical and behave finitely under the algebra $\mathbb{D}(\mathrm{X}_Q)$ of invariant differential operators on X_Q. The space $\mathcal{A}_{2,Q}$ is finite dimensional, and inherits the Hilbert structure from the bigger space $L^2(\mathrm{X}_Q : \tau_Q)$. Without the simplifying assumption, $\mathcal{A}_{2,Q}$ is defined as a finite direct sum of similar function spaces for $\mathrm{X}_{Q,v}$, as $v \in {}^Q\mathcal{W}$.

Let P_0 be a fixed minimal element of $\mathcal{P}_\sigma$. Then the Eisenstein integral $E^\circ(P_0 : \lambda)$ is essentially obtained as a (sum of) matrix coefficient(s) of a K-finite vector with an H-fixed distribution vector of a σ-minimal principal series representation of the form (1.1) with $Q = P_0$, see [4] and [7].

In contrast, for non-minimal $Q \in \mathcal{P}_\sigma$ the Eisenstein integral $E^\circ(Q : \nu)$ is obtained from $E^\circ(P_0 : \lambda)$ by means of a residual calculus in the variable $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$, see [12], Eqn. (8.7) and Lemmas 13.18 and 13.12. In particular, for such Q it is a priori not clear that the normalized Eisenstein integral $E^\circ(Q : \nu)$ is a matrix coefficient of the generalized principal series representation (1.1).

In terms of the Eisenstein integral, the spherical Fourier transform is defined by the formula

$$\mathcal{F}_Q f(\nu) = \int_{\mathrm{X}} E^\circ(Q : -\bar{\nu} : x)^* f(x)\, dx \in \mathcal{A}_{2,Q},$$

for $f \in \mathcal{C}(\mathrm{X} : \tau)$ and $\nu \in i\mathfrak{a}^*_{Q\mathrm{q}}$; see [12], § 19. The star indicates that the adjoint of an endomorphism in $\mathrm{Hom}(\mathcal{A}_{2,Q}, V_\tau)$ is taken. The transform $\mathcal{F}_Q$ is a continuous linear map from $\mathcal{C}(\mathrm{X} : \tau)$ into the space $\mathcal{S}(i\mathfrak{a}^*_{Q\mathrm{q}}) \otimes \mathcal{A}_{2,Q}$ of Euclidean Schwartz functions on $i\mathfrak{a}^*_{Q\mathrm{q}}$ with values in the finite dimensional Hilbert space $\mathcal{A}_{2,Q}$. The wave packet transform $\mathcal{J}_Q$ is defined as the adjoint of the Fourier transform with respect to the natural L^2-type inner products on the spaces involved; see [12], § 20. It is a continuous linear map $\mathcal{S}(i\mathfrak{a}^*_{Q\mathrm{q}}) \otimes \mathcal{A}_{2,Q} \to \mathcal{C}(\mathrm{X} : \tau)$, given by the formula

$$\mathcal{J}_Q\varphi(x) = \int_{i\mathfrak{a}^*_{Q\mathrm{q}}} E^\circ(Q : \nu : x)\, \varphi(\nu)\, d\nu,$$

for $\varphi \in \mathcal{S}(i\mathfrak{a}^*_{Q\mathrm{q}}) \otimes \mathcal{A}_{2,Q}$ and $x \in \mathrm{X}$. Here $d\nu$ is Lebesgue measure on $i\mathfrak{a}^*_{Q\mathrm{q}}$, suitably normalized.

Two parabolic subgroups $P, Q \in \mathcal{P}_\sigma$ are called associated if their σ-split components $\mathfrak{a}_{P\mathrm{q}}$ and $\mathfrak{a}_{Q\mathrm{q}}$ are conjugate under the Weyl group W of the root system of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}$. The notion of associatedness defines an equivalence relation $\sim$ on $\mathcal{P}_\sigma$. Let $\mathbf{P}_\sigma$ be a choice of representatives in $\mathcal{P}_\sigma$ for the classes in $\mathcal{P}_\sigma/\sim$. Then the Plancherel formula for functions in $\mathcal{C}(\mathrm{X} : \tau)$ takes the form

$$f = \sum_{Q \in \mathbf{P}_\sigma} [W : W^*_Q]\, \mathcal{J}_Q \mathcal{F}_Q f, \qquad (f \in \mathcal{C}(\mathrm{X} : \tau)),$$

with W^*_Q the normalizer in W of $\mathfrak{a}_{Q\mathrm{q}}$. The operator $[W : W^*_Q]\mathcal{J}_Q\mathcal{F}_Q$ is a continuous projection operator onto a closed subspace $\mathcal{C}_Q(\mathrm{X} : \tau)$ of $\mathcal{C}(\mathrm{X} : \tau)$. Moreover,

$$\mathcal{C}(\mathrm{X} : \tau) = \oplus_{Q \in \mathbf{P}_\sigma} \quad \mathcal{C}_Q(\mathrm{X} : \tau),$$

with orthogonal summands. It follows from the above that $[W : W^*_Q]^{1/2}\mathcal{F}_Q$ extends to a partial isometry from $L^2(\mathrm{X} : \tau)$ to $L^2(i\mathfrak{a}^*_{Q\mathrm{q}}) \otimes \mathcal{A}_{2,Q}$. Its adjoint extends $[W : W^*_Q]^{1/2}\mathcal{J}_Q$ to a partial isometry in the opposite direction.

In the present paper, we build the Plancherel decomposition for $(L, L^2(\mathrm{X}))$ from the above results for all τ. For this it is crucial to relate the Eisenstein integral $E^\circ(Q : \nu)$ to the generalized principal series representations $\pi_{Q,\xi,-\nu}$.

In [19], Delorme has *defined* a normalized Eisenstein integral ${}^\backprime E^\circ(Q : \nu)$ essentially as a matrix coefficient of the generalized principal series. One way to establish the wanted relationship of $E^\circ(Q : \nu)$ with the generalized principal series would thus be to prove the following identity of meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{Q\mathfrak{q}\mathbb{C}}$:

$$E^\circ(Q : \nu) = {}^\backprime E^\circ(Q : -\nu). \tag{1.2}$$

In view of the vanishing theorem of [11], the Eisenstein integral $E^\circ(Q : \nu)$ can be uniquely characterized in terms of its annihilating ideal in $\mathbb{D}(\mathrm{X})$ and its asymptotic behavior towards infinity on X; see [12], Def. 13.7 and Prop. 13.6. The identity (1.2) would follow if not only the Eisenstein integral on the left-hand side but also the Eisenstein integral on the right-hand side satisfied these characterizing conditions. For the latter to be true one needs that, for $\psi \in \mathcal{A}_{2,Q}$, the family $\nu \mapsto {}^\backprime E^\circ(Q : -\nu)\psi$ belongs to the space $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X} : \tau)$ of [12], Prop. 13.6. For this in turn, the full set of exponents of the family ${}^\backprime E^\circ(Q : -\nu)\psi$ in its asymptotic expansion along P_0 must be of a certain form; see [12], Defs. 6.3 and 6.1. We have not been able to deduce this type of information from Delorme's work. Nevertheless, by following a different strategy we have been able to establish (1.2), but only at the end of the paper, in Corollary 11.21, after a relation of our Eisenstein integrals with the principal series has been established.

More precisely, the mentioned characterization of the Eisenstein integral $E^\circ(Q : \nu)$ is used to construct certain embeddings of $(\mathfrak{g}, K)$-modules

$$\pi_{Q,\xi,-\nu} \hookrightarrow (L, C^\infty(\mathrm{X})). \tag{1.3}$$

The existence of these embeddings, on the level of $(\mathfrak{g}, K)$-modules, is sufficient to establish the Plancherel decomposition in the sense of representation theory, Theorem 10.9. Further details will be given at a later stage in this introduction.

At the end of the paper we invoke the automatic continuity theorem, Theorem 11.1, due to W. Casselman and N.R. Wallach, see [18] and [29], to show that the embedding (1.3) extends to a G-homomorphism. This implies that our Eisenstein integrals are essentially generalized matrix coefficients of K-finite and H-fixed distribution vectors of principal series representations. From this information combined with results of [16], the identity (1.2) can then be established.

After this motivation, we shall now give an outline of the paper, in particular describing how the Eisenstein integrals give rise to the embeddings (1.3).

In Sect. 3 we show that the discrete part $L^2_d(\mathrm{X} : \tau)$ of $L^2(\mathrm{X} : \tau)$ is finite dimensional. This fact can be derived from the description of the discrete series by T. Oshima and T. Matsuki in [26]. We show that it can be obtained from [12] and weaker information on the discrete series, also due to [26], namely the rank condition and the fact that the $\mathbb{D}(\mathrm{X})$-characters of $L^2_d(\mathrm{X})$ are real and regular. The mentioned result implies that the parameter space $\mathcal{A}_{2,Q}(\tau)$ of the Eisenstein integral equals $L^2_d(\mathrm{X}_Q : \tau_Q)$. Accordingly, it may be decomposed in an orthogonal finite dimensional sum of isotypical subspaces $\mathcal{A}_{2,Q}(\tau)_\xi$, where $\xi \in \mathrm{X}^\wedge_{Q,ds}$, the collection of discrete series represetations for X_Q.

In Sect. 4 we explain the connection of the Eisenstein integrals with the principal series. Let $\widehat{K}$ be the unitary dual of K, i.e., the collection of equivalence classes of irreducible unitary representations of K. If V is a locally convex space equipped with a continuous representation of K, then by V_K we denote the subspace of K-finite vectors; for $\vartheta \subset \widehat{K}$ a finite subset we denote by V_ϑ the subspace of V_K consisting of vectors whose K-types belong to ϑ. Let $\vartheta \subset \widehat{K}$ be a finite subset. We define $\mathbf{V}_\vartheta$ to be the space of continuous functions $K \to \mathbb{C}$ that are left K-finite with types contained in the set ϑ. Moreover, we define τ_ϑ to be the restriction of the right regular representation of K to $\mathbf{V}_\vartheta$. Let $\delta_e\colon \mathbf{V}_\vartheta \to \mathbb{C}$ be evaluation in e. Then $F \mapsto \delta_e \circ F$ is a natural isomorphism from $L^2(X : \tau_\vartheta)$ onto $L^2(X)_\vartheta$. Its inverse, called sphericalization, is denoted by ς_ϑ.

For $\xi \in \mathrm{X}^\wedge_{Q,ds}$, we denote by $\bar{V}(\xi)$ the space of continuous linear M_Q-equivariant maps $\mathcal{H}_\xi \to L^2(\mathrm{X}_Q)$. This space is a finite dimensional Hilbert space. We denote by $L^2(K : \xi)$ the space of the induced representation $\mathrm{Ind}^K_{K\cap M_Q}(\xi|_{K\cap M_Q})$. It is well known that the induced representation (1.1) may be realized as a ν-dependent representation in $L^2(K : \xi)$, which we shall denote by $\pi_{Q,\xi,\nu}$ as well; this is the so-called compact picture of (1.1).

If $\vartheta \subset \widehat{K}$ is a finite subset, there is a natural isometry from $\bar{V}(\xi) \otimes L^2(K : \xi)_\vartheta$ into $\mathcal{A}_{2,Q}(\tau_\vartheta)$, denoted $T \mapsto \psi_T$. We show in Sect. 4 that we may use the Eisenstein integrals to define a map $J_{Q,\xi,\nu}\colon \bar{V}(\xi) \otimes L^2(K : \xi)_K \to C^\infty(\mathrm{X})_K$ by the formula

$$J_{Q,\xi,\nu}(T)(x) = \delta_e\big[E^\circ_\vartheta(Q : \nu : x)\psi_T\big]. \tag{1.4}$$

Here $\vartheta \subset \widehat{K}$ is any finite subset such that $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta$ and E°_ϑ denotes the Eisenstein integral with $\tau = \tau_\vartheta$. The map $J_{Q,\xi,\nu}$ is a priori well-defined for ν in the complement of the union of a certain set $\mathcal{H}(Q, \xi)$ of hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. This union is disjoint from $i\mathfrak{a}^*_{Q\mathrm{q}}$.

The main result of the section is Theorem 4.6. It asserts that $\mathcal{H}(Q, \xi)$ is locally finite and that, for ν in the complement of $\cup\mathcal{H}(Q, \xi)$, the map $J_{Q,\xi,\nu}$ is $(\mathfrak{g}, K)$-equivariant for the infinitesimal representations associated with $1 \otimes \pi_{Q,\xi,-\nu}$ and L. The proof of this result is given in the next two sections. In the first of these we prepare for the proof by showing that $\pi_{Q,\xi,\nu}$ is finitely generated, with local uniformity in the parameter ν, see Proposition 5.1. This result is needed for the proof of the local finiteness of $\mathcal{H}(Q, \xi)$.

In Sect. 6 the $(\mathfrak{g}, K)$-equivariance of the map $J_{Q,\xi,\nu}$ is established. The K-equivariance readily follows from the definitions. For the $\mathfrak{g}$-equivariance it is necessary to compute derivatives of the Eisenstein integral of the form $L_X E^\circ_\tau(Q : \nu)\psi$, for $\psi \in \mathcal{A}_{2,Q}(\tau)$ and $X \in \mathfrak{g}$. The computation is achieved by introducing a meromorphic family of spherical functions $\widetilde{F}\colon \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \times \mathrm{X} \to \mathfrak{g}^*_{\mathbb{C}} \otimes V_\tau$ by the formula

$$\widetilde{F}_\nu(x)(Z) = L_Z\big[E^\circ_\tau(Q : \nu : \,\cdot\,)\psi\big](x),$$

for $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, $x \in \mathrm{X}$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. The function $\widetilde{F}_\nu$ is $\widetilde{\tau}$-spherical, with $\widetilde{\tau} := \mathrm{Ad}^\vee_K \otimes \tau$ and $\mathrm{Ad}_K := \mathrm{Ad}|_K$. It has the same annihilating ideal in $\mathbb{D}(\mathrm{X})$ as the Eisenstein integral $E^\circ_\tau(Q : \nu)\psi$. Moreover, its asymptotic behavior on X can be expressed in terms of that of $E^\circ_\tau(Q : \nu)$. By the mentioned characterization of Eisenstein integrals this enables us to show that $\widetilde{F}_\nu$ equals an Eisenstein integral of the form $E^\circ_{\widetilde{\tau}}(Q : \nu)\partial_Q(\nu)\psi$, with $\partial_Q(\nu)$ an explicitly given differential operator $\mathcal{A}_{2,Q}(\tau) \to \mathcal{A}_{2,Q}(\widetilde{\tau})$, see Theorem 6.12. The $\mathfrak{g}$-equivariance of $J_{Q,\xi,\nu}$ is then obtained by computing the action of $\partial_Q(\nu)$ on ψ_T, for $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta$; see Lemma 6.13 and Proposition 6.15. At the end of the section we complete the proof of Theorem 4.6 by establishing the local finiteness of $\mathcal{H}(Q, \xi)$, combining the results of Sects. 5 and 6; see Proposition 6.16.

