# The Plancherel theorem for a reductive symmetric space

Lectures for the European School of Group Theory,

August 14–26, 2000, SDU–Odense University

Erik P. van den Ban

1/11–2004

## Contents

# 1 Introduction

This chapter is based is on a series of lectures given at the meeting of the European School of Group Theory in August 2000, Odense, Denmark. The purpose of the lectures was to explain the structure of the Plancherel decomposition for a reductive symmetric space, as well as many of the main ideas involved in the proof found in joint work with Henrik Schlichtkrull. [1]

**Reductive symmetric spaces** The purpose of this exposition is to explain the structure of the Plancherel decomposition for a reductive symmetric space.

Throughout the text we will assume that $G$ is a reductive Lie group, i.e., a Lie group whose Lie algebra $\mathfrak{g}$ is a real reductive Lie algebra. We adopt the convention to denote Lie groups by Roman capitals and their Lie algebras by the corresponding German lower case letters. At a later stage we shall impose the restrictive condition that $G$ belongs to *Harish-Chandra's class* of reductive groups. This class contains all connected semisimple groups with finite center, and was introduced by Harish-Chandra [56], Sect. 3, in order to accommodate a certain type of inductive argument that pervades his papers [56]–[58]. We briefly recall the definition and main properties of this class in an appendix.

We assume $\sigma$ to be an involution of $G$, i.e., $\sigma \in \mathrm{Aut}\,(G)$ and $\sigma^2 = I$. Moreover, $H$ is an open subgroup of the group $G^\sigma$ of fixed points for $\sigma$. Equivalently, $H$ is a subgroup with the property

$$(G^\sigma)_e \subset H \subset G^\sigma\,.$$

The pair $(G, H)$ is called a *reductive symmetric pair,* and the associated homogeneous space $\mathrm{X} := G/H$ a *reductive symmetric space.* If $G$ is of the Harish-Chandra class, then both pair and space are said to be of this class as well.

The reason for the terminology symmetric space is the following. Let the derivative of $\sigma$ at the identity element $e$ be denoted by the same symbol. Then $\sigma$ is an involution of the Lie algebra $\mathfrak{g}$, which therefore decomposes as the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \tag{1.1}$$

where $\mathfrak{h}, \mathfrak{q}$ are the $+1$ and $-1$ eigenspaces for $\sigma$. We note that $\mathfrak{h}$ equals the Lie algebra of $H$ and that the decomposition (1.1) is invariant under the adjoint action by $H$. It can be shown that there exists a nondegenerate indefinite inner product $\beta_e$ on $\mathfrak{q}$, which is $H$-invariant. Indeed, if $\mathfrak{g}$ is semisimple, then the restriction of the Killing form has this property; in general one may take $\beta_e$ to be a suitable extension to $\mathfrak{q}$ of the Killing form's restriction to $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{q}$. From $T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{q}$ and the $H$-invariance of $\beta_e$ it follows that $\beta_e$ induces a $G$-invariant pseudo-Riemannian metric on $G/H$ by the formula

$$\beta_{gH} := (\ell_g^{-1})^* \beta_e \qquad (g \in G)\,.$$

[1] It is my great pleasure to thank Jean-Philippe Anker and Bent Ørsted for inviting me to give these lectures. I thank the Mathematics Department of the University of Copenhagen for providing assistance with typing the first version of this exposition.

The natural map $\overline{\sigma} : G/H \to G/H$, $gH \mapsto \sigma(g)H$ can be shown to be the geodesic reflection in the origin $eH$ for the metric $\beta$. By homogeneity it follows that the (locally defined) geodesic reflection $S_x$ at any point $x \in \mathrm{X}$ extends to a global isometry. A space with this property is called symmetric. For a more general definition of symmetric space we refer the reader to [82], p. 98.

The following are motivating and guiding examples of symmetric spaces.

**Example 1.1** (The Riemannian case) Assume that $G^\sigma$ is a maximal compact subgroup of $G$ and let $H = G^\sigma$. The Killing form's restriction to $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{q}$ extends to an $H$-invariant positive definite inner product on $\mathfrak{q}$, so that $\mathrm{X} = G/H$ is a Riemannian symmetric space. In this case the involution $\sigma$ is called a Cartan involution and it is customary to write $H = K$ and $\sigma = \theta$. By the work of E. Cartan, it is well known that every Riemannian symmetric space of noncompact type arises in this fashion, see [63] for details.

**Example 1.2** (The case of the group) Let ${}^{\backprime}G$ be a reductive group; then $G = {}^{\backprime}G \times {}^{\backprime}G$ is reductive as well. The group $G$ acts transitively on ${}^{\backprime}G$ by the left times right action given by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$. The stabilizer of ${}^{\backprime}e$ in $G$ equals the diagonal subgroup $H = \mathrm{diagonal}\,({}^{\backprime}G \times {}^{\backprime}G)$ of $G$. Hence, the map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ induces an isomorphism of $G$-spaces

$$G/H \simeq {}^{\backprime}G\,.$$

Moreover, $H = G^\sigma$, where $\sigma$ is the involution of $G$ defined by $(g_1, g_2) \mapsto (g_2, g_1)$.

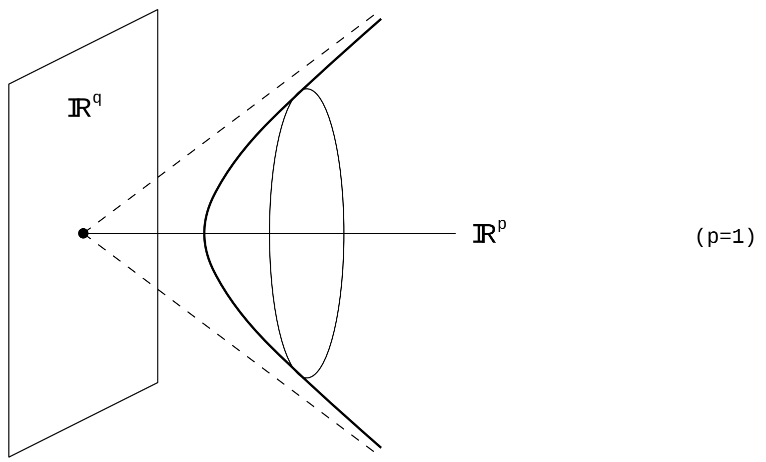


**Figure 1.** $\mathrm{X}_{p,q}$ for $p = 1$

**Example 1.3** (The real hyperbolic spaces) Let $p, q \geq 1$ be integers, and put $n = p + q$. We agree to write $x = (x', x'')$ according to $\mathbb{R}^n \simeq \mathbb{R}^p \times \mathbb{R}^q$. Let

$(\cdot\,,\cdot)$ denote the standard inner products on $\mathbb{R}^p$ and $\mathbb{R}^q$ and define the indefinite inner product $\beta$ on $\mathbb{R}^n$ by

$$\beta(x,y) = (x',y') - (x'',y'')\,.$$

The real hyperbolic space $\mathrm{X}_{p,q}$ is defined to be the submanifold of $\mathbb{R}^n$ consisting of points $x$ with $\beta(x,x) = 1$, or, written out in coordinates,

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 = 1\,.$$

Moreover, if $p = 1$, we impose the additional condition $x_1 > 0$ to ensure that $\mathrm{X}_{p,q}$ is connected. In this case we may visualize $\mathrm{X}_{p,q}$ as in Figure 1. In case $p > 1$, we may visualize the space $\mathrm{X}_{p,q}$ as in Figure 2.

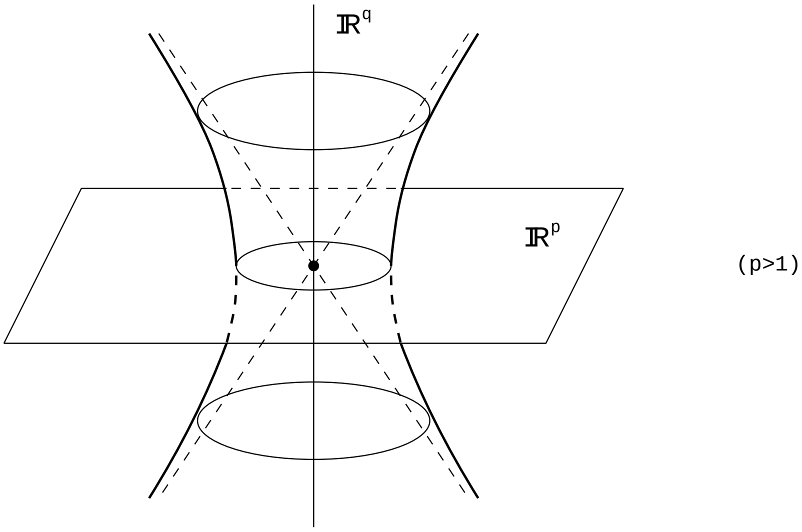


**Figure2**. $\mathrm{X}_{p,q}$ for $p > 1$

The stabilizer of $\beta$ in $\mathrm{SL}\,(n,\mathbb{R})$ is denoted by $\mathrm{SO}\,(p,q)$. Its identity component $\mathrm{SO}_e(p,q)$ acts transitively on $\mathrm{X}_{p,q}$. Moreover, the stabilizer of $e_1 = (1,0,\ldots,0)$ equals $\mathrm{SO}_e(p-1,q)$, so that

$$\mathrm{X}_{p,q} \simeq \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)\,.$$

We define a pseudo-Riemannian structure $\overline{\beta}$ on $\mathrm{X}_{p,q}$ by

$$\overline{\beta}_x = \beta\,\Big|_{T_x\mathrm{X}_{p,q}}\,.$$

Clearly, $\overline{\beta}$ is $\mathrm{SO}_e(p,q)$-invariant. Moreover, from

$$T_{e_1}\mathrm{X}_{p,q} \simeq \mathbb{R}^{p-1} \times \mathbb{R}^q$$

we read that $\overline{\beta}$ has signature $(p-1,q)$. Thus, if $p=1$, then $\mathrm{X}_{p,q}$ is Riemannian; if $p>1$, then $\mathrm{X}_{p,q}$ is pseudo-Riemannian, and one can show that $\mathrm{X}_{p,q}$ lies outside the range of Examples 1.1 and 1.2 in case $p=q=2$.