In Sect. 7 we define a Fourier transform $f \mapsto \hat{f}(Q : \xi : \nu)$ from $C^\infty_c(\mathrm{X})_K$ to $\bar{V}(\xi) \otimes L^2(K : \xi)_K$ by transposition of the map $J_{Q,\xi,-\bar{\nu}}$. It is given by the formula

$$\langle \hat{f}(Q : \xi : \nu) \,|\, T\rangle = \int_X f(x)\overline{J_{Q,\xi,-\bar{\nu}}(T)(x)}\, dx$$

and intertwines the $(\mathfrak{g}, K)$-module of L with that of $1 \otimes \pi_{Q,\xi,-\nu}$. In view of (1.4), the transform $f \mapsto \hat{f}$ is related to the spherical Fourier transform by the formula

$$\langle \hat{f}(Q : \xi : \nu) \,|\, T\rangle = \langle \mathcal{F}_Q(\varsigma_\vartheta f)(\nu) \,|\, \psi_T\rangle, \tag{1.5}$$

for $f \in C^\infty_c(X)_\vartheta$.

The established relation (1.5) combined with the spherical Plancherel formula implies that the Fourier transform $f \mapsto \hat{f}(Q : \xi : \nu)$ defines an isometry from $L^2(\mathrm{X})$ into the direct integral

$$\pi = \sum_{Q \in \mathbf{P}_\sigma} \sum_{\xi \in \mathrm{X}^\wedge_{Q,ds}} [W : W^*_Q] \int_{i\mathfrak{a}^*_{Q\mathrm{q}}} 1 \otimes \pi_{Q,\xi,-\nu}\, d\nu, \tag{1.6}$$

realized in a Hilbert space $\mathfrak{L}^2$. The continuous parts of this direct integral are studied in Sect. 8. In Sect. 9 it is first shown, in Theorem 9.5, that the Fourier transform $f \mapsto \hat{f}$ extends to an isometry $\mathfrak{F}$ from $L^2(\mathrm{X})$ into $\mathfrak{L}^2$. Moreover, its restriction to $C^\infty_c(\mathrm{X})_K$ is a $(\mathfrak{g}, K)$-module map into $\mathfrak{L}^{2\infty}$. By an argument involving continuity and density, it is then shown that $\mathfrak{F}$ is

G-equivariant, see Theorem 9.6. At this stage we have established that $\mathfrak{F}$ maps the regular representation L isometrically into a direct integral decomposition. For this to give the Plancherel decomposition, we need to show that the image of $\mathfrak{F}$ is a direct integral with representations that are irreducible and mutually inequivalent outside a set of Plancherel measure zero. This is done in Lemma 10.5 and Proposition 10.8. In the process we use results of F. Bruhat and Harish-Chandra on irreducibility and equivalence of unitarily parabolically induced representations, see Theorem 10.7. The Plancherel theorem is formulated in Theorem 10.9. Finally, in Theorem 10.11 a precise description of the image of $\mathfrak{F}$ is given.

At this point it is still not clear that our description of the Plancherel decomposition uses the same parametrizations as the one in Delorme's paper [21]. It is the object of the last section to show that this is indeed the case. As said, a key idea is to use the automatic continuity theorem, Theorem 11.1, due to Casselman and Wallach, see [18] and [29]. It implies that the map $J_{Q,\xi,\nu}$ has a continuous linear extension, hence can be realized by taking the matrix coefficient with an H-fixed distribution vector of $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$. By means of the description of such vectors in [16], combined with an asymptotic analysis, it is shown that our Eisenstein integral is related to Delorme's by the identity (1.2), see Corollary 11.21.

Finally, the constants $[W\colon W_Q^*]$ occurring in our formula (1.6) differ from those in the similar formula of Delorme. This is due to different choices of normalizations of measures, as is explained in the final part of the paper.

2 Notation and preliminaries

Throughout this paper, we use all notation and preliminaries from [12], Sect. 2. In particular, G is a group of Harish-Chandra's class, σ an involution of G and H an open subgroup of G^σ, the group of fixed points for σ. The associated reductive symmetric space is denoted by

$$\mathrm{X} = G/H.$$

All occurring measures will be normalized according to the conventions described in [12], end of Sect. 5.

Apart from the references just given, we shall give precise references to [12] for additional notation, definitions and results.

3 A property of the discrete series

In this section we discuss an important result on the discrete part of $L^2(\mathrm{X})$, which is a consequence of the classification of the discrete series by T. Oshima and T. Matsuki in [26]. In our approach to the Plancherel formula via the residue calculus, we obtain it as a consequence of the rank condition and the regularity of the infinitesimal character, also due to [26], see [12], Rem. 16.2.

In the rest of this section we assume that (τ, V_τ) is a finite dimensional unitary representation of K. A function $f\colon \mathrm{X} \to V_\tau$ is called τ-spherical if $f(kx) = \tau(k) f(x)$, for all $x \in \mathrm{X}$ and $k \in K$. The Hilbert space of square integrable τ-spherical functions is denoted by $L^2(\mathrm{X}\colon \tau)$. Its discrete part, denoted $L^2_d(\mathrm{X}\colon \tau)$, is defined as in [12], § 12. The Fréchet space of τ-spherical Schwartz functions, denoted $\mathcal{C}(\mathrm{X}\colon \tau)$, is defined as in [12], Eqn. (12.1). The subspace of $\mathbb{D}(\mathrm{X})$-finite functions in $\mathcal{C}(\mathrm{X}\colon \tau)$ is denoted by $\mathcal{A}_2(\mathrm{X}\colon \tau)$.

Proposition 3.1 *Let (τ, V_τ) be a finite dimensional unitary representation of K. Then*

$$L^2_d(\mathrm{X}\colon \tau) = \mathcal{A}_2(\mathrm{X}\colon \tau). \tag{3.1}$$

Moreover, each of the spaces above is finite dimensional.

Proof: By the reasoning at the end of the proof of Lemma 12.6 in [12] it follows that the space on the right-hand side of (3.1) is contained in the space on the left-hand side. If the center of G is not compact modulo H, then it follows from [26], see [12], Thm. 16.1, that X has no discrete series. Hence, $L^2_d(\mathrm{X}) = 0$ and we obtain (3.1).

On the other hand, if G has a compact center modulo H the result is part of [12], Lemma 12.6. □

If $(\xi, \mathcal{H}_\xi)$ is an irreducible unitary representation of G, let $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2(\mathrm{X}))$ denote the space of G-equivariant continuous linear maps from $\mathcal{H}_\xi$ into $L^2(\mathrm{X})$. This space is non-trivial if and only if (the class of) ξ belongs to $\mathrm{X}^\wedge_{ds}$, the collection of equivalence classes of discrete series representations of X. If $\xi \in \mathrm{X}^\wedge_{ds}$, then the mentioned space is finite dimensional, by the finite multiplicity of the discrete series, see [1], Thm. 3.1.

For any irreducible unitary representation ξ, the canonical map from the tensor product $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2(\mathrm{X})) \otimes \mathcal{H}_\xi$ to $L^2(\mathrm{X})$ is an embedding, which is G-equivariant for the representations $1 \otimes \xi$ and L, respectively. We denote its image by $L^2(\mathrm{X})_\xi$ and equip the space $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2(\mathrm{X}))$ with the unique inner product that turns the mentioned embedding into an isometric G-equivariant isomorphism

$$m_\xi\colon\ \mathrm{Hom}_G\big(\mathcal{H}_\xi, L^2(\mathrm{X})\big) \otimes \mathcal{H}_\xi \xrightarrow{\simeq} L^2(\mathrm{X})_\xi. \tag{3.2}$$

Obviously the space on the right-hand side of (3.2) depends on ξ through its class $[\xi]$, and will therefore also be indicated with index $[\xi]$ in place of ξ.

With the notation just introduced, it follows that

$$L^2_d(\mathrm{X}) = \widehat{\bigoplus}_{\omega \in \mathrm{X}^\wedge_{ds}} L^2(\mathrm{X})_\omega, \tag{3.3}$$

with orthogonal summands. Here and elsewhere, the hat over the summation symbol indicates that the closure of the algebraic direct sum is taken.

If ω is an equivalence class of an irreducible unitary representation of G, we write $L^2(\mathrm{X}\colon \tau)_\omega := L^2(\mathrm{X}\colon \tau) \cap [L^2(\mathrm{X})_\omega \otimes V_\tau]$. It is readily seen that this space is non-trivial if and only if ω belongs to $\mathrm{X}^\wedge_{ds}$ and has a K-type in common with the contragredient of τ. The collection of ω with this property is denoted by $\mathrm{X}^\wedge_{ds}(\tau)$.

Lemma 3.2 *The collection* $\mathrm{X}^\wedge_{ds}(\tau)$ *is finite. Moreover,*

$$L^2_d(\mathrm{X}\colon \tau) = \oplus_{\omega \in \mathrm{X}^\wedge_{ds}(\tau)} \; L^2(\mathrm{X}\colon \tau)_\omega, \tag{3.4}$$

where the direct sum is orthogonal and all the summands are finite dimensional.

Proof: That the direct sum decomposition is orthogonal and has closure $L^2_d(\mathrm{X}\colon \tau)$ follows from the similar properties of (3.3). The space on the left-hand side of (3.4) is finite dimensional, by Proposition 3.1. Since all summands on the right-hand side are non-trivial, the collection parametrizing these summands is finite. □

Remark 3.3 It follows from Proposition 3.1 that the spaces $L^2(\mathrm{X}\colon \tau)_\omega$, for $\omega \in \mathrm{X}^\wedge_{ds}$, are contained in $\mathcal{A}_2(\mathrm{X}\colon \tau)$; we therefore also denote them by $\mathcal{A}_2(\mathrm{X}\colon \tau)_\omega$. Note that $L^2(\mathrm{X}\colon \tau)_\omega = 0$ for ω an irreducible unitary representation of G that does not belong to $\mathrm{X}^\wedge_{ds}$. Accordingly, we put $\mathcal{A}_2(\mathrm{X}\colon \tau)_\omega = 0$ for such ω. In view of what has been said, the decomposition (3.4) may be rewritten as

$$\mathcal{A}_2(\mathrm{X}\colon \tau) = \oplus_{\omega \in \mathrm{X}^\wedge_{ds}(\tau)} \; \mathcal{A}_2(\mathrm{X}\colon \tau)_\omega. \tag{3.5}$$

Let $C(K)_K$ denote the space of right K-finite continuous functions on K. If ϑ is a finite subset of $\widehat{K}$, the unitary dual of K, then by $C(K)_\vartheta$ we denote the subspace of $C(K)_K$ consisting of functions with right K-types contained in the set ϑ. If $\delta \in \widehat{K}$, then $\delta^\vee$ denotes the contragredient representation. Accordingly, we put $\vartheta^\vee := \{\delta^\vee \mid \delta \in \vartheta\}$. We define

$$\mathbf{V}_\vartheta := C(K)_{\vartheta^\vee} \tag{3.6}$$

and equip this space with the restriction of the right regular representation of K; this restriction is denoted by τ_ϑ. We endow $\mathbf{V}_\vartheta$ with the $L^2(K)$-inner product defined by means of normalized Haar measure. By δ_e we denote the map $\mathbf{V}_\vartheta \to \mathbb{C}$, $\varphi \mapsto \varphi(e)$.

Lemma 3.4 *Let E be a complete locally convex space equipped with a continuous representation of K. Then the map $I \otimes \delta_e$ restricts to a topological linear isomorphism from $(E \otimes \mathbf{V}_\vartheta)^K$ onto E_ϑ. If E is equipped with a continuous pre-Hilbert structure for which K acts unitarily, then the isomorphism is an isometry. In particular, this yields natural isometries*

$$L^2(\mathrm{X}\colon \tau_\vartheta) \simeq L^2(\mathrm{X})_\vartheta, \quad C_c^\infty(\mathrm{X}\colon \tau_\vartheta) \simeq C_c^\infty(\mathrm{X})_\vartheta,$$

where the last two spaces are equipped with the inner products inherited from the first two spaces.

Proof: This is well known and easy to prove. □

The inverse of the isomorphism $I \otimes \delta_e$ will be denoted by $\varsigma = \varsigma_\vartheta$; see [7], text before Lemma 5, for similar notation. Given a finite subset $\vartheta \subset \widehat{K}$ we shall write $\mathrm{X}^\wedge_{ds}(\vartheta)$ for $\mathrm{X}^\wedge_{ds}(\tau_\vartheta)$, the set of discrete series representations that have a K-type contained in ϑ. The following result is now an immediate consequence of Lemma 3.2.

Corollary 3.5 *Let $\vartheta \subset \widehat{K}$ be a finite set of K-types. Then $\mathrm{X}^\wedge_{ds}(\vartheta)$ is a finite set.*

We end this section with two simple relations between ς_ϑ and $\varsigma_{\vartheta'}$, for finite subsets $\vartheta, \vartheta' \subset \widehat{K}$ with $\vartheta \subset \vartheta'$. Let E be a complete locally convex space equipped with a continuous representation of K. We denote by $\mathrm{i}_{\vartheta',\vartheta}\colon E_\vartheta \to E_{\vartheta'}$ the natural inclusion map and by $P_{\vartheta,\vartheta'}\colon E_{\vartheta'} \to E_\vartheta$ the K-equivariant projection map. Likewise, the inclusion map $\mathbf{V}_\vartheta \to \mathbf{V}_{\vartheta'}$ and the K-equivariant projection $\mathbf{V}_{\vartheta'} \to \mathbf{V}_\vartheta$ (relative to $\tau_{\vartheta'}, \tau_\vartheta$) are denoted by $\mathrm{i}_{\vartheta',\vartheta}$ and $P_{\vartheta,\vartheta'}$, respectively. By K-equivariance, the maps $I \otimes \mathrm{i}_{\vartheta',\vartheta}$ and $I \otimes P_{\vartheta,\vartheta'}$ induce maps

$$I \otimes \mathrm{i}_{\vartheta',\vartheta}\colon\ (E \otimes \mathbf{V}_\vartheta)^K \to (E \otimes \mathbf{V}_{\vartheta'})^K, \quad I \otimes P_{\vartheta,\vartheta'}\colon\ (E \otimes \mathbf{V}_{\vartheta'})^K \to (E \otimes \mathbf{V}_\vartheta)^K.$$

Lemma 3.6 *Let notation be as above. Then*

$$\varsigma_{\vartheta'} \circ (I \otimes \mathrm{i}_{\vartheta',\vartheta}) = \mathrm{i}_{\vartheta',\vartheta} \circ \varsigma_\vartheta, \qquad \varsigma_\vartheta \circ (I \otimes P_{\vartheta,\vartheta'}) = P_{\vartheta,\vartheta'} \circ \varsigma_{\vartheta'}.$$

Proof: The first identity is immediate from the definitions. The second identity follows from the first by using that the maps $P_{\vartheta,\vartheta'}\colon E_{\vartheta'} \to E_\vartheta$ and $P_{\vartheta,\vartheta'}\colon \mathbf{V}_{\vartheta'} \to \mathbf{V}_\vartheta$ may both be characterized by the identities $P_{\vartheta,\vartheta'} \circ \mathrm{i}_{\vartheta',\vartheta} = I$ and $P_{\vartheta,\vartheta'} \circ \mathrm{i}_{\vartheta',\vartheta'\setminus\vartheta} = 0$. □

4 Eisenstein integrals and induced representations

Let $Q \in \mathscr{P}_\sigma$. We denote by $\mathrm{X}^\wedge_{Q,*,ds}$ the collection of equivalence classes of unitary irreducible representations $\xi \in M_Q$ such that ξ is a discrete series representation of $\mathrm{X}_{Q,v}$, for some $v \in N_K(\mathfrak{a}_\mathrm{q})$.