We leave it to the reader to check that the geodesics on $\mathrm{X}_{p,q}$ are the intersections of $\mathrm{X}_{p,q}$ with two-dimensional linear subspaces of $\mathbb{R}^n$. This is readily seen for the geodesics through the origin $e_1$; the other geodesics are obtained under the action of $\mathrm{SO}_e(p,q)$. The geodesic reflection in the origin $e_1$ is given by the restriction to $\mathrm{X}_{p,q}$ of the map $S: x \mapsto (x_1, -x_2, \dots, -x_n)$.

Finally, we mention that the hyperbolic spaces can also be defined over the fields of complex and quaternion numbers, in which case they correspond to the symmetric pairs $(\mathrm{SU}\,(p,q), \mathrm{S}\,(\mathrm{U}\,(1)\times\mathrm{U}\,(p-1,q))$ and $(\mathrm{Sp}\,(p,q), \mathrm{Sp}\,(1)\times\mathrm{Sp}\,(p-1,q))$, respectively.

**The Plancherel decomposition** Being reductive, the groups $G$ and $H$ are unimodular. Therefore, the symmetric space $\mathrm{X}=G/H$ carries a $G$-invariant measure, which we denote by $dx$. The associated space of square integrable functions on X is denoted by $L^2(\mathrm{X}) = L^2(\mathrm{X}, dx)$. This space is invariant under left translation, by invariance of the measure. Accordingly we define the so-called left regular representation $L$ of $G$ in $L^2(\mathrm{X})$ by

$$L_g f(x) = f(g^{-1}x), \tag{1.2}$$

for $f \in L^2(\mathrm{X})$, $x \in \mathrm{X}$, $g \in G$. This representation is unitary, again by invariance of the measure $dx$.

The Plancherel theorem for X describes the decomposition of $(L, L^2(\mathrm{X}))$ as a direct integral of unitary representations

$$(L, L^2(\mathrm{X})) \simeq \int_{\widehat{G}}^{\oplus} m_\pi\, \pi\, d\mu(\pi)\,. \tag{1.3}$$

Here $\widehat{G}$ denotes the set of equivalence classes of irreducible unitary representations of $G$, equipped with a certain topology. Moreover, $d\mu$ is a Borel measure on $\widehat{G}$, called the Plancherel measure. Finally, $\pi \mapsto m_\pi$ is a measurable function on $\widehat{G}$ with values in $\mathbb{N}\cup\{\infty\}$, describing the multiplicities by which the representations $\pi$ enter the decomposition. In the next section we will describe the meaning of the above formula in more detail. It amounts to a far reaching generalization of the Plancherel theorems for both Fourier series and Fourier transform in Euclidean space.

From Examples 3.1 and 3.2 one sees that the Plancherel theorem for reductive symmetric spaces includes both the Plancherel theorem for Riemannian symmetric spaces and the Plancherel theorem for real reductive groups. In the Riemannian case the Plancherel theorem was established by Harish-Chandra [50], [51] up to two conjectures, the first one concerning a property of the Plancherel measure and the second involving a certain completeness result (injectivity of the associated Fourier transform). The first of these conjectures was established by S. G. Gindikin and S. Karpelevič [48], who in fact explicitly determined the Plancherel measure. The completeness result was established by Harish-Chandra as a byproduct of the theory of the discrete series, [53].

Later, the completeness was also differently obtained in connection with the Paley-Wiener theorem, [60] and [47].

In the case of the group, see Example 1.2, the Plancherel theorem was established by Harish-Chandra, in a monumental series of papers, including those on the discrete series, [53] and [54], and culminating in [56]–[58].

For the hyperbolic spaces, see Example 1.3, the Plancherel formula was obtained by several authors, of whom we mention V. Molchanov [72], W. Rossmann [81] and J. Faraut [43]. In other special cases the Plancherel formula was obtained by G. van Dijk and M. Poel [79] and by N. Bopp and P. Harinck [27]. For the general class of symmetric spaces of type $G_{\mathbb{C}}/G$ the Plancherel theorem was established by Harinck [49].

The theory of harmonic analysis on general symmetric spaces, in terms of their general structure theory, gained momentum in the beginning of the 1980's with the appearance of the wonderful papers [78], by T. Oshima and J. Sekiguchi on the continuous spectrum for a general class of symmetric spaces, and [45], by M. Flensted-Jensen on the discrete series for symmetric spaces. The ideas of the latter paper inspired the fundamental paper [77] by T. Oshima and T. Matsuki on the classification of the discrete series. At that point it became clear that the determination of the full Plancherel decomposition was a reasonable goal to strive for. Such a result was announced by Oshima in the 1980's, see [75], p. 608, but the details have not appeared.

Starting from the papers [5] and [6] on the so-called minimal principal series, E.P. van den Ban and H. Schlichtkrull determined the most-continuous part of the Plancherel decomposition in the early 1990's, see [16]. A survey of this work can be found in [82]. In the meantime, P. Delorme, partly in collaboration with J. Carmona, developed the theory of the generalized principal series, see [34], [38], [35], [39]. In all papers mentioned in this paragraph the influence of Harish-Chandra's work in the case of the group is very strong.