In this section we describe the relation of the normalized Eisenstein integral $E^\circ(Q : \nu)$ with the induced representations $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$, where $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$. In the rest of this section we assume $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$ to be fixed.

Let ${}^Q\mathcal{W} \subset N_K(\mathfrak{a}_\mathrm{q})$ be a choice of representatives for $W_Q\backslash W/W_{K\cap H}$, see [12], text after Eqn. (2.2). For $v \in {}^Q\mathcal{W}$, we equip $\mathrm{X}_{Q,v}$ with the left M_Q-invariant measure $dx_{Q,v}$, specified at the end of [12], Sect. 5. Moreover, we define $\bar{V}(Q, \xi, v) = \bar{V}(\xi, v)$ by

$$\bar{V}(\xi, v) := \mathrm{Hom}_{M_Q}\big(\mathcal{H}_\xi, L^2(\mathrm{X}_{Q,v})\big). \tag{4.1}$$

As mentioned in Sect. 3, this space is finite dimensional. In accordance with the mentioned section, we equip it with the unique inner product that turns the natural map

$$m_{\xi,v}\colon \quad \bar{V}(\xi, v) \otimes \mathcal{H}_\xi \quad \xrightarrow{\simeq} \quad L^2(\mathrm{X}_{Q,v})_\xi, \tag{4.2}$$

into an isometric M_Q-equivariant isomorphism. We define the formal direct sums

$$\bar{V}(\xi) := \oplus_{v \in {}^Q\mathcal{W}} \ \bar{V}(\xi, v), \qquad L^2_{Q,\xi} := \oplus_{v \in {}^Q\mathcal{W}} \ L^2(\mathrm{X}_{Q,v})_\xi \tag{4.3}$$

and equip them with the direct sum inner products. The first of these direct sums will also be denoted by $\bar{V}(Q, \xi)$. The second of these direct sums is a unitary M_Q-module. The direct sum of the maps $m_{\xi,v}$ as v ranges over ${}^Q\mathcal{W}$, is an isometric isomorphism

$$m_\xi\colon \ \bar{V}(\xi) \otimes \mathcal{H}_\xi \ \xrightarrow{\simeq} \ L^2_{Q,\xi} \tag{4.4}$$

that intertwines the natural M_Q-representations.

Remark 4.1 If Q is minimal, then $\mathrm{X}^\wedge_{Q,*,ds}$ coincides with the set $\widehat{M}_{\mathrm{ps}}$, defined in [3], p. 368. Moreover, ${}^Q\mathcal{W} = \mathcal{W}$ is a choice of representatives for $W/W_{K\cap H}$ in $N_K(\mathfrak{a}_\mathrm{q})$. If $v \in \mathcal{W}$, and $\eta \in \mathcal{H}_\xi^{M\cap vHv^{-1}}$, then the map $j_\eta\colon \mathcal{H}_\xi \to L^2(M/M \cap vHv^{-1})$, defined by $j_\eta(v)(m) = \langle v \,|\, \xi(m)\eta)\rangle$, is an M-equivariant map. Moreover, $\eta \mapsto j_\eta$ defines an anti-linear map from $V(\xi, v)$ onto $\mathrm{Hom}_M(\mathcal{H}_\xi, L^2(M/M \cap vHv^{-1}))$. This gives an identification of $\overline{V(\xi, v)}$ with $\bar{V}(\xi, v)$. We recall from [3], p. 378, that we equipped $V(\xi, v) = \mathcal{H}_\xi^{M\cap vHv^{-1}}$ with the restriction of the inner product from $\mathcal{H}_\xi$. By the Schur orthogonality relations this implies that the inner product on $\overline{V(\xi, v)}$ coincides with $\dim(\xi)$ times the inner product on $\bar{V}(\xi, v)$. Let $V(\xi)$ be defined as in [3], Eqn. (5.1). Then $\overline{V(\xi)} \simeq \bar{V}(\xi)$ and the inner product on $\overline{V(\xi)}$ coincides with $\dim(\xi)$ times the inner product on $\bar{V}(\xi)$.

For $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, let $L^2(Q : \xi : \nu)$ denote the space of measurable functions $G \to \mathcal{H}_\xi$, transforming according to the rule

$$\varphi(manx) = a^{\nu+\rho_Q}\, \xi(m)\, \varphi(x), \qquad (x \in G, \ (m, a, n) \in M_Q \times A_Q \times N_Q),$$

and satisfying $\int_K \|\varphi(k)\|^2_\xi \, dk < \infty$. As usual we identify measurable functions that are equal almost everywhere. The space $L^2(Q : \xi : \nu)$ is a Hilbert space for the inner product given by

$$\langle \varphi \,|\, \psi \rangle = \int_K \langle \varphi(k) \,|\, \psi(k) \rangle_\xi \, dk. \tag{4.5}$$

The restriction of the right regular representation of G to this space is denoted by $\mathrm{Ind}_Q^G(\xi \otimes \nu \otimes 1)$, or more briefly by $\pi_{Q,\xi,\nu} = \pi_{\xi,\nu}$.

Let $C^\infty(Q:\xi:\nu)$ denote the subspace of $L^2(Q:\xi:\nu)$ consisting of functions that are smooth $G \to \mathcal{H}_\xi^\infty$. This subspace is G-invariant; the associated G-representation in it is continuous for the usual Fréchet topology.

Remark 4.2 It follows from [13], § III.7, that the Fréchet G-module $C^\infty(Q:\xi:\nu)$ equals the G-module of smooth vectors for the representation $\pi_{Q,\xi,\nu}$, equipped with its natural Fréchet topology.

It will be convenient to work with the compact picture of the induced representation $\pi_{\xi,\nu}$. Let $L^2(K:\xi)$ denote the space of square integrable functions $\varphi: K \to \mathcal{H}_\xi$ that transform according to the rule

$$\varphi(mk) = \xi(m)\varphi(k), \qquad (k \in K, m \in K_Q). \tag{4.6}$$

Multiplication induces a diffeomorphism $Q \times_{K_Q} K \simeq G$. Hence, restriction to K induces an isometry from $L^2(Q:\xi:\nu)$ onto $L^2(K:\xi)$. This isometry restricts to a topological linear isomorphism from $C^\infty(Q:\xi:\nu)$ onto the subspace $C^\infty(K:\xi)$ of functions in $L^2(K:\xi)$ that are smooth $K \to \mathcal{H}_\xi^\infty$, where the latter space is equipped with the usual Fréchet topology. Via the isometric restriction map we transfer $\pi_{\xi,\nu}$ to a G-representation in $L^2(K:\xi)$, also denoted by $\pi_{Q,\xi,\nu} = \pi_{\xi,\nu}$.

Let (τ, V_τ) be a finite dimensional unitary representation of K. We define

$$L^2(K:\xi:\tau) := \left[L^2(K:\xi) \otimes V_\tau\right]^K. \tag{4.7}$$

By finite dimensionality of τ, the space in (4.7) is finite dimensional and contained in $C(K, \mathcal{H}_\xi) \otimes V_\tau$.

Let ev_e denote the evaluation map $C(K, \mathcal{H}_\xi) \to \mathcal{H}_\xi$, $\varphi \mapsto \varphi(e)$, and let $\mathrm{ev}_e \otimes I$ denote the induced map $L^2(K:\xi:\tau) \to \mathcal{H}_\xi \otimes V_\tau$.

Lemma 4.3

(a) *The map* $\mathrm{ev}_e \otimes I$ *defines an isometric isomorphism from* $L^2(K:\xi:\tau)$ *onto the space* $(\mathcal{H}_\xi \otimes V_\tau)^{K_Q}$.
(b) *The space* $L^2(K:\xi:\tau)$ *equals its subspace* $C^\infty(K:\xi:\tau) := [C^\infty(K:\xi) \otimes V_\tau]^K$.

Proof: Observe that $L^2(K:\xi)$ is the representation space for $\mathrm{Ind}_{K_Q}^K(\xi|_{K_Q})$. Hence (a) follows by Frobenius reciprocity. It is readily checked that $\mathrm{ev}_e \otimes I$ maps $C^\infty(K:\xi:\tau)$ onto $(\mathcal{H}_\xi^\infty \otimes V_\tau)^{K_Q}$. The latter space equals $(\mathcal{H}_{\xi K_Q} \otimes V_\tau)^{K_Q} = (\mathcal{H}_\xi \otimes V_\tau)^{K_Q}$; hence (b) follows. □

Given $T \in \bar{V}(\xi) \otimes L^2(K:\xi:\tau)$ we may now define the element $\psi_T \in L^2_{Q,\xi} \otimes V_\tau$ by

$$\psi_T = [m_\xi \otimes I] \circ [I \otimes \mathrm{ev}_e \otimes I](T).$$

We agree to denote the map $\mathrm{ev}_e \otimes I\colon\ L^2(K : \xi : \tau) \to (\mathcal{H}_\xi \otimes V_\tau)^{K_Q}$ also by $\varphi \mapsto \varphi(e)$. With this notation, if $T = \eta \otimes \varphi$, with $\eta \in \bar{V}(\xi)$ and $\varphi \in L^2(K : \xi : \tau)$, then

$$\psi_{T,v} = [\eta_v \otimes I](\varphi(e)), \qquad (v \in {}^Q\mathcal{W}). \tag{4.8}$$

We recall from Remark 3.3, applied to the space $\mathrm{X}_{Q,v}$ in place of X, for $v \in {}^Q\mathcal{W}$, that $[L^2(\mathrm{X}_{Q,v})_\xi \otimes V_\tau]^{K_Q} \simeq \mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q)_\xi$, naturally and isometrically. The space

$$\mathcal{A}_{2,Q}(\tau)_\xi := \oplus_{v \in {}^Q\mathcal{W}} \quad \mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q)_\xi \tag{4.9}$$

is a subspace of the space $\mathcal{A}_{2,Q}(\tau)$, defined in [12], Eqn. (13.1), as the similar direct sum without the indices ξ on the summands. It follows from the above discussion combined with (4.3) that summation over ${}^Q\mathcal{W}$ naturally induces an isometric isomorphism

$$\left(L^2_{Q,\xi} \otimes V_\tau\right)^{K_Q} \simeq \mathcal{A}_{2,Q}(\tau)_\xi, \tag{4.10}$$

via which we shall identify.

Lemma 4.4 *The map $T \mapsto \psi_T$ is an isometry from $\bar{V}(\xi) \otimes L^2(K : \xi : \tau)$ onto $\mathcal{A}_{2,Q}(\tau)_\xi$.*

Proof: It follows from Lemma 4.3 that

$$I \otimes \mathrm{ev}_e \otimes I\colon \quad \bar{V}(\xi) \otimes L^2(K : \xi : \tau) \ \to\ \bar{V}(\xi) \otimes [\mathcal{H}_\xi \otimes V_\tau]^{K_Q} \tag{4.11}$$

is an isometric isomorphism. The map $m_\xi \otimes I$ is an isometry from $\bar{V}(\xi) \otimes \mathcal{H}_\xi \otimes V_\tau$ onto $L^2_{Q,\xi} \otimes V_\tau$, which intertwines the K_Q-actions $1 \otimes \xi|_{K_Q} \otimes \tau_Q$ and $L|_{K_Q} \otimes \tau_Q$. Therefore, it induces an isometry between the subspaces of K_Q-invariants, which by (4.10) is identified with an isometry

$$m_\xi \otimes I\colon \quad \bar{V}(\xi) \otimes [\mathcal{H}_\xi \otimes V_\tau]^{K_Q} \xrightarrow{\simeq} \mathcal{A}_{2,Q}(\tau)_\xi. \tag{4.12}$$

Since $T \mapsto \psi_T$ is the composition of (4.11) with (4.12), the result follows. □

It follows from Lemma 3.4 that

$$L^2(K : \xi : \tau_\vartheta) \simeq L^2(K : \xi)_\vartheta,$$

with an isometric isomorphism. The latter space is equal to $C^\infty(K : \xi)_\vartheta$, in view of Lemmas 4.3 (b) and 3.4. Accordingly, the map $T \mapsto \psi_T$, defined for $\tau = \tau_\vartheta$, may naturally be viewed as an isometric isomorphism

$$T \mapsto \psi_T, \qquad \bar{V}(Q, \xi) \otimes C^\infty(K : \xi)_\vartheta \ \xrightarrow{\simeq}\ \mathcal{A}_{2,Q}(\tau_\vartheta)_\xi. \tag{4.13}$$

Moreover, it is given by the following formula, for $T = \eta \otimes \varphi \in \bar{V}(Q, \xi) \otimes C^\infty(K : \xi)_\vartheta$;

$$\mathrm{pr}_v \psi_T = \eta_v(\varphi(e)), \qquad (v \in {}^Q\mathcal{W}).$$

We now come to the connection with the normalized Eisenstein integral $E^\circ_\tau(Q : \nu) = E^\circ(Q : \nu)$, defined as in [12], Def. 13.7. The Eisenstein integral is meromorphic in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, as a function with values in $C^\infty(\mathrm{X}) \otimes \mathrm{Hom}(\mathcal{A}_{2,Q}, V_\tau)$. If $\psi \in \mathcal{A}_{2,Q}$, we agree to write $E^\circ(Q : \psi : \nu : \cdot) = E^\circ(Q : \nu : \cdot)\psi$. Then $E^\circ(Q : \psi : \nu) \in C^\infty(\mathrm{X} : \tau)$, for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

We need a 'functorial' property of the normalized Eisenstein integral that we shall now describe. Let $(\tau', V_{\tau'})$ be a second finite dimensional unitary representation of K, and let S: $V_\tau \to V_{\tau'}$ be a K-equivariant linear map. Then via action on the last tensor component, S naturally induces linear maps $C^\infty(K : \xi : \tau) \to C^\infty(K : \xi : \tau')$, $\mathcal{A}_{2,Q}(\tau)_\xi \to \mathcal{A}_{2,Q}(\tau')_\xi$ and $C^\infty(\mathrm{X} : \tau) \to C^\infty(\mathrm{X} : \tau')$ that we all denote by $I \otimes S$.