In the fall of 1995, during the special year at the Mittag-Leffler Institute near Stockholm, Sweden, Delorme on the one hand and van den Ban and Schlichtkrull on the other, independently announced a proof for the general Plancherel theorem. At the same time van den Ban and Schlichtkrull announced a proof of the Paley–Wiener theorem as well. It should be mentioned that in their original proof of the Plancherel theorem they needed Delorme's results from [39] and [38] on the so-called Maass–Selberg relations. In the meantime they have found an independent proof of these relations.

The two now existing proofs of the Plancherel theorem are very different. Delorme's proof, which has appeared in [40], builds on the above mentioned theory of the representations of the generalized principal series, in turn based on the theory of the discrete series, and on a detailed study of the associated Eisenstein integrals. In Delorme's work, the elberg relations are obtained through a technique called truncation of inner products, see [39], which in turn is inspired by work of J. Arthur [2]. The completeness part of the proof relies on an idea of J. Bernstein [25]. We refer the reader to Delorme's exposition, elsewhere in this volume, for more information on his strategy of proof.

The proofs of the Plancherel and Paley–Wiener theorem by van den Ban and Schlichtkrull are based on a Fourier inversion theorem, published in [17]. The proofs have now appeared in [21], [22] and [23]. In the present exposition the strategy of their proof of the Plancherel theorem will be explained. Elsewhere in this volume, Schlichtkrull discusses the Paley–Wiener theorem.

For other surveys of the general theory we refer the reader to the papers [11], [19], [9] and [41].

**Outline of the exposition** In the next section we will first give a description of the general idea of what a Plancherel decomposition amounts to. In particular we shall indicate the interaction with invariant differential operators that plays such an important role in the theory. These ideas will be illustrated with the the classical examples of Fourier series, the Peter–Weyl theorem for compact groups and the Plancherel decomposition for compact symmetric spaces.

We then proceed, in Section 3, to discussing the structure theory for reductive symmetric spaces in terms of the structure theory of reductive algebras. In Section 4 we discuss the structure of the algebra of invariant operators and its interaction with the discrete series of reductive symmetric spaces. The necessary preparations for the description of the Plancherel decompositions are continued with the description in Section 6 of the structure of the so-called $\sigma$-parabolic subgroups of $G$. These are of importance for the definition of the generalized principal series of representations in Section 7. Finally, in Sections 8 and 9, the preparations are finished with the description of the $H$-fixed generalized vectors of the principal series and the action of the algebra of invariant differential operators on them.

In Section 10 we give the precise formulation of the Plancherel theorem in the sense of representation theory, both in unnormalized and normalized form. In the subsequent Section 11 we show that reduction to $K$-finite functions leads to the equivalent Plancherel theorem for spherical Schwartz functions. In particular we motivate and give the definition of Eisenstein integrals. In Section 12 the most continuous part of the Plancherel decomposition is characterized by the help of certain differential operators. At that point the exposition will have covered the description of the Plancherel decomposition in an order that is transparent from the point of view of exposition. In contrast, the logical order of the proof is very different.

In the final three sections we give a sketch of the main arguments in the proof. First, in Section 13, we sketch the proof of the most continuous part of the Plancherel decomposition, based on a Paley–Wiener shift argument. In this shift, certain residual contributions are cancelled out by the action of invariant differential operators. However, it turns out that the residues can be controlled by means of a residue calculus for root systems that we briefly explain in the next section. This leads to a full Fourier inversion theorem. In the final section we explain how the Plancherel theorem can be deduced from this Fourier inversion theorem. At the very end, the associated Fourier transforms that enter the analysis through the residue calculus are related to representation theory.

## 2 Direct integral decomposition

**Introduction** In this section we will discuss direct integral decompositions of the type mentioned in (1.3). We will avoid the machinery of the general representation theory of locally compact groups or $C^*$-algebras in which this notion is defined in a precise way, see, e.g., [42] and [89]. To avoid these technicalities we have opted for a somewhat naive presentation. Its sole purpose is to provide motivation for the constructions, definitions and results that will be presented later in the particular setting of reductive symmetric spaces. Let us first consider some motivating examples.

**Fourier series** From the representation theoretic point of view the theory of Fourier series may be described as follows. Let $G = \mathbb{R}/2\pi\mathbb{Z}$, $H = \{0\}$; then $\mathrm{X} = G/H \simeq \mathbb{R}/2\pi\mathbb{Z}$. Let $dx/2\pi$ denote translation invariant measure on X, normalized by $\int_{\mathrm{X}} \frac{dx}{2\pi} = 1$. There is a natural unitary representation $L$ of $G$ on $L^2(\mathrm{X})$ given by $L_g f(x) = f(-g + x)$.

For $n \in \mathbb{Z}$, let $L^2(\mathrm{X})_n$ denote the one-dimensional complex linear space spanned by the exponential function $x \mapsto e^{inx}$. Then

$$L^2(\mathrm{X}) = \widehat{\oplus}_{n\in\mathbb{Z}}\, L^2(\mathrm{X})_n\,,$$

the sum being orthogonal and $G$-invariant. The projection operator onto $L^2(\mathrm{X})_n$ is given by $f \mapsto \hat{f}(n)e^{in\,\cdot}$, wherein $f \mapsto \hat{f}$, the Fourier transform, is given by

$$\hat{f}(n) = \langle f\,,\, e^{in\,\cdot}\rangle_{L^2(\mathrm{X})} = \int_0^{2\pi} f(x)e^{-inx}\frac{dx}{2\pi}\,.$$

Here and in the following, complex positive definite inner products will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$, and will be assumed to be antilinear in the second variable.

The Fourier transform maps $L^2(\mathrm{X})$ into the space $\mathbb{C}^{\mathbb{Z}}$ of functions $\mathbb{Z} \to \mathbb{C}$ and intertwines the $G$-action on the first of these spaces with the $G$-action on the second given by $x \cdot (c_n)_{n\in\mathbb{Z}} = (e^{-inx}c_n)_{n\in\mathbb{Z}}$. The Plancherel theorem asserts that the Fourier transform is an isometry from $L^2(\mathrm{X})$ onto $\ell^2(\mathbb{Z})$, whence the Parseval identities. Equivalently, the Fourier transform is inverted by its adjoint $\mathcal{J}$, which is given by

$$(c_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} c_n e^{in\,\cdot}\,.$$

**The Peter–Weyl theorem** This theorem generalizes the theory of Fourier series to the case of a compact group $G$. We fix a choice of bi-invariant Haar measure $dx$ on $G$ by requiring it to be normalized, i.e., $\int_G dx = 1$. The left regular representation $L$ and the right regular representation $R$ of $G$ in the associated space of square integrable representations are defined by

$$L_g f(x) = f(g^{-1}x) \quad \text{and} \quad R_g f(x) = f(xg), \tag{2.1}$$

for $f \in L^2(G)$, $g \in G$ and $x \in G$. These representations are unitary, by bi-invariance of the measure. Accordingly, the exterior tensor product $L \otimes R$ defines a unitary representation of $G \times G$ in $L^2(G)$.

Let $\widehat{G}$ be the set of (equivalence classes of) irreducible unitary representations of $G$. According to the Peter–Weyl theorem the following is a $G \times G$-invariant orthogonal direct sum decomposition,

$$L^2(G) = \widehat{\oplus}_{\delta\in\widehat{G}}\; L^2(G)_\delta\,, \tag{2.2}$$

where each space $L^2(G)_\delta$ can be described as follows. Let $V_\delta$ be a finite-dimensional Hilbert space in which $\delta$ is unitarily realized. Then $L^2(G)_\delta$ is the image of the map $\mathrm{M}_\delta : \mathrm{End}\,(V_\delta) \to C^\infty(G)$ given by

$$\mathrm{M}_\delta(T)(x) = \mathrm{tr}\,(\delta(x)^{-1} \circ T) \qquad (T \in \mathrm{End}\,(V_\delta), x \in G)\,.$$

The map $\mathrm{M}_\delta$ intertwines the representation $\delta \otimes \delta^*$ of $G \times G$ in $\mathrm{End}\,(V_\delta) \simeq V_\delta \otimes V_\delta^*$ with the representation $L \otimes R$ of $G \times G$ in $L^2(G)$. The latter is unitary because $dx$ is bi-$G$-invariant. We equip $\mathrm{End}\,(V_\delta)$ with the Hilbert-Schmid (or tensor) inner product $\langle \cdot\,,\,\cdot \rangle_{\mathrm{HS}}$ and denote the associated norm by $\|\cdot\|_{\mathrm{HS}}$. Then by the Schur orthogonality relations, the map $\sqrt{\dim \delta}\,\mathrm{M}_\delta$ is an isometry, for every $\delta \in \widehat{G}$.

A straightforward calculation shows that the adjoint of $\mathrm{M}_\delta : \mathrm{End}\,(V_\delta) \to L^2(G)$ is given by the map $L^2(G) \to \mathrm{End}\,(V_\delta)$, $f \mapsto \delta(f)$, where, as usual,

$$\delta(f) := \int_G f(x)\delta(x)\,dx. \tag{2.3}$$

It follows that for every $\delta \in \widehat{G}$ the map $f \mapsto \sqrt{\dim \delta}\ \delta(f)$ is an isometry $L^2(G)_\delta \to \mathrm{End}\,(V_\delta)$. Accordingly, if $f \in L^2(G)$, then

$$\|f\|^2_{L^2(G)} = \sum_{\delta \in \widehat{G}} \dim(\delta)\ \|\delta(f)\|^2_{\mathrm{HS}}.$$

We equip the algebraic direct sum of the spaces $\mathrm{End}\,(V_\delta)$, for $\delta \in \widehat{G}$, with the direct sum of the inner products $\dim(\delta)\langle \cdot\,,\,\cdot \rangle_{\mathrm{HS}}$. The completion of this pre-Hilbert space is denoted by

$$\mathfrak{H} := \widehat{\bigoplus}_{\delta \in \widehat{G}}\ \mathrm{End}\,(V_\delta)\,. \tag{2.4}$$

The direct sum $\pi$ of the representations $\delta \otimes \delta^*$ is a unitary representation of $G \times G$ in $\mathfrak{H}$.