Lemma 4.5 *Let S: $V_\tau \to V_{\tau'}$ be a K-equivariant map as above.*

(a) *Let $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi : \tau)$. Then $\psi_{[I\otimes I\otimes S]T} = [I \otimes S]\psi_T$.*
(b) *Let $\psi \in \mathcal{A}_{2,Q}(\tau)$. Then*

$$[I \otimes S]E^\circ_\tau(Q : \psi : \nu) = E^\circ_{\tau'}(Q : [I \otimes S]\psi : \nu),$$

as a meromorphic $C^\infty(\mathrm{X} : \tau)$-valued identity in the variable $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Proof: (a) is a straightforward consequence of the definitions. Assertion (b) follows from the characterization of the Eisenstein integral in [12], Def. 13.7. More precisely, it follows from the mentioned definition and [12], Prop. 13.6 (a), that the family $f = E^\circ(Q : \psi)$ belongs to $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X} : \tau)$. See [12], Def. 6.6, for the definition of the latter space. Moreover, still by [12], Prop. 13.6, for ν in a non-empty open subset Ω of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, each $v \in {}^Q\mathcal{W}$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\nu-\rho_Q}(Q, v \mid f_\nu, X, m) = \psi_v(m). \tag{4.14}$$

It readily follows from the definitions that g: $(\nu, x) \mapsto S(f(\nu, x))$ belongs to $\mathcal{E}^{\mathrm{hyp}}_Q(\mathrm{X} : \tau')$; moreover, (4.14) implies that

$$q_{\nu-\rho_Q}(Q, v \mid g_\nu, X, m) = S(\psi_v(m)) = [\mathrm{pr}_v[I \otimes S]\psi](m),$$

for all $\nu \in \Omega$, each $v \in {}^Q\mathcal{W}$, and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$. In view of [12], Def. 13.7 and Prop. 13.6 (a), this implies that $g = E^\circ(Q : [I \otimes S]\psi)$. □

If $\vartheta \subset \widehat{K}$ is a finite subset and $\psi \in \mathcal{A}_{2,Q}(\tau_\vartheta)$, we denote the associated normalized Eisenstein integral $E^\circ_{\tau_\vartheta}(Q : \psi : \nu)$ also by $E^\circ_\vartheta(Q : \psi : \nu)$. This Eisenstein integral is a smooth τ_ϑ-spherical function, depending meromorphically on the parameter $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Lemma 4.5 implies an obvious relation between the Eisenstein integrals $E^\circ_\vartheta(Q : \psi : \nu)$ for different subsets ϑ. If $\vartheta \subset \vartheta'$ are finite subsets of $\widehat{K}$, then $\mathbf{V}_\vartheta \subset \mathbf{V}_{\vartheta'}$. The associated inclusion map is denoted by $\mathrm{i}_{\vartheta',\vartheta}$; it intertwines τ_ϑ with $\tau_{\vartheta'}$. From Lemmas 3.6 and 4.5 (a) it follows that

$$\begin{aligned}\psi_{[I\otimes \mathrm{i}_{\vartheta',\vartheta}]T} &= \psi_{[I\otimes I\otimes \mathrm{i}_{\vartheta',\vartheta}][I\otimes\varsigma_\vartheta]T}\\ &= [I\otimes \mathrm{i}_{\vartheta',\vartheta}]\psi_T, \qquad \big(T\in \bar{V}(\xi)\otimes C^\infty(K:\xi)_\vartheta\big). \quad (4.15)\end{aligned}$$

Moreover, from Lemma 4.5 (b) it follows that

$$E^\circ_{\vartheta'}(Q : [I\otimes \mathrm{i}_{\vartheta',\vartheta}]\psi : \nu) = [I\otimes \mathrm{i}_{\vartheta',\vartheta}]E^\circ_\vartheta(Q : \psi : \nu), \quad (\psi\in \mathcal{A}_{2,Q}(\tau_\vartheta)). \quad (4.16)$$

We have similar formulas for the K-equivariant projection operator $P_{\vartheta,\vartheta'}$: $\mathbf{V}_{\vartheta'} \to \mathbf{V}_\vartheta$. From Lemmas 3.6 and 4.5 it follows that

$$\psi_{[I\otimes I\otimes P_{\vartheta,\vartheta'}]T} = [I\otimes P_{\vartheta,\vartheta'}]\psi_T, \qquad \big(T\in \bar{V}(\xi)\otimes C^\infty(K:\xi)_{\vartheta'}\big), \quad (4.17)$$

and

$$E^\circ_\vartheta(Q : [I\otimes P_{\vartheta,\vartheta'}]\psi : \nu) = [I\otimes P_{\vartheta,\vartheta'}]E^\circ_{\vartheta'}(Q : \psi : \nu), \quad (\psi\in \mathcal{A}_{2,Q}(\tau_{\vartheta'})). \quad (4.18)$$

We recall from [12], § 4, that a $\Sigma_r(Q)$-hyperplane in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a hyperplane of the form $(\alpha^\perp)_\mathbb{C} + \xi$, with $\alpha\in\Sigma_r(Q)$ and $\xi\in\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. The hyperplane is said to be real if ξ may be chosen from $\mathfrak{a}^*_{Q\mathrm{q}}$. If $\vartheta\subset\widehat{K}$ is a finite subset, then by [12], Prop. 13.14, there exists a locally finite collection $\mathcal{H}$ of real $\Sigma_r(Q)$-hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that for each $T\in\bar{V}(\xi)\otimes C^\infty(K:\xi)_\vartheta$ the function $\nu\mapsto E^\circ_\vartheta(Q : \psi_T : \nu)$ has a singular locus contained in $\cup\mathcal{H}$. We denote by $\mathcal{H}(Q,\xi,\vartheta)$ the minimal collection with this property. It follows from the definition just given that $\vartheta\subset\vartheta' \Rightarrow \mathcal{H}(Q,\xi,\vartheta)\subset\mathcal{H}(Q,\xi,\vartheta')$. Let $\mathcal{H}(Q,\xi)$ denote the union of the collections $\mathcal{H}(Q,\xi,\vartheta)$, as ϑ ranges over the collection of finite subsets of $\widehat{K}$. Then

$$i\mathfrak{a}^*_{Q\mathrm{q}} \cap \cup\mathcal{H}(Q,\xi) = \emptyset, \quad (4.19)$$

by the regularity theorem for the normalized Eisenstein integral, see [12], Thm. 18.8.

For $\nu\in\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}\setminus\cup\mathcal{H}(Q,\xi)$, we define the linear map

$$J_{Q,\xi,\nu} = J_{\xi,\nu} \colon\ \bar{V}(\xi)\otimes C^\infty(K:\xi)_K \to C^\infty(\mathrm{X})_K$$

by

$$J_{\xi,\nu}(T)(x) = E^\circ_\vartheta(Q : \psi_T : \nu : x)(e), \qquad (x\in\mathrm{X}), \quad (4.20)$$

for $\vartheta\subset\widehat{K}$ a finite subset and $T\in\bar{V}(\xi)\otimes C^\infty(K:\xi)_\vartheta$. This definition is unambiguous in view of (4.15) and (4.16).

Theorem 4.6 *Let $Q \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$. The collection $\mathcal{H}(Q,\xi)$ consists of real $\Sigma_r(Q)$-hyperplanes and is locally finite. Its union is disjoint from $i\mathfrak{a}^*_{Q\mathrm{q}}$. Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ be in the complement of this union. Then $J_{Q,\xi,\nu}$ is a $(\mathfrak{g}, K)$-intertwining map from $\bar{V}(\xi) \otimes C^\infty(K:\xi)_K$, equipped with the induced representation $1 \otimes \pi_{Q,\xi,-\nu}$, to $C^\infty(\mathrm{X})_K$, equipped with the $(\mathfrak{g}, K)$-module structure induced by the left regular representation of G in $C^\infty(\mathrm{X})$.*

The proof of this theorem will be given in the next two sections. In Sect. 5 we investigate uniformity of generators for $\pi_{Q,\xi,\nu}$ relative to the parameter ν. In Sect. 6 we shall investigate the effect of left differentiations on left spherical functions.

5 Generators of induced representations

In this section we show that the parabolically induced representations, introduced in Sect. 4, are generated by finitely many K-finite vectors, with local uniformity in the continuous induction parameter.

Proposition 5.1 *Let $Q \in \mathcal{P}_\sigma$ and let ξ be a unitary representation of M_Q of finite length. Assume that $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a bounded subset. Then there exists a finite subset $\vartheta \subset \widehat{K}$ such that, for all $\nu \in \Omega$,*

$$\pi_{Q,\xi,\nu}(U(\mathfrak{g}))C^\infty(K:\xi)_\vartheta = C^\infty(K:\xi)_K. \tag{5.1}$$

Remark 5.2 In particular, the result holds for $\sigma = \theta$; then Q is an arbitrary parabolic subgroup of G and $\mathfrak{a}_{Q\mathrm{q}}$ equals its usual Langlands split component $\mathfrak{a}_Q$.

Proof: It suffices to prove the result for ξ irreducible. We shall do this by a method given for ξ tempered in [28], § 5.5.5. Let

$$\omega := \{\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \mid \quad \langle \mathrm{Re}\,\nu - \rho_Q\,,\,\alpha\rangle > 0, \quad \forall\, \alpha \in \Delta_r(Q)\}.$$

Then for $\nu \in \omega$ we may define the standard intertwining operator $A(\nu) = A(\bar{Q} : Q : \xi : \nu)$ from $C^\infty(Q:\xi:\nu)$ to $C^\infty(\bar{Q}:\xi:\nu)$, by

$$A(\nu)f(x) = \int_{\bar{N}_Q} f(\bar{n}\,x)\,d\bar{n}, \qquad (x \in G),$$

where $d\bar{n}$ denotes a choice of Haar measure on $\bar{N}_Q$. The integral is absolutely convergent; this follows by an argument that involves estimates completely analogous to the ones given for Q minimal in [4], proof of Lemma 15.6. It also follows from these estimates that, for $f \in C^\infty(K:\xi)$, the function $A(\nu)f \in C^\infty(K:\xi)$ depends holomorphically on $\nu \in \omega$.

Lemma 5.3 *Let $f, g \in C^\infty(K:\xi)$, $\nu \in \omega$ and $X \in \mathfrak{a}^+_{Q\mathrm{q}}$. Then*

$$\lim_{t\to\infty} e^{t(-\nu+\rho_Q)(X)}\langle \pi_{Q,\xi,\nu}(m\exp tX)f \mid g\rangle = \langle \xi(m)[A(\nu)f](e) \mid g(e)\rangle_\xi. \tag{5.2}$$

Proof: See [29], Lemma 10.5.1. □

Completion of the proof of Prop. 5.1: From (5.2) it can be deduced, by an argument due to Langlands [24], Lemma 3.13, see also Milicic [25], Proof of Thm. 1, that if $f \in C^\infty(Q : \xi : \nu)_K$ and $A(\nu)f \neq 0$, then f is a cyclic vector for $\pi_{\xi,\nu}$ in the sense that the $(\mathfrak{g}, K)$-module generated by f equals $C^\infty(Q : \xi : \nu)_K$. See also [29], Cor. 10.5.2. We can now prove the result in the case that the closure of Ω is contained in ω. Indeed, assume this to be the case and let $\nu_0 \in \overline{\Omega}$. Since $f \mapsto A(\nu_0)f(e)$ can be expressed as a convolution operator with non-trivial kernel, there exists a finite set $\vartheta \subset \widehat{K}$ and a function $f \in C^\infty(K : \xi)_\vartheta$ such that $A(\nu_0)f(e) \neq 0$; by continuity in the parameter ν there exists an open neighborhood ω_0 of ν_0 in ω such that $A(\nu)f(e) \neq 0$ for all $\nu \in \omega_0$. From what we said above, it follows that (5.1) holds for all $\nu \in \omega_0$. By compactness of the set $\overline{\Omega}$, the result now readily follows in case $\overline{\Omega}$ is contained in ω.

We shall now use tensoring with a finite dimensional representation to extend the result to an arbitrary bounded subset $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Let $P \in \mathscr{P}^{\min}_\sigma$ be such that $P \subset Q$. Let $\Delta_Q(P) := \{\alpha \in \Delta(P) \mid \alpha|_{\mathfrak{a}_{Q\mathrm{q}}} = 0\}$ and put $\Delta(Q) = \Delta(P) \setminus \Delta_Q(P)$. We fix $n \in \mathbb{N}$ such that $\langle \mathrm{Re}\,\nu - \rho_Q\,,\, \alpha\rangle / \langle \alpha\,,\, \alpha\rangle > -8n$ for all $\nu \in \bar{\Omega}$ and $\alpha \in \Delta(Q)$. We fix $\mu \in \mathfrak{a}^*_\mathrm{q}$ with the property that $\langle \mu\,,\, \alpha\rangle / \langle \alpha\,,\, \alpha\rangle$ equals $8n$ for all $\alpha \in \Delta(Q)$ and zero for all $\alpha \in \Delta_Q(P)$. Then $\mu + \overline{\Omega} \subset \omega$. Hence there exists a finite subset $\vartheta' \subset \widehat{K}$ such that $\pi_{\xi,\nu+\mu}(U(\mathfrak{g}))C^\infty(K : \xi)_{\vartheta'} = C^\infty(K : \xi)_K$, for all $\nu \in \overline{\Omega}$.

It follows from the condition on μ that $\langle \mu\,,\, \alpha\rangle / 2\langle \alpha\,,\, \alpha\rangle \in 4\mathbb{Z}$ for all $\alpha \in \Delta(P)$. Since Σ is a possibly non-reduced root system, this implies that $\langle \mu\,,\, \alpha\rangle / 2\langle \alpha\,,\, \alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in \Sigma$. According to [4], Cor. 5.7 and Prop. 5.5, there exists a class one finite dimensional irreducible G-module (F, π) of $\Delta(P)$-highest $\mathfrak{a}_\mathrm{q}$-weight μ; the highest weight space F_μ is one dimensional, and $M_\sigma = M_{P\sigma}$ acts trivially on it. Since $M_{Q\sigma}$ centralizes $\mathfrak{a}_{Q\mathrm{q}}$, it normalizes F_μ. By compactness it follows that $(K_Q)_e$ acts trivially on F_μ. Since μ vanishes on ${}^*\mathfrak{a}_{Q\mathrm{q}} = \mathfrak{a}_\mathrm{q} \cap \mathfrak{m}_Q$, it follows that ${}^*A_{Q\mathrm{q}}$ also acts trivially on F_μ. Finally, since $M_{Q\sigma}$ is generated by M_σ, $(K_Q)_e$ and ${}^*A_{Q\mathrm{q}}$, it follows that $M_{Q\sigma}$ acts by the identity on F_μ.