For $f \in L^2(G)$ we define the Fourier transform $\hat{f} \in \mathfrak{H}$ by $\hat{f}(\delta) = \delta(f) \in \mathrm{End}\,(V_\delta)$, for every $\delta \in \widehat{G}$. Then the Peter–Weyl theorem implies that the Fourier transform $f \mapsto \hat{f}$ defines an isometry $L^2(G) \simeq \mathfrak{H}$, intertwining the unitary representations $L \otimes R$ and $\pi$ of $G \times G$. This result is called the Plancherel theorem for the group $G$. The associated decomposition

$$L \otimes R \simeq \widehat{\bigoplus}_{\delta \in \widehat{G}}\ \delta \otimes \delta^* \tag{2.5}$$

as a representation of $G \times G$ is called the Plancherel decomposition. Its constituents $\delta \otimes \delta^*$ are mutually inequivalent irreducible representations of $G \times G$. For this reason, the decomposition (2.5) is said to be multiplicity free with respect to the action of $G \times G$. We thus see that it is very natural to view the group $G$ as equipped with the left times right action of $G \times G$. This amounts to viewing the group as a symmetric space for $G \times G$, as explained in Example 1.2.

By the Plancherel theorem, the inverse $\mathcal{J}$ of the Fourier transform equals its transpose, hence is given by the formula

$$\mathcal{J}(T) = \sum_{\delta \in \widehat{G}} \dim \delta\ \mathrm{M}_\delta(T_\delta)\,,$$

for $T = (T_\delta \mid \delta \in \widehat{G}) \in \mathfrak{H}$. In particular, the orthogonal projection $P_\delta : L^2(G) \to L^2(G)_\delta$ is given by

$$P_\delta(f) = \dim\delta\, \mathrm{M}_\delta(\hat{f}(\delta)). \tag{2.6}$$

We end this discussion with a slightly different description of the map $\mathrm{M}_\delta$. If $V$ is a complex linear space, then by $\overline{V}$ we denote its conjugate. Thus, as a real linear space $\overline{V}$ equals $V$, but the complex multiplication is given by $(z, v) \mapsto \bar{z}v$, $\mathbb{C} \times \overline{V} \to \overline{V}$.

A sesquilinear inner product $\langle \cdot\,,\,\cdot \rangle_V$ on $V$ may now be viewed as a complex bilinear map $V \times \overline{V} \to \mathbb{C}$. If $V$ is a Hilbert space for $\langle \cdot\,,\,\cdot \rangle$, then the map $\eta \mapsto \langle \cdot\,,\,\eta \rangle_V$ induces an isomorphism from $\overline{V}$ onto the dual Hilbert space $V^*$, via which we shall identify these spaces. Note that the dual inner product on $V^*$ corresponds with the inner product on $\overline{V}$ given by $\langle v\,,\,w \rangle_{\overline{V}} = \langle w\,,\,v \rangle_V$ for $v, w \in \overline{V}$.

The map $\mathrm{M}_\delta$ may now also be described as the matrix coefficient map $V_\delta \otimes \overline{V}_\delta \to C^\infty(G)$ given by

$$\mathrm{M}_\delta(v \otimes \eta)(x) = \langle v\,,\,\delta(x)\eta \rangle_{V_\delta}\,, \tag{2.7}$$

for $v \in V_\delta$, $\eta \in \overline{V}_\delta$ and $x \in G$.

**Compact homogeneous spaces** Let $G$ be a compact Lie group and $H$ a closed subgroup. Put $\mathrm{X} = G/H$ and let $dx$ be normalized invariant measure of X. Then we may identify $L^2(\mathrm{X})$ with the subspace $L^2(G)^H$ of right-$H$-invariant functions in $L^2(G)$. Accordingly, the left regular representation $L_\mathrm{X}$ of $G$ in $L^2(\mathrm{X})$ coincides with the restriction of $L$.

Let the matrix coefficient map $M_\delta : V_\delta \otimes \overline{V}_\delta \to C^\infty(G)$ be defined as in (2.7), and put

$$\mathrm{M}_{\mathrm{X},\delta} := \mathrm{M}_\delta|_{V_\delta \otimes \overline{V}_\delta^H}.$$

Then by right equivariance of $\mathrm{M}_\delta$ it follows that $\mathrm{M}_{\mathrm{X},\delta}$ maps $V_\delta \otimes \overline{V}_\delta^H$ bijectively onto

$$L^2(\mathrm{X})_\delta := L^2(G)_\delta \cap L^2(G)^H\,.$$

Moreover, the map $\mathrm{M}_{\mathrm{X},\delta}$ intertwines the $G$-representations $\delta \otimes 1$ and $L$. The adjoint of the map $\mathrm{M}_{\mathrm{X},\delta}$ is readily seen to be given by

$$f \mapsto \hat{f}_X(\delta) := \hat{f}(\delta)|_{V_\delta^H} \in V_\delta \otimes \overline{V}_\delta^H\,.$$

Let $\widehat{G}_H$ denote the set of $\delta \in \widehat{G}$ with the property that $\overline{V}_\delta$ has nontrivial $H$-invariant elements. Then it follows from the invariance of the decomposition (2.2) that

$$L^2(\mathrm{X}) = \widehat{\bigoplus}_{\delta \in \widehat{G}_H}\ L^2(\mathrm{X})_\delta. \tag{2.8}$$

Moreover, the orthogonal projection $P_\delta$ from $L^2(\mathrm{X})$ onto $L^2(\mathrm{X})_\delta$ is now given by the formula

$$P_\delta(f) = \dim\delta\, \mathrm{M}_{\mathrm{X},\delta}(\hat{f}_\mathrm{X}(\delta)).$$

Next, let $\mathfrak{H}$, $\pi$ be defined as in (2.4) and let $\mathfrak{H}_X$ be the closed subspace of $\mathfrak{H}$ consisting of elements that are $\pi(e,h)$-invariant for all $h \in H$. Then

$$\mathfrak{H}_X = \widehat{\oplus}_{\delta \in \widehat{G}} \; V_\delta \otimes \overline{V}_\delta^H .$$

Let $\pi_X$ be the representation of $G$ in $\mathfrak{H}_X$ given by $\pi_X(g) = \pi(g,e)$, for $g \in G$. Thus, $\pi_X$ is the orthogonal direct sum of the representations $\delta \otimes 1$, for $\delta \in \widehat{G}$. The above reasoning leads to the following Plancherel theorem for the compact homogeneous space X.

The Fourier transform $f \mapsto \hat{f}_X$ defines an isometry from $L^2(\mathrm{X})$ onto $\mathfrak{H}_X$, which intertwines the representations $L_{\mathrm{X}}$ and $\pi_X$ of $G$. Thus, we have the unitary equivalence

$$L_{\mathrm{X}} \simeq \widehat{\oplus}_{\delta \in \widehat{G}_H} \; m_\delta \, \delta, \tag{2.9}$$

where $m_\delta = \dim(\overline{V}_\delta^H)$. Moreover, the inverse transform $\mathcal{J}_{\mathrm{X}}$ is the adjoint of $f \mapsto \hat{f}_{\mathrm{X}}$ and given by

$$\mathcal{J}_{\mathrm{X}}(T) = \sum_{\delta \in \widehat{G}_H} \dim \delta \; \mathrm{M}_{\mathrm{X},\delta}(T_\delta)$$

for $T \in \mathfrak{H}_{\mathrm{X}}$.

**Compact symmetric spaces** We retain the notation of the previous subsection, and assume in addition that $G$ is a compact connected semisimple Lie group and that the subgroup $H$ is the group $G^\sigma$ of fixed points for an involution $\sigma$ of $G$. Then the associated homogeneous space $\mathrm{X} = G/H$ is a compact symmetric space. In this case it is known that

$$\dim \overline{V}_\delta^H = 1 \tag{2.10}$$

for $\delta \in \widehat{G}_H$. Thus, it follows from (2.9) that $(L_{\mathrm{X}}, L^2(\mathrm{X}))$ admits the multiplicity free decomposition

$$L_{\mathrm{X}} \sim \oplus_{\delta \in \widehat{G}_H} \; \delta \, . \tag{2.11}$$

If $G = \mathrm{SO}\,(n)$, $H = \mathrm{SO}\,(n-1)$, then $\mathrm{X} = S^n$ and the decomposition corresponds to the one known from the theory of spherical harmonics.

**The compact group as a symmetric space** We now assume that ${}^{\backprime}G$ is a compact Lie group. Then by the Peter–Weyl theorem for the group ${}^{\backprime}G$ we have the Plancherel decomposition (2.5) which now becomes the following decomposition of the exterior tensor product representation $L \otimes R$ of ${}^{\backprime}G \times {}^{\backprime}G$ in $L^2({}^{\backprime}G)$,

$$L \otimes R \simeq \widehat{\oplus}_{\delta \in {}^{\backprime}\widehat{G}} \; \delta \otimes \delta^*. \tag{2.12}$$

As said earlier, this shows that it is very natural to view ${}^{\backprime}G$ as a homogeneous space for $G := {}^{\backprime}G \times {}^{\backprime}G$ via the left times right action. As in Example 1.2 this viewpoint leads to the natural identification of the $G$-space ${}^{\backprime}G$ with the symmetric $G$-space $\mathrm{X} := G/H$, where $H$ is the diagonal subgroup of ${}^{\backprime}G \times {}^{\backprime}G$. The identification naturally induces an isometry $L^2({}^{\backprime}G) \simeq L^2(\mathrm{X})$, via which

$L \otimes R$ corresponds with the left regular representation $L_{\mathrm{X}}$ of $G$ in $L^2(\mathrm{X})$. Thus, (2.12) amounts to the Plancherel decomposition for the space X.

On the other hand, since X is a compact symmetric space for $G$, the Plancherel decomposition (2.11) can be obtained as a consequence of the Peter–Weyl theorem for $G$. We will proceed to identify it with the decomposition (2.12). The irreducible representations of $G$ are the representations of the form $\delta \otimes \rho$, with $\delta$ and $\rho$ irreducible representations of ${}^{\backprime}G$. Let $V_\delta$ and $V_\rho$ be (finite-dimensional) Hilbert spaces in which $\delta$ and $\rho$ are realized, respectively. Then

$$(V_\delta \otimes V_\rho)^H \simeq \mathrm{Hom}_{{}^{\backprime}G}(V_\rho^*, V_\delta),$$

naturally. It follows that the map $\delta \mapsto \delta \otimes \delta^*$ induces a bijection

$${}^{\backprime}\widehat{G} \simeq \widehat{G}_H.$$

Moreover, if $\delta \in {}^{\backprime}\widehat{G}$, then $(V_\delta \otimes V_\delta^*)^H \simeq \mathbb{C}I$, by Schur's lemma, which is in agreement with the more general assertion (2.10). The decomposition (2.11) is thus seen to coincide with (2.12).