Let $e_\mu \in F_\mu$ be a non-trivial highest weight vector. Then the map $m: F^* \to C^\infty(G)$ defined by $m(v)(x) = v(\pi(x^{-1})e_\mu)$ is readily seen to be an equivariant map from F^* into $C^\infty(Q : 1 : -\mu)$. The map $M_\nu: C^\infty(Q : \xi : \nu + \mu) \otimes F^* \to C^\infty(Q : \xi : \nu)$ given by $M_\nu(\varphi \otimes v) = m(v)\varphi$ is G-equivariant, for every $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Let $v_K \in F^*$ be a non-trivial K-fixed vector. Then, since $G = QK$, the function $m(v_K)$ is nowhere vanishing. From this we see that M_ν is surjective, for every $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. It follows that the $U(\mathfrak{g})$-module generated by $V_\nu := M_\nu(C^\infty(Q : \xi : \nu + \mu)_{\vartheta'} \otimes F^*)$ equals $C^\infty(Q : \xi : \nu)_K$, for all $\nu \in \overline{\Omega}$. Let $\vartheta \subset \widehat{K}$ be the collection of all K-types occurring in $\delta \otimes F^*$ for some $\delta \in \vartheta'$. Then ϑ is a finite set and $V_\nu \subset C^\infty(Q : \xi : \nu)_\vartheta$, for all $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Hence, (5.1) follows for all $\nu \in \Omega$. □

6 Differentiation of spherical functions

In this section we assume (τ, V_τ) to be a finite dimensional unitary representation of K. We shall investigate the action of L_Z, for $Z \in \mathfrak{g}$, on the Eisenstein integral $E^\circ(Q : \nu)$. Here L denotes the infinitesimal left regular representation. As a preparation, we shall first investigate the action of L_Z on functions from the space $C^\infty(\mathrm{X}_+ : \tau)$, defined in [12], § 6. Secondly, we shall investigate the action of L_Z on families from $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \tau)$, defined in [12], Def. 6.6.

Given a function $F \in C^\infty(\mathrm{X}_+ : \tau)$, we define the function $\widetilde{F}: \mathrm{X}_+ \to \mathfrak{g}_\mathbb{C}^* \otimes V_\tau \simeq \mathrm{Hom}(\mathfrak{g}_\mathbb{C}, V_\tau)$ by

$$\widetilde{F}(x)(Z) = L_Z F(x), \qquad (x \in \mathrm{X}_+,\ Z \in \mathfrak{g}_\mathbb{C}).$$

One readily checks that

$$\widetilde{F}(kx)(Z) = \tau(k)\widetilde{F}(x)(\mathrm{Ad}(k^{-1})Z), \qquad (x \in \mathrm{X}_+,\ k \in K,\ Z \in \mathfrak{g}_\mathbb{C}).$$

Hence, $\widetilde{F}$ is a spherical function of its own right. In fact, let $\mathrm{Ad}_K^\vee$ denote the restriction to K of the coadjoint representation of G in $\mathfrak{g}_\mathbb{C}^*$ and put $\widetilde{\tau} := \mathrm{Ad}_K^\vee \otimes \tau$. Then

$$\widetilde{F} \in C^\infty(\mathrm{X}_+ : \widetilde{\tau}).$$

Our first objective is to show that if F has a certain converging expansion towards infinity along (Q, v), for $Q \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$, then $\widetilde{F}$ has a similar expansion, which can be computed in terms of that of F. As a preparation, we study sets consisting of points of the form mav, where $v \in N_K(\mathfrak{a}_\mathrm{q})$, $m \in M_{Q\sigma}$ and $a \to \infty$ in $A_{Q\mathrm{q}}^+$. They describe regions of convergence for the expansions involved, in the spirit of [11], § 3. We will also describe decompositions of elements of $\mathfrak{g}$ along such sets, in a fashion similar to [11], § 4. These will be needed to compute the expansion of $\widetilde{F}$.

Let $Q \in \mathcal{P}_\sigma$. We define the function $R_{Q,v}: M_{1Q} \to]0, \infty[$ as in [11], Sect. 3. Recall that $R_{Q,v}$ is left K_Q- and right $M_{1Q} \cap vHv^{-1}$-invariant; thus, it may be viewed as a function on $\mathrm{X}_{1Q,v}$. If $Q = G$, then $R_{Q,v}$ equals the constant function 1 and if $Q \neq G$, then according to [11], Lemma 3.2, it is given by

$$R_{Q,v}(au) = \max_{\alpha \in \Sigma(Q)} a^{-\alpha},$$

for $a \in A_\mathrm{q}$ and $u \in N_{K_Q}(\mathfrak{a}_\mathrm{q})$. The inclusion map $M_Q \to M_{1Q}$ induces an embedding via which we may identify $\mathrm{X}_{Q,v}$ with a sub M_Q-manifold of $\mathrm{X}_{1Q,v}$. From [11], Lemma 3.2, we recall that $R_{Q,v} \geq 1$ on $\mathrm{X}_{Q,v}$.

Lemma 6.1 *Let $v \in N_K(\mathfrak{a}_\mathrm{q})$ and put $Q' = v^{-1}Qv$. Then*

$$R_{Q,v}(m) = R_{Q',1}(v^{-1}mv), \qquad (m \in M_{1Q}).$$

Proof: This follows immediately from the characterization of $R_{Q,v}$ given above. □

In accordance with [11], Eqn. (3.7), we define, for $v \in N_K(\mathfrak{a}_q)$ and $R > 0$,

$$M_{1Q,v}[R] := \{m \in M_{1Q} \mid R_{Q,v}(m) < R\},$$

and $M_{Q\sigma,v}[R] := M_{Q\sigma} \cap M_{1Q,v}[R]$. Note that $M_{1Q,1}[R]$ and $M_{Q\sigma,1}[R]$ equal the sets $M_{1Q}[R]$ and $M_{Q\sigma}[R]$, defined in [11], text preceding Lemma 4.7, respectively. Finally, for $R > 0$ we define

$$A^+_{Q\mathrm{q}}(R) := \left\{a \in A_{Q\mathrm{q}} \mid a^{-\alpha} < R \quad \text{for all} \quad \alpha \in \Delta_r(Q)\right\}. \tag{6.1}$$

Lemma 6.2 *Let $v \in N_K(\mathfrak{a}_q)$ and put $Q' = v^{-1}Qv$. Let $R > 0$.*

(a) $M_{1Q,v}[R] = vM_{1Q'}[R]v^{-1}, \quad M_{Q\sigma,v}[R] = vM_{Q'\sigma}[R]v^{-1}$.
(b) $A^+_{Q\mathrm{q}}(R) = vA^+_{Q'\mathrm{q}}(R)v^{-1}$.

Proof: Assertion (a) follows readily from combining Lemma 6.1 with the definitions of the sets involved. Assertion (b) is clear from (6.1). □

We define the open dense subset M'_{1Q} of M_{1Q} as in [11], Eqn. (4.3). Write $\mathfrak{g}^{\pm} := \ker(-I \pm \theta\sigma)$ and put $\mathfrak{g}^{\pm}_{\alpha} := \mathfrak{g}_{\alpha} \cap \mathfrak{g}^{\pm}$, for $\alpha \in \Sigma$. Write $H_{1Q} := M_{1Q} \cap H$. Then by [11], Cor. 4.2,

$$M'_{1Q} = K_Q\, [M'_{1Q} \cap A_\mathrm{q}]\, H_{1Q},$$
$$M'_{1Q} \cap A_\mathrm{q} = \left\{a \in A_\mathrm{q} \mid a^{\alpha} \neq 1 \text{ for all}\, \alpha \in \Sigma(Q) \text{ with } \mathfrak{g}^+_{\alpha} \neq 0\right\}. \tag{6.2}$$

In particular, M'_{1Q} is a left K_Q- and right H_{1Q}-invariant open dense subset of M_{1Q}. If $v \in N_K(\mathfrak{a}_q)$, then by $M'_{1Q,v}$ we denote the analogue of the set M'_{1Q} for the pair (G, vHv^{-1}).

Lemma 6.3 *Let $v \in N_K(\mathfrak{a}_q)$ and put $Q' = v^{-1}Qv$. Then $M'_{1Q,v} := vM'_{1Q'}v^{-1}$.*

Proof: This readily follows from the definition. □

Lemma 6.4 *Let $v \in N_K(\mathfrak{a}_q)$.*

(a) $M_{1Q,v}[1] \subset M'_{1Q,v}$.
(b) *Let $R_1, R_2 > 0$. Then $M_{Q\sigma,v}[R_1]A^+_{Q\mathrm{q}}(R_2) \subset M_{1Q,v}[R_1R_2]$.*

Proof: For $v = 1$, the results are given in [11], Lemma 4.7. Let now v be arbitrary and put $Q' = v^{-1}Qv$. Using Lemma 6.2 (a) with $R = 1$ and Lemma 6.3 we obtain (a) from the similar statement with Q', 1 in place of Q, v. Likewise, assertion (b) follows by application of Lemma 6.2. □

We now come to the investigation of decompositions in $\mathfrak{g}$, needed for the study of the asymptotic behavior of $\widetilde{F}$. Write $\mathfrak{k}_{(Q)} := \mathfrak{k} \cap (\mathfrak{n}_Q + \bar{\mathfrak{n}}_Q)$. Then $I + \theta: X \mapsto X + \theta X$ is a linear isomorphism from $\bar{\mathfrak{n}}_Q$ onto $\mathfrak{k}_{(Q)}$. For $\alpha \in \Sigma$ we put $\mathfrak{k}^{\pm}_{\alpha} := (I + \theta)(\mathfrak{g}^{\pm}_{-\alpha})$. Then $\mathfrak{k}_{(Q)}$ is the direct sum of the spaces $\mathfrak{k}^{\pm}_{\alpha}$, for $\alpha \in \Sigma(Q)$.

Lemma 6.5 *Let* $v \in N_K(\mathfrak{a}_q)$. *If* $m \in M'_{1Q,v}$, *then* $\mathfrak{n}_Q \subset \mathfrak{k}_{(Q)} \oplus \mathrm{Ad}(mv)\mathfrak{h}$.

Proof: For $v = 1$ this follows from [11], Lemma 4.3 (b), with $\bar{Q}$ in place of Q. If v is arbitrary, put $Q' = v^{-1}Qv$. Then for $m \in M'_{1Q,v}$ we have $v^{-1}mv \in M'_{1Q'}$, hence $\mathrm{Ad}(v^{-1})\mathfrak{n}_Q = \mathfrak{n}_{Q'} \subset \mathfrak{k}_{(Q')} \oplus \mathrm{Ad}(v^{-1}mv)\mathfrak{h}$, and the result follows by application of $\mathrm{Ad}(v)$. □

By the above lemma, for $m \in M'_{1Q,v}$ we may define a linear map $\Phi(m) = \Phi_{Q,v}(m) \in \mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$ by

$$X \in \Phi(m)X + \mathrm{Ad}(mv)\mathfrak{h}, \qquad (X \in \mathfrak{n}_Q). \tag{6.3}$$

It is readily seen that $\Phi_{Q,v}$ is an analytic $\mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$-valued function on $M'_{1Q,v}$.

Lemma 6.6 *If* $m \in M'_{1Q,v}$, $k \in K_Q$ *and* $h \in M_{1Q} \cap vHv^{-1}$, *then*

$$\Phi(kmh) = \mathrm{Ad}(k) \circ \Phi(m) \circ \mathrm{Ad}(k)^{-1}.$$

Proof: Since M_{1Q} normalizes $\mathfrak{n}_Q$ and K_Q normalizes $\mathfrak{k}_{(Q)}$ the result is an immediate consequence of the definition in equation (6.3). □

Lemma 6.7 *Let* $v \in N_K(\mathfrak{a}_q)$ *and put* $Q' = v^{-1}Qv$. *Then, for all* $m \in M'_{1Q,v}$,

$$\Phi_{Q,v}(m) = \mathrm{Ad}(v) \circ \Phi_{Q',1}(v^{-1}mv) \circ \mathrm{Ad}(v)^{-1}.$$

Proof: This follows from (6.3), by the same reasoning as in the proof of Lemma 6.5. □

Let $\Psi = \Psi_Q : M'_{1Q} \to \mathrm{Hom}(\bar{\mathfrak{n}}_Q, \mathfrak{k}_{(Q)})$ be defined as in [11], Eqn. (4.4). Then, for $X \in \bar{\mathfrak{n}}_Q$ and $m \in M'_{1Q}$,

$$X \in \mathrm{Ad}(m)^{-1}\Psi(m)X + \mathfrak{h}. \tag{6.4}$$

Lemma 6.8 *Let* $m \in M'_{1Q}$. *Then*

$$\Phi_{Q,1}(m) = -\Psi(m) \circ \sigma \circ \mathrm{Ad}(m^{-1}). \tag{6.5}$$

Proof: If $X \in \mathfrak{n}_Q$ and $m \in M'_{1Q}$, then $\sigma\mathrm{Ad}(m^{-1})X \in \bar{\mathfrak{n}}_Q$, so that $\sigma\mathrm{Ad}(m^{-1})X$ belongs to $\mathrm{Ad}(m^{-1})\Psi(m)\sigma\mathrm{Ad}(m^{-1})X + \mathfrak{h}$. Since $\mathrm{Ad}(m^{-1})X \in -\sigma\mathrm{Ad}(m^{-1})X + \mathfrak{h}$, this implies that

$$\mathrm{Ad}(m^{-1})X \in -\mathrm{Ad}(m^{-1})\Psi(m)\sigma\mathrm{Ad}(m^{-1})X + \mathfrak{h}.$$

Comparing with the definition of $\Phi_{Q,1}(m)$ given in (6.3) with $v = 1$, we obtain the desired identity. □

In the formulation of the next result we use the terminology of neat convergence of exponential polynomial series, introduced in [11], § 1.

Proposition 6.9 *Let $v \in N_K(\mathfrak{a}_{\mathrm{q}})$. There exist unique real analytic $\mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$-valued functions $\Phi_\mu = \Phi_{Q,v,\mu}$ on $M_{Q\sigma}$, for $\mu \in \mathbb{N}\Delta_r(Q)$, such that, for every $m \in M_{Q\sigma}$ and all $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$,*

$$\Phi_{Q,v}(ma) = \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu}\Phi_\mu(m), \tag{6.6}$$

with absolutely convergent series. Moreover, $\Phi_0 = 0$. Finally, for every $R > 1$ the series in (6.6) converges neatly on $A^+_{Q\mathrm{q}}(R^{-1})$ as a $\Delta_r(Q)$-power series with coefficients in $C^\infty(M_{Q\sigma,v}[R]) \otimes \mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$.

Proof: We first assume that $v = 1$. Let $\Psi_\mu : M_{Q\sigma} \to \mathrm{End}(\bar{\mathfrak{n}}_Q)$ be as in [11], Prop. 4.8. Then it follows from combining the mentioned proposition with (6.5) that, for $m \in M_{Q\sigma}$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,1}(m)^{-1})$,

$$\Phi(ma) = -(I+\theta) \circ \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu}\Psi_\mu(m) \circ \sigma \circ \mathrm{Ad}(ma)^{-1},$$

with absolutely convergent series. We now see that the restriction of $\Phi(ma)$ to $\mathfrak{g}_\alpha$, for $\alpha \in \Sigma(Q)$, equals

$$-(I+\theta) \circ \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu-\alpha}\Psi_\mu(m) \circ \sigma \circ \mathrm{Ad}(m)|_{\mathfrak{g}_\alpha}.$$

Put $\Phi_0 = 0$ and, for $\nu \in \mathbb{N}\Delta_r(Q) \setminus \{0\}$, define $\Phi_\nu(m) \in \mathrm{Hom}(\mathfrak{n}_Q, \mathfrak{k}_{(Q)})$ by

$$\Phi_\nu(m)|_{\mathfrak{g}_\alpha} := -(I+\theta) \circ \Psi_{\nu-\alpha}(m) \circ \sigma \circ \mathrm{Ad}(m)|_{\mathfrak{g}_\alpha}$$

if $\nu - \alpha \in \mathbb{N}\Delta_r(Q)$, and by $\Phi_\nu(m)|_{\mathfrak{g}_\alpha} = 0$ otherwise. Then (6.6) follows with absolute convergence. All remaining assertions about convergence follow from the analogous assertions in [11], Prop. 4.8.

We now turn to the case that v is general. Let $Q' = v^{-1}Qv$, and define

$$\Phi_{Q,v,\mu}(m) = \mathrm{Ad}(v) \circ \Phi_{Q',1,\mathrm{Ad}(v)^{-1}\mu}(v^{-1}mv) \circ \mathrm{Ad}(v)^{-1},$$

for $\mu \in \mathbb{N}\Delta_r(Q)$ and $m \in M_{Q\sigma}$. Then all assertions follow from the similar assertions with Q', 1 in place of Q, v, by application of Lemmas 6.7 and 6.2. □

We now come to the behavior of L_Z, for $Z \in \mathfrak{g}_\mathbb{C}$, at points of the form mav, with $v \in N_K(\mathfrak{a}_{\mathrm{q}})$, $m \in M_{Q\sigma}$ and $a \to \infty$ in $A^+_{Q\mathrm{q}}$. We start by observing that

$$\mathfrak{g} = \mathfrak{n}_Q \oplus \mathfrak{a}_{Q\mathrm{q}} \oplus (\mathfrak{m}_{Q\sigma} \cap \mathfrak{p}) \oplus \mathfrak{k}, \tag{6.7}$$

as a direct sum of linear spaces. Accordingly, we write, for $Z \in \mathfrak{g}_\mathbb{C}$,

$$Z = Z_{\mathrm{n}} + Z_{\mathrm{a}} + Z_{\mathrm{m}} + Z_{\mathrm{k}}, \tag{6.8}$$

with terms in the complexifications of the summands in (6.7), respectively. If $\mathfrak{l}$ is a real Lie algebra, then by $U(\mathfrak{l})$ we denote the universal enveloping algebra of its complexification, and by $U_k(\mathfrak{l})$, for $k \in \mathbb{N}$, the subspace of elements of order at most k. For $Z \in \mathfrak{g}_{\mathbb{C}}$ we define the element $D_0(Z) = D_{Q,v,0}(Z)$ of $U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ by

$$D_0(Z) := Z_{\mathrm{m}} \otimes I \otimes I + I \otimes Z_{\mathrm{a}} \otimes I + I \otimes I \otimes \tau(\check{Z}_{\mathrm{k}}), \tag{6.9}$$

where $X \mapsto \check{X}$ denotes the canonical anti-automorphism of $U(\mathfrak{g})$. If, moreover, $m \in M_{Q\sigma}$, we define, for $\mu \in \mathbb{N}\Delta_r(Q) \setminus \{0\}$, the element $D_\mu(Z, m) = D_{Q,v,\mu}(Z, m)$ of $U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ by

$$D_\mu(Z, m) := I \otimes I \otimes \tau(\Phi_{Q,v,\mu}(m)\check{Z}_{\mathrm{n}}).$$

Finally, if $m \in M_{Q\sigma}$ and $a \in A^+_{Q\mathrm{q}}(R_{Q,v}(m)^{-1})$, we define the element $D_{Q,v}(Z, a, m) \in U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ by

$$D_{Q,v}(Z, a, m) = \sum_{\mu \in \mathbb{N}\Delta_r(Q)} a^{-\mu} D_\mu(Z, m), \tag{6.10}$$

where we have put $D_0(Z, m) = D_0(Z)$. We also agree to write

$$D^+_{Q,v}(Z, a, m) := D_{Q,v}(Z, a, m) - D_0(Z).$$

It follows from Prop. 6.9 that, for each $R > 1$, the series (6.10) is neatly convergent on $A^+_{Q\mathrm{q}}(R^{-1})$ as a $\Delta_r(Q)$-exponential series with values in $C^\infty(M_{Q\sigma,v}[R]) \otimes U_1(\mathfrak{m}_{Q\sigma}) \otimes U_1(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$. Moreover,

$$D^+_{Q,v}(Z, a, m) = I \otimes I \otimes \tau(\Phi_{Q,v}(ma)\check{Z}_{\mathrm{n}}). \tag{6.11}$$

In the formulation of the following result we use the notation of the paper [11], Sects. 1–3. Via the left regular representation, we view $U(\mathfrak{m}_{Q\sigma}) \otimes U(\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathrm{End}(V_\tau)$ as the algebra of right-invariant differential operators on $M_{1Q} \simeq M_{Q\sigma} \times A_{Q\mathrm{q}}$, with coefficients in $\mathrm{End}(V_\tau)$.

Proposition 6.10 *Let $F \in C^{\mathrm{ep}}(\mathrm{X}_+ : \tau)$. Then $\widetilde{F} \in C^{\mathrm{ep}}(\mathrm{X}_+ : \widetilde{\tau})$. Moreover, if $Q \in \mathcal{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$, then $\mathrm{Exp}\,(Q, v \mid \widetilde{F}) \subset \mathrm{Exp}\,(Q, v \mid F) - \mathbb{N}\Delta_r(Q)$. Finally, for every $Z \in \mathfrak{g}_{\mathbb{C}}$, the $\Delta_r(Q)$-exponential expansion*

$$\widetilde{F}(mav)(Z) = \sum_\xi a^\xi\, q_\xi(Q, v \mid \widetilde{F}, \log a, m)(Z) \tag{6.12}$$

along (Q, v) arises from the similar expansion

$$F(mav) = \sum_\xi a^\xi\, q_\xi(Q, v \mid F, \log a, m) \tag{6.13}$$

by the formal application of the expansion (6.10). In particular, if ξ is a leading exponent of F along (Q, v), then, for every $Z \in \mathfrak{g}_{\mathbb{C}}$,

$$q_\xi(Q, v \mid \widetilde{F}, \log(\cdot), \cdot)(Z) = [D_{Q,v,0}(Z) - \xi(Z_{\mathrm{a}})]\, q_\xi(Q, v \mid F, \log(\cdot), \cdot). \tag{6.14}$$

Proof: It is obvious that $\widetilde{F} \in C^\infty(\mathrm{X}_+ : \tau)$. We shall investigate its expansion along (Q, v), for $Q \in \mathscr{P}_\sigma$ and $v \in N_K(\mathfrak{a}_\mathrm{q})$. We start by observing that, for $R > 1$, the expansion (6.13) converges neatly on $A_\mathrm{q}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a, with coefficients in the space $C^\infty(\mathrm{X}_{Q,v,+}[R] : \tau_Q)$, see [11], Thm. 3.4.

If φ is a smooth function on a Lie group L, with values in a complete locally convex space, then for $X \in \mathfrak{l}$ and $x \in L$ we put $\varphi(X; x) := d/dt\, \varphi(\exp tXx)|_{t=0}$. Accordingly, it follows from (6.8) that for $Z \in \mathfrak{g}_\mathbb{C}$, and $m \in M_{Q\sigma}$ and $a \in A_{Q\mathrm{q}}$ with $mav \in \mathrm{X}_+$, we have

$$\begin{aligned}\tilde{F}(mav)(Z) &= F(\check{Z}; mav)\\ &= \tau(\check{Z}_\mathrm{k})F(mav) + F(\check{Z}_\mathrm{m}; mav)\\ &\quad + F(\check{Z}_\mathrm{a}; mav) + F(\check{Z}_\mathrm{n}; mav).\end{aligned} \tag{6.15}$$

The sum of the first three terms allows an expansion that is obtained by the termwise formal application of $D_{Q,v,0}(Z)$ to the expansion (6.13), by [11], Lemmas 1.9 and 1.10. Moreover, the resulting expansion converges on $A_\mathrm{q}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a, with coefficients in the space $C^\infty(\mathrm{X}_{Q,v,+}[R], V_\tau)$. Thus, it remains to discuss the last term in (6.15). Since F is right H-invariant and left τ-spherical, we see by application of (6.3) and (6.11) that the mentioned term may be rewritten as

$$\begin{aligned}F(\check{Z}_\mathrm{n}; mav) &= F(\Phi_{Q,v}(ma)\check{Z}_\mathrm{n}; mav)\\ &= \tau(\Phi_{Q,v}(ma)\check{Z}_\mathrm{n})F(mav)\\ &= D_{Q,v}^+(Z, a, m)F(\,\cdot\, v)(ma).\end{aligned}$$

It follows from Proposition 6.9 that the series for $D_{Q,v}^+(Z)$ converges neatly on $A_\mathrm{q}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a, with coefficients in the space $C^\infty(M_{Q,\sigma,v}[R]) \otimes \mathrm{End}(V_\tau)$. From [11], Lemma 1.10, it now follows that $F(\check{Z}_\mathrm{n}; mav)$ admits a $\Delta_r(Q)$-exponential polynomial expansion that is obtained by the obvious formal application of the series for $D_{Q,v}^+(Z, a, m)$ to the series for $F(mav)$. The resulting series converges neatly on $A_{Q\mathrm{q}}^+(R^{-1})$ as a $\Delta_r(Q)$-exponential polynomial expansion in the variable a with coefficients in $C^\infty(M_{Q\sigma,v}[R], V_\tau)$. It follows that $\tilde{F}(mav)(Z)$ has an expansion of the type asserted along (Q, v), with exponents as indicated.

In particular, if Q is minimal, it follows that $\tilde{F}(mav)$ allows a neatly converging $\Delta(Q)$-exponential polynomial expansion in the variable $a \in A_\mathrm{q}^+(Q)$, with coefficients in $C^\infty(\mathrm{X}_{Q,v}) \otimes \mathfrak{g}_\mathbb{C}^* \otimes V_\tau$. This implies that $\widetilde{F}$ belongs to the space $C^{\mathrm{ep}}(\mathrm{X}_+ : \tilde{\tau})$, defined in [11], Def. 2.1.

It remains to prove the assertion about the leading exponent ξ for F along (Q, v). From the above discussion we readily see that the term in the expansion (6.12) with exponent ξ is obtained from the application of the constant term $D_{Q,v,0}(Z)$ of $D_{Q,v}(Z,a,m)$ to the term in the expansion (6.13) with exponent ξ. This yields

$$a^{\xi} q_{\xi}(Q, v \mid \widetilde{F}, \log a, m)(Z) \\ = D_{Q,v,0}(Z)\big[(m,a) \mapsto a^{\xi} q_{\xi}(Q, v \mid F, \log a, m)\big].$$

Now use that $a^{-\xi} \circ D_{Q,v,0}(Z) \circ a^{\xi} = D_{Q,v,0}(Z) + \xi(\check{Z}_{\mathrm{a}})$ to obtain (6.14). □

We can now describe the action of L_Z, for $Z \in \mathfrak{g}_{\mathbb{C}}$, on families from the space $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}: \tau)$, defined in [12], Def. 6.6.

Theorem 6.11 *Let $F \in \mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}: \tau)$. Then the family $\widetilde{F}\colon \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \times \mathrm{X} \to \mathfrak{g}^*_{\mathbb{C}} \otimes V_\tau$, defined by $(\widetilde{F})_\nu = (F_\nu)^\sim$ belongs to $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}: \widetilde{\tau})$. Moreover, for every $Z \in \mathfrak{g}_{\mathbb{C}}$ and all ν in an open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$,*

$$q_{\nu-\rho_Q}(Q, v \mid \widetilde{F}_\nu : \log(\,\cdot\,) :\ \cdot\,)(Z) \\ = [D_{Q,v,0}(Z) - (\nu - \rho_Q)(Z_{\mathrm{a}})]\, q_{\nu-\rho_Q}(Q, v \mid F_\nu : \log(\,\cdot\,) :\ \cdot\,). \tag{6.16}$$

Proof: There exist $\delta \in \mathrm{D}_Q$ and a finite subset $Y \subset {}^*\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that $F \in \mathcal{E}_{Q,Y}^{\mathrm{hyp}}(\mathrm{X}: \tau : \delta)$. Let $\mathcal{H} = \mathcal{H}_F$, $d = d_F$ and $k = \deg_a F$ be defined as in the text following [12], Def. 6.1. Then F satisfies all conditions of the mentioned definition. It follows from the characterization of the expansions for $\widetilde{F}$ in Proposition 6.10 that $\widetilde{F}$ satisfies the hypotheses of [12], Def. 6.1 with $\widetilde{\tau}$ in place of τ, with the same $Y, \mathcal{H}, d, k$. In particular, $\widetilde{F}$ belongs to $C_{Q,Y}^{\mathrm{ep,hyp}}(\mathrm{X}_+ : \widetilde{\tau})$.

Since F_ν is annihilated by the ideal $I_{\delta,\nu}$ for generic $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the same holds for $\widetilde{F}_\nu$, and we see that $\widetilde{F} \in \mathcal{E}_{Q,Y}^{\mathrm{hyp}}(\mathrm{X}_+ : \widetilde{\tau} : \delta)$, see [12], Def. 6.3.

Let now $s \in W$, $P \in \mathcal{P}^1_\sigma$ such that $s(\mathfrak{a}_{Q\mathrm{q}}) \not\subset \mathfrak{a}_{P\mathrm{q}}$ and $v \in N_K(\mathfrak{a}_{\mathrm{q}})$. Then there exists an open dense subset $\Omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ such that F satisfies the condition stated in [12], Def. 6.4. It follows from Proposition 6.10 and the fact that the functions $m \mapsto D_{P,v,\mu}(Z, m)$ are smooth on all of $M_{P\sigma}$, for $Z \in \mathfrak{g}_{\mathbb{C}}$, $\mu \in \mathbb{N}\Delta_r(P)$, that $\tilde{F}$ also satisfies the condition of [12], Def. 6.4, with the same set Ω. We conclude that $\widetilde{F} \in \mathcal{E}_{Q,Y}^{\mathrm{hyp}}(\mathrm{X}_+ : \widetilde{\tau} : \delta)_{\mathrm{glob}}$. In view of [12], Lemma 6.9, $\nu \mapsto F_\nu$ is a meromorphic $C^\infty(\mathrm{X}: \tau)$-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Hence, $\nu \mapsto \widetilde{F}_\nu$ is a meromorphic $C^\infty(\mathrm{X}: \widetilde{\tau})$-valued function on $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. In view of [12], Def. 6.6, we now infer that $\widetilde{F} \in \mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X}: \widetilde{\tau})$.

Finally, for ν in an open dense subset of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the element $\nu - \rho_Q$ is a leading exponent for F along (Q, v). Thus, (6.16) follows from (6.14). □

Next, we apply the above result to the normalized Eisenstein integral $E^\circ(Q : \psi : \nu)$, defined for $\psi \in \mathcal{A}_{2,Q}$. Let $v \in {}^Q\mathcal{W}$. Given a function $\psi_v \in \mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q)$ and an element $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ we define the function

$$\partial_{Q,v}(\nu)\psi_v\colon\ \mathrm{X}_{Q,v} \to \mathfrak{g}^*_{\mathbb{C}} \otimes V_\tau \tag{6.17}$$

by

$$\partial_{Q,v}(\nu)\psi_v(x)(Z) = [L_{Z_{\mathrm{m}}} - (\nu - \rho_Q)(Z_{\mathrm{a}})]\psi_v(x) - \tau(Z_{\mathrm{k}})[\psi_v(x)],$$

for $x \in \mathrm{X}_{Q,v}$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. Clearly, the function $\partial_{Q,v}(\nu)\psi_v$ is a $\mathbb{D}(\mathrm{X}_{Q,v})$-finite Schwartz function with values in $\mathfrak{g}_{\mathbb{C}}^* \otimes V_\tau$. Since K_Q normalizes the decomposition (6.7) and centralizes $\mathfrak{a}_{Q\mathrm{q}}$, one readily checks that the function is $\widetilde{\tau}_Q$-spherical, with $\widetilde{\tau}_Q := \widetilde{\tau}|_{K_Q}$. Hence,

$$\partial_{Q,v}(\nu)\psi_v \in \mathcal{A}_2(\mathrm{X}_{Q,v} : \widetilde{\tau}_Q).$$

We define the map $\partial_Q(\nu): \mathcal{A}_{2,Q}(\tau) \to \mathcal{A}_{2,Q}(\widetilde{\tau})$ as the direct sum, for $v \in {}^Q\mathcal{W}$, of the maps $\partial_{Q,v}(\nu): \mathcal{A}_2(\mathrm{X}_{Q,v} : \tau_Q) \to \mathcal{A}_2(\mathrm{X}_{Q,v} : \widetilde{\tau}_Q)$.

Theorem 6.12 *Let $\psi \in \mathcal{A}_{2,Q}(\tau)$ and let the family $F: \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \times \mathrm{X} \to V_\tau$ be defined by*

$$F(\nu, x) = E_\tau^\circ(Q : \psi : \nu : x).$$

Then the family $\widetilde{F}: \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^* \times \mathrm{X} \to \mathfrak{g}_{\mathbb{C}}^* \otimes V_\tau$, defined by $(\widetilde{F})_\nu = (F_\nu)^\sim$, is given by

$$\widetilde{F}(\nu, x) = E_{\widetilde{\tau}}^\circ(Q : \partial_Q(\nu)\psi : \nu : x).$$

Proof: It follows from [12], Def. 13.7 and Prop. 13.6, that the family F belongs to $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \tau)$ and that the family $G := E^\circ(Q : \partial_Q(\nu)\psi)$ belongs to $\mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \widetilde{\tau})$. Let $v \in {}^Q\mathcal{W}$. Then it follows from the mentioned proposition, combined with [11], Thm. 7.7, Eqn. (7.14), that, for ν in an open dense subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\nu-\rho_Q}(Q, v \mid F_\nu, X, m) = \psi_v(m), \tag{6.18}$$
$$q_{\nu-\rho_Q}(Q, v \mid G_\nu, X, m) = \partial_{Q,v}(\nu)\psi_v(m). \tag{6.19}$$

From Theorem 6.11 we see that $\widetilde{F} \in \mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \widetilde{\tau})$. Moreover, combining (6.18) and (6.16) we infer that, for $Z \in \mathfrak{g}_{\mathbb{C}}$, ν in an open dense subset of $\mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$ and all $X \in \mathfrak{a}_{Q\mathrm{q}}$ and $m \in \mathrm{X}_{Q,v,+}$,

$$q_{\nu-\rho_Q}(Q, v \mid \widetilde{F}_\nu, X, m)(Z) = [D_{Q,v,0}(Z) - (\nu - \rho_Q)(Z_{\mathrm{a}})][(m, a) \mapsto \psi_v(m)].$$

From (6.9) we see that the expression on the right-hand side of this equation equals $[\partial_{Q,v}(\nu)\psi_v(m)](Z)$; hence

$$q_{\nu-\rho_Q}(Q, v \mid \widetilde{F}_\nu, X, m) = \partial_{Q,v}(\nu)\psi_v(m). \tag{6.20}$$

Comparing (6.20) with (6.19) we deduce that the family $\widetilde{F} - G \in \mathcal{E}_Q^{\mathrm{hyp}}(\mathrm{X} : \widetilde{\tau})$ satisfies the hypothesis of the vanishing theorem, [12], Thm. 6.11. Hence, $\widetilde{F} = G$. □

Given $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and $\varphi \in C^\infty(K : \xi : \tau)$ we define the function

$$d(Q, \xi, \nu)\, \varphi \in C^\infty(K : \xi) \otimes \mathfrak{g}^*_{\mathbb{C}} \otimes V_\tau$$

by

$$[d(Q, \xi, \nu)\, \varphi](k, Z) = ([\pi_{\xi,-\nu}(Z) \otimes I]\, \varphi)(k), \tag{6.21}$$

for $k \in K$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. One readily verifies that $d(Q, \xi, \nu)\varphi \in C^\infty(K : \xi : \widetilde{\tau})$.

Lemma 6.13 *Let $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi : \tau)$. Then, for all $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$,*

$$\psi_{[I \otimes d(Q,\xi,\nu)]T} = \partial_Q(\nu)\psi_T.$$

Proof: By linearity it suffices to prove this for $T = \eta \otimes \varphi$, with $\eta \in \bar{V}(\xi)$ and $\varphi \in C^\infty(K : \xi : \tau)$. Let $v \in {}^Q\mathcal{W}$ and $Z \in \mathfrak{g}_{\mathbb{C}}$. Then combining (6.21) with the decomposition (6.8) we infer that

$$[d(Q, \xi, \nu)\varphi](e)(Z) = [\xi(Z_{\mathrm{m}}) \otimes I - I \otimes \tau(Z_{\mathrm{k}})]\varphi(e) - (\nu - \rho_Q)(Z_{\mathrm{a}})\varphi(e).$$

By equivariance, η_v maps $\mathcal{H}^\infty_\xi$ into $L^2(\mathrm{X}_{Q,v})^\infty_\xi \subset C^\infty(\mathrm{X}_{Q,v})$, intertwining the $(\mathfrak{m}_Q, K_Q)$-actions. Using formula (4.8) we now obtain that

$$\begin{aligned}
&\psi_{[I \otimes d(Q,\xi,\nu)]T,v}(\,\cdot\,)(Z) \\
&\quad = [\eta_v \otimes I]([\xi(Z_{\mathrm{m}}) \otimes I]\varphi(e)) \\
&\qquad -[(\nu - \rho_Q)(Z_{\mathrm{a}}) I \otimes I + I \otimes \tau(Z_{\mathrm{k}})](\eta_v \otimes I)(\varphi(e)) \\
&\quad = [L_{Z_{\mathrm{m}}} - (\nu - \rho_Q)(Z_{\mathrm{a}})](\psi_{T,v})(\,\cdot\,) - \tau(Z_{\mathrm{k}})[\psi_{T,v}(\,\cdot\,)] \\
&\quad = (\partial(Q, \nu)\psi_T)_v(\,\cdot\,)(Z).
\end{aligned}$$

□

Corollary 6.14 *Let $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi : \tau)$ and let the family $F\colon \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \times \mathrm{X} \to V_\tau$ be defined by*

$$F_\nu = E^\circ(Q : \psi_T : \nu).$$

Then the family $\widetilde{F}\colon \nu \mapsto (F_\nu)^\sim$ is given by

$$\widetilde{F}_\nu = E^\circ(Q : \psi_{[I \otimes d(Q,\xi,\nu)]T} : \nu). \tag{6.22}$$

Proof: This follows from Theorem 6.12 and Lemma 6.13. □

As a consequence of the above, we can now express derivatives of the normalized Eisenstein integral in a form needed for the proof of Theorem 4.6.

Proposition 6.15 *Let $\vartheta \subset \widehat{K}$ be a finite subset, and let $\vartheta' \subset \widehat{K}$ be the union of the collections of K-types occurring in $\mathrm{Ad}_K \otimes \delta$, as $\delta \in \vartheta$. Let $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta$. Then $(I \otimes \pi_{\xi,-\nu}(Z))T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_{\vartheta'}$, for all $Z \in \mathfrak{g}_{\mathbb{C}}$ and $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Moreover, for all $Z \in \mathfrak{g}_{\mathbb{C}}$, $x \in \mathrm{X}$ and $k \in K$,*

$$L_{\mathrm{Ad}(k)^{-1}Z} E^\circ_\vartheta(Q : \psi_T : \nu)(x)(k) = E^\circ_{\vartheta'}(Q : \psi_{[I \otimes \pi_{\xi,-\nu}(Z)]T} : \nu)(x)(k), \tag{6.23}$$

as a meromorphic identity in $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.

Proof: Let $\tau = \tau_\vartheta$. We shall use the natural identification $C^\infty(K : \xi)_\vartheta \simeq C^\infty(K : \xi : \tau)$ of Lemma 3.4, so that ψ_T may be viewed as an element of $\bar{V}(\xi) \otimes C^\infty(K : \xi : \tau)$.

Define the family F as in Corollary 6.14. We shall derive the identity (6.23) from (6.22) by using the functorial properties of Lemma 4.5.

Fix $Z \in \mathfrak{g}_\mathbb{C}$. We define the matrix coefficient map $m_Z\colon \mathfrak{g}_\mathbb{C}^* \to C^\infty(K)$ by

$$m_Z(\zeta)(k) = \zeta(\mathrm{Ad}(k^{-1})Z), \qquad (\zeta \in \mathfrak{g}_\mathbb{C}^*,\ k \in K).$$

The map m_Z intertwines the representation $\mathrm{Ad}_K^\vee$ of K in $\mathfrak{g}_\mathbb{C}^*$ with the right regular representation of K in $C(K)$. In particular, it maps into the finite dimensional space $C(K)_{\vartheta_0^\vee}$, with $\vartheta_0 \subset \widehat{K}$ the set of K-types in Ad_K. We define the equivariant map

$$S_1\colon = m_Z \otimes I\colon\ \ \mathfrak{g}_\mathbb{C}^* \otimes \mathbf{V}_\vartheta \to C(K)_{\vartheta_0^\vee} \otimes \mathbf{V}_\vartheta.$$

On the other hand, we define the map $S_2\colon C(K)_{\vartheta_0^\vee} \otimes \mathbf{V}_\vartheta \to C(K)$ by $\phi \otimes \psi \mapsto \phi\psi$. This map intertwines $\tau_{\vartheta_0} \otimes \tau_\vartheta$ with the right regular representation of K in $C(K)$, hence maps into $C(K)_{\vartheta'^\vee}$. The space $C(K)_{\vartheta_0^\vee} \otimes \mathbf{V}_\vartheta$ may be naturally identified with a finite dimensional K-submodule of $C(K \times K)$, the latter being equipped with the diagonal K-action from the right. Under this identification the map S_2 corresponds with the restriction of the map $\Delta^*\colon C(K \times K) \to C(K)$ given by $\Delta^*\varphi(k) = \varphi(k, k)$.

The map $S = S_2 \circ S_1\colon\ \mathfrak{g}_\mathbb{C}^* \otimes V_\tau \to \mathbf{V}_{\vartheta'}$ is K-equivariant. We shall apply $I \otimes S$ to both sides of the identity (6.22). Application of $I \otimes S_1$ to the left-hand side yields $(I \otimes S_1)[\widetilde{F}_\nu(\,\cdot\,)](k) = \widetilde{F}_\nu(\,\cdot\,, \mathrm{Ad}(k^{-1})Z)$, which in turn equals $L_{\mathrm{Ad}(k^{-1})Z} F_\nu$. By application of $I \otimes S_2$ to the latter function we find

$$\begin{aligned}(I \otimes S)[\widetilde{F}_\nu(\,\cdot\,)](k) &= L_{\mathrm{Ad}(k^{-1})Z} F_\nu(\,\cdot\,)(k)\\ &= L_{\mathrm{Ad}(k^{-1})Z} E^\circ(Q : \psi_T : \nu)(\,\cdot\,)(k).\end{aligned} \tag{6.24}$$

On the other hand, from Lemma 4.5 we see that application of $I \otimes S$ to the expressions on both sides of (6.22) yields

$$(I \otimes S)\widetilde{F}_\nu = E^\circ(Q : \psi_{[I \otimes I \otimes S][I \otimes d(Q,\xi,\nu)]T} : \nu). \tag{6.25}$$

We observe that $(I \otimes S) \circ d(Q, \xi, \nu)$ is a linear map from $C^\infty(K : \xi : \tau_\vartheta)$ to $C^\infty(K : \xi : \tau_{\vartheta'})$ and claim that the following diagram commutes, for every $\nu \in \mathfrak{a}_{Q\mathrm{q}\mathbb{C}}^*$,

$$\begin{array}{ccc} C^\infty(K : \xi : \tau_\vartheta) & \xrightarrow{(I \otimes S) \circ d(Q,\xi,\nu)} & C^\infty(K : \xi : \tau_{\vartheta'}) \\ \downarrow & & \downarrow \\ C^\infty(K : \xi)_\vartheta & \xrightarrow{\pi_{\xi,-\nu}(Z)} & C^\infty(K : \xi)_{\vartheta'}\,. \end{array} \tag{6.26}$$

Here the vertical arrows represent the natural isomorphisms of Lemma 3.4. We denote both of these isomorphisms by $\varphi \mapsto \varphi'$. It suffices to prove the claim, since its validity implies that $\pi_{\xi,-\nu}(Z)$ maps $C^\infty(K : \xi)_\vartheta$ into

$C^\infty(K:\xi)_{\vartheta'}$ and that the expression on the right-hand side of (6.25) equals the one on the right-hand side of (6.23). Combining this with (6.24) we obtain (6.23).

To see that the claim holds, let $\varphi \in C^\infty(K:\xi:\tau_\vartheta) = (C^\infty(K:\xi)\otimes\mathbf{V}_\vartheta)^K$. The associated element $\varphi' \in C^\infty(K:\xi)_\vartheta$ is given by

$$\varphi'(k) = \varphi(k)(e), \qquad (k \in K).$$

The element $(I\otimes S_1)d(Q,\xi,\nu)\varphi$ of $[C^\infty(K:\xi)\otimes C(K)_{\vartheta_0^\vee}\otimes\mathbf{V}_\vartheta]^K$ is given by

$$\begin{aligned}[(I\otimes S_1)d(Q,\xi,\nu)\varphi](k)(k_1) &= [d(Q,\xi,\nu)\varphi(k)]\big(\mathrm{Ad}\big(k_1^{-1}\big)Z\big)\\ &= \big[I\otimes\pi_{\xi,-\nu}\big(\mathrm{Ad}\big(k_1^{-1}\big)Z\big)\big]\varphi(k);\end{aligned}$$

see (6.21). Hence, the element $(I\otimes S)d(Q,\xi,\nu)\varphi \in [C^\infty(K:\xi)\otimes\mathbf{V}_{\vartheta'}]^K$ is given by

$$(I\otimes S)d(Q,\xi,\nu)\varphi(k)(k_1) = \big(I\otimes\pi_{\xi,-\nu}\big(\mathrm{Ad}\big(k_1^{-1}\big)Z\big)\big)\varphi(k)(k_1).$$

The natural isomorphism from $[C^\infty(K:\xi)\otimes\mathbf{V}_{\vartheta'}]^K$ onto $C^\infty(K:\xi)_{\vartheta'}$ is induced by the map $I\otimes\delta_e$, where $\delta_e\colon\mathbf{V}_{\vartheta'}\to\mathbb{C}$ denotes evaluation at e (see (3.6)). Hence,

$$((I\otimes S)d(Q,\xi,\nu)\varphi)'(k) = [(I\otimes\pi_{\xi,-\nu}(Z))\varphi](k)(e) = [\pi_{\xi,-\nu}(Z)\varphi'](k).$$

This establishes the claim. □

We shall apply the above result in combination with Proposition 5.1 to obtain the assertion of Theorem 4.6 about finiteness. If $H \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ is a $\Sigma_r(Q)$-hyperplane, we denote by α_H the shortest root of $\Sigma_r(Q)$ such that H is a translate of $(\alpha_H^\perp)_\mathbb{C}$. Thus, $\langle\alpha_H\,,\,\cdot\,\rangle$ equals a constant c on H; we denote by l_H the linear polynomial function $\langle\alpha_H\,,\,\cdot\,\rangle - c$. In accordance with [12], Eqn. (4.3), given a locally finite collection $\mathcal{H}$ of $\Sigma_r(Q)$-hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ and a map $d\colon\mathcal{H}\to\mathbb{N}$, we define, for every bounded subset ω of $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the polynomial $\pi_{\omega,d}$ by

$$\pi_{\omega,d}(\nu) = \prod_{\substack{H\in\mathcal{H}\\ H\cap\omega\neq\emptyset}} l_H^{d(H)}. \tag{6.27}$$

Proposition 6.16 *Let $Q\in\mathcal{P}_\sigma$, $\xi\in \mathrm{X}^\wedge_{Q,*,ds}$. Then $\mathcal{H}(Q,\xi)$ is a locally finite collection of real $\Sigma_r(Q)$-hyperplanes. Moreover, there exists a map $d\colon\mathcal{H}(Q,\xi)\to\mathbb{N}$ such that, for every finite dimensional unitary representation τ of K, every $T\in\bar{V}(\xi)\otimes C^\infty(K:\xi:\tau)$ and every bounded open subset $\omega\subset\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the $C^\infty(\mathrm{X}:\tau)$-valued function*

$$\nu\mapsto\pi_{\omega,d}(\nu)E^\circ(Q:\psi_T:\nu) \tag{6.28}$$

is holomorphic on ω. Here $\pi_{\omega,d}$ is defined by (6.27) with $\mathcal{H}=\mathcal{H}(Q,\xi)$.

Proof: Select any bounded open subset $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Let $\vartheta \subset \widehat{K}$ be a finite set associated with $Q, \xi, -\omega$ as in Proposition 5.1. According to [12], Prop. 13.14, there exists a map $d\colon \mathcal{H}(Q, \xi, \vartheta) \to \mathbb{N}$ with the property that, for every $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta$, the map $\nu \mapsto E^\circ_\vartheta(Q : \psi_T : \nu)$ belongs to $\mathcal{M}(\mathfrak{a}^*_{Q\mathrm{q}}, \mathcal{H}(Q, \xi, \vartheta), d, C^\infty(\mathrm{X} : \tau_\vartheta))$. See [12], § 4, for the definition of the latter space.

Let $\vartheta' \subset \widehat{K}$ be an arbitrary finite subset. Fix $\nu_0 \in \omega$. Then by Proposition 5.1 there exists $k \in \mathbb{N}$ such that the map

$$M_\nu\colon\ U_k(\mathfrak{g}) \otimes C(K : \xi)_\vartheta \to C(K : \xi)_K,\ \ u \otimes \varphi \mapsto \pi_{\xi,-\nu}(u)\varphi$$

has image containing $C(K : \xi)_{\vartheta'}$ for $\nu = \nu_0$. On the other hand, let $\vartheta'' \subset \widehat{K}$ be the finite collection of K-types occurring in $\delta_1 \otimes \delta_2$, with $\delta_1 \in \widehat{K}$ a K-type occurring in the adjoint representation of K in $U_k(\mathfrak{g})$ and with $\delta_2 \in \vartheta$. Then the image of M_ν is contained in $C^\infty(K : \xi)_{\vartheta''}$ for all $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. Let $P_{\vartheta',\vartheta''}$ denote the K-equivariant projection from $C(K : \xi)_{\vartheta''}$ onto $C(K : \xi)_{\vartheta'}$. Then $P_{\vartheta',\vartheta''} \circ M_{\nu_0}$ is surjective. Hence there exists a finite dimensional subspace $E \subset U_k(\mathfrak{g}) \otimes C(K : \xi)_\vartheta$ such that $R_\nu := P_{\vartheta',\vartheta''} \circ M_\nu|_E\colon E \to C(K : \xi)_{\vartheta'}$ is a bijection for $\nu = \nu_0$. By continuity and finite dimensionality, there exists an open neighborhood ω_0 of ν_0 in ω such that R_ν is a bijection for all $\nu \in \omega_0$. By Cramer's rule, the inverse $S_\nu := R_\nu^{-1} \in \mathrm{Hom}(C(K : \xi)_{\vartheta'}, E)$ depends holomorphically on $\nu \in \omega_0$. Let $(u_i \mid 1 \leq i \leq I)$ be a basis of $U_k(\mathfrak{g})$ and $(\varphi_j \mid 1 \leq j \leq J)$ a basis of $C(K : \xi)_\vartheta$. Then there exist holomorphic $[C^\infty(K : \xi)_{\vartheta'}]^*$-valued functions s_{ij} on ω_0, for $1 \leq i \leq I,\ 1 \leq j \leq J$, such that

$$S_\nu(\varphi) = \sum_{\substack{1 \leq i \leq I \\ 1 \leq j \leq J}} s_{ij}(\nu, \varphi) u_i \otimes \varphi_j, \qquad (\nu \in \omega_0),$$

for $\varphi \in C^\infty(K : \xi)_{\vartheta'}$. Let $\varphi \in C(K : \xi)_{\vartheta'}$. Then $\varphi = P_{\vartheta',\vartheta''} \circ M_\nu \circ S_\nu(\varphi)$, hence

$$\varphi = \sum_{i,j} s_{ij}(\nu, \varphi)\, P_{\vartheta',\vartheta''} \pi_{\xi,-\nu}(u_i)\varphi_j.$$

Let $\eta \in \bar{V}(\xi)$. Then it follows from the above by application of (4.17) and (4.18) that

$$\begin{aligned} &\pi_{\omega,d}(\nu) E^\circ_{\vartheta'}(Q : \psi_{\eta\otimes\varphi} : \nu) \\ &\qquad = \sum_{i,j} s_{ij}(\nu, \varphi) \pi_{\omega,d}(\nu) E^\circ_{\vartheta'}(Q : \psi_{\eta \otimes P_{\vartheta',\vartheta''} \pi_{\xi,-\nu}(u_i)\varphi_j} : \nu) \\ &\qquad = \sum_{i,j} s_{ij}(\nu, \varphi)(I \otimes P_{\vartheta',\vartheta''}) \big[\pi_{\omega,d}(\nu) E^\circ_{\vartheta''}(Q : \psi_{\eta \otimes \pi_{\xi,-\nu}(u_i)\varphi_j} : \nu)\big]. \end{aligned}$$

Applying $I \otimes \delta_e = \varsigma_{\vartheta'}^{-1}$ and using Lemma 3.6 and Proposition 6.15 we infer that

$$\begin{aligned}
&\pi_{\omega,d}(\nu) E^\circ_{\vartheta'}(Q : \psi_{\eta\otimes\varphi} : \nu : \,\cdot\,)(e) \\
&\qquad = \sum_{i,j} s_{ij}(\nu, \varphi) P_{\vartheta',\vartheta''}\big[\pi_{\omega,d}(\nu) E^\circ_{\vartheta''}(Q : \psi_{\eta\otimes\pi_{\xi,-\nu}(u_i)\varphi_j} : \nu : \,\cdot\,)(e)\big] \\
&\qquad = \sum_{i,j} s_{ij}(\nu, \varphi) P_{\vartheta',\vartheta''} L_{u_i}\big[\pi_{\omega,d}(\nu) E^\circ_{\vartheta}(Q : \psi_{\eta\otimes\varphi_j} : \nu : \,\cdot\,)(e)\big].
\end{aligned} \tag{6.29}$$

From this we conclude that the expression on the left-hand side of the above equation depends holomorphically on $\nu \in \omega_0$, as a function with values in $C^\infty(\mathrm{X})$. Since ν_0 was arbitrary, it follows that the expression on the left-hand side of (6.29) in fact depends holomorphically on $\nu \in \omega$. Hence, every $H \in \mathcal{H}(Q, \xi, \vartheta')$ with $H \cap \omega \neq \emptyset$ must be contained in $\mathcal{H}(Q, \xi, \vartheta)$. This shows that the collection $\mathcal{H}(Q, \xi)$ is locally finite. The argument also shows that there exists a map d: $\mathcal{H}(Q, \xi) \to \mathbb{N}$ such that the assertion of the proposition holds for every τ of the form $\tau = \tau_{\vartheta'}$, with $\vartheta' \subset \widehat{K}$ a finite subset. The general result now follows by application of the functorial property of Lemma 4.5. □

Corollary 6.17 *Let d: $\mathcal{H}(Q, \xi) \to \mathbb{N}$ be as in Proposition 6.16. Then, for every $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_K$ and every bounded open subset $\omega \subset \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, the function*

$$\nu \mapsto \pi_{\omega,d}(\nu) J_{Q,\xi,\nu}(T)$$

extends to a holomorphic $C^\infty(\mathrm{X})$-valued function on ω.

Proof: This follows from Proposition 6.16 and the definition of $J_{Q,\xi,\nu}$, see (4.20). □

We can now finally give the promised proof.

Proof of Theorem 4.6: The properties of $\mathcal{H}(Q, \xi)$ have been established in Proposition 6.16 and (4.19). Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. That $J_{Q,\xi,\nu}$ is a $\mathfrak{g}$-equivariant map follows from formula (4.20) combined with formula (6.23) with $k = e$. It remains to establish the K-equivariance of $J_{Q,\xi,\nu}$. Let $\vartheta \subset \widehat{K}$ be a finite subset and let $T = \eta \otimes \varphi \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_\vartheta$. We denote the natural isomorphism $C^\infty(K : \xi)_\vartheta \to C^\infty(K : \xi : \tau_\vartheta)$ of Lemma 3.4 by $\varsigma = \varsigma_\vartheta$. Let $k_1 \in K$. Then one readily checks that $\varsigma \circ \pi_{\xi,-\nu}(k_1) = (I \otimes S) \circ \varsigma$, with S the endomorphism of $\mathbf{V}_\vartheta = C(K)_{\vartheta^\vee}$ given by restriction of the left translation L_{k_1}. In particular, S intertwines $\tau_\vartheta = R|_{\mathbf{V}_\vartheta}$ with itself. By the identification discussed in the text before (4.13) we have

$$\begin{aligned}
\psi_{[I\otimes\pi_{\xi,-\nu}(k_1)]T} &= \psi_{[I\otimes\varsigma\pi_{\xi,-\nu}(k_1)]T} \\
&= \psi_{(I\otimes I\otimes S)(I\otimes\varsigma)T}.
\end{aligned}$$

By Lemma 4.5 (a), combined with the identification mentioned above, the latter expression equals

$$(I \otimes S)\psi_{(I\otimes\varsigma)T} = (I \otimes S)\psi_T.$$

Applying Lemma 4.5 (b) we now find that

$$\begin{aligned} J_{Q,\xi,\nu}([I \otimes \pi_{\xi,-\nu}(k_1)]T) &= E^\circ_\vartheta(Q : \psi_{[I\otimes\pi_{\xi,-\nu}(k_1)]T} : \nu)(\,\cdot\,)(e) \\ &= \big[[I \otimes S]E^\circ_\vartheta(Q : \psi_T : \nu)(\,\cdot\,)\big](e) \\ &= E^\circ_\vartheta(Q : \psi_T : \nu)(\,\cdot\,)\big(k_1^{-1}\big) \\ &= L_{k_1} J_{Q,\xi,\nu}(T). \end{aligned}$$

□

7 The Fourier transform

Let $Q \in \mathscr{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{Q,*,ds}$. We will use the map $J_{Q,\xi}$, introduced in (4.20), to define a $(\mathfrak{g}, K)$-equivariant Fourier transform for functions from $C^\infty_c(\mathrm{X})_K$.

We define the collection $\mathscr{H}^\vee(Q, \xi)$ of hyperplanes in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ by

$$\mathscr{H}^\vee(Q, \xi) := \{-H \mid H \in \mathscr{H}(Q, \xi)\}.$$

Since $\mathscr{H}(Q, \xi)$ is a locally finite collection of real $\Sigma_r(Q)$-hyperplanes, see Theorem 4.6, the same holds for $\mathscr{H}^\vee(Q, \xi)$. It also follows from the mentioned theorem that $\cup\mathscr{H}^\vee(Q, \xi)$ is disjoint from $i\mathfrak{a}^*_{Q\mathrm{q}}$.

Since $\mathscr{H}(Q, \xi)$ consists of real $\Sigma_r(Q)$-hyperplanes, every hyperplane of $\mathscr{H}(Q, \xi)$ is invariant under the complex conjugation $\lambda \mapsto \bar{\lambda}$ in $\mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$, defined with respect to the real form $\mathfrak{a}^*_{Q\mathrm{q}}$. Hence, $\mathscr{H}^\vee(Q, \xi) = \{-\bar{H} \mid H \in \mathscr{H}(Q, \xi)\}$. This justifies the following definition.

Definition 7.1 Let $f \in C^\infty_c(\mathrm{X})_K$. For $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathscr{H}^\vee(Q, \xi)$, the Fourier transform $\hat{f}(Q : \xi : \nu)$ is defined to be the element of $\bar{V}(\xi) \otimes C^\infty(K : \xi)_K$ determined by

$$\langle \hat{f}(Q : \xi : \nu) \mid T \rangle = \int_{\mathrm{X}} f(x)\overline{J_{Q,\xi,-\bar{\nu}}(T)(x)}\, dx, \tag{7.1}$$

for all $T \in \bar{V}(\xi) \otimes C^\infty(K : \xi)_K$.

Lemma 7.2 *Let $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}} \setminus \cup\mathscr{H}^\vee(Q, \xi)$. Then the map $f \mapsto \hat{f}(Q : \xi : \nu)$ from $C^\infty_c(\mathrm{X})_K$ to $\bar{V}(\xi) \otimes C^\infty(K : \xi)_K$ intertwines the $(\mathfrak{g}, K)$-module structure on $C^\infty(\mathrm{X})_K$ coming from the left regular representation with the $(\mathfrak{g}, K)$-module structure on $\bar{V}(\xi) \otimes C^\infty(K : \xi)_K$ coming from $1 \otimes \pi_{Q,\xi,-\nu}$.*