**Harmonic analysis on noncompact spaces** If the reductive symmetric space $\mathrm{X} = G/H$ is compact, then the Plancherel decomposition corresponds to the decomposition of $L^2(\mathrm{X})$ into invariant *subspaces,* which are finite multiples of irreducible representations, see (2.8).

In contrast, this cannot be expected when X is noncompact. This is already apparent from the classical example $G = \mathbb{R}^n$, $H = \{0\}$, $\mathrm{X} = \mathbb{R}^n$. The irreducible unitary representations of $G$ are all 1-dimensional and given by $\pi_\xi : G \times \mathbb{C} \to \mathbb{C}$, $(x, z) \mapsto e^{-\xi(x)}z$, with $\xi \in i\mathbb{R}^{n*} = i(\mathbb{R}^n)^*$.

Fix a choice of Lebesgue measure $dx$ on $\mathbb{R}^n$. Then there is a Fourier transform $f \mapsto \hat{f}$ given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, \overline{e^{\xi(x)}}\, dx\,,$$

for functions $f$ in $C^\infty(\mathbb{R}^n)$ with sufficiently rapid decay at infinity. Let $L$ be the natural unitary representation of $G$ on $L^2(\mathbb{R}^n)$ given by $L_a f(x) = f(-a + x)$. Then the Fourier transform has the intertwining property

$$(L_a f)^\wedge(\xi) = \pi_\xi(a)\hat{f}(\xi) \qquad (\xi \in i\mathbb{R}^{n*}, a \in \mathbb{R}^n)\,.$$

The Plancherel theorem asserts that there exists a (unique) normalization $d\xi$ of Lebesgue measure on $i(\mathbb{R}^n)^* \simeq \widehat{G}$ such that $f \mapsto \hat{f}$ extends to an isometry

$$L^2(\mathbb{R}^n, dx) \simeq L^2(i\mathbb{R}^{n*}, d\xi)\,.$$

In particular, the inverse of the Fourier transform is given by its adjoint $\mathcal{J}$. In view of the fact that the Fourier transform is a continuous linear map from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to the Schwartz space $\mathcal{S}(i\mathbb{R}^{n*})$ it readily follows that

$$\mathcal{J}\varphi(x) = \int_{i(\mathbb{R}^n)^*} \varphi(\xi)\, e^{\xi(x)}\, d\xi\,, \tag{2.13}$$

for $\varphi \in \mathcal{S}(i(\mathbb{R}^n)^*)$. The identity $\mathcal{J} \circ \mathfrak{F} = I$ combined with (2.13) leads to the inversion formula

$$f(x) = \int_{i(\mathbb{R}^n)^*} \hat{f}(\xi)\, e^{\xi(x)}\, d\xi\,, \quad (x \in \mathbb{R}^n)\,,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. It exhibits each Schwartz function $f$ as a superposition of the functions $f_\xi : x \mapsto \hat{f}(\xi)e^\xi$, for $\xi \in i\mathbb{R}^{n*}$. However, none of the components $f_\xi$ is contained in $L^2(\mathbb{R}^n)$. For each $\xi$, let $\mathcal{H}_\xi$ be the one-dimensional linear span of the function $e^\xi$ in $C^\infty(\mathbb{R}^n)$. Then $\mathcal{H}_\xi$ is an invariant subspace for the left regular representation of $G$ in $C^\infty(\mathbb{R}^n)$. The restriction of the left regular representation to $\mathcal{H}_\xi$ is equivalent to $\pi_\xi$. Thus, the Plancherel decomposition for the Euclidean Fourier transform yields a decomposition of $L$ into irreducible unitary representations that may be realized on invariant subspaces $\mathcal{H}_\xi$ of $C^\infty(\mathbb{R}^N)$.

For a general reductive symmetric space $\mathrm{X} = G/H$ of the noncompact type there exist analogues of the components $f_\xi \in \mathcal{H}_\xi$ mentioned above, with $\xi$ ranging over the irreducible unitary representations of $G$. However, due to the fact that these representations generally are infinite dimensional, we shall only require that the so-called subspaces of smooth vectors of $\mathcal{H}_\xi$ are realized as invariant subspaces of $C^\infty(\mathrm{X})$.

**The abstract Plancherel theorem** Let $\mathrm{X} = G/H$ be a reductive symmetric space of the Harish-Chandra class and let $dx$ be a choice of invariant measure on $\mathrm{X} := G/H$. In this subsection we shall give a naive description of the 'abstract' Plancherel theorem.

We begin by observing that in the Riemannian case, with $H = K$ a maximal compact subgroup, the Plancherel decomposition for $G/K$ can be derived from the similar decomposition for $G$, since $L^2(G/K)$ may be identified with the space of right $K$-invariant functions in $L^2(G)$. Accordingly, the irreducible unitary representations entering the decomposition of $L^2(G/K)$ must possess a $K$-fixed vector. Thus, in this case the situation is similar to that of the compact symmetric spaces.

In the general situation, where $H$ is noncompact, such a simple relation between the Plancherel decompositions for $G$ and $G/H$ does not exist. Nevertheless, $H$-fixed vectors do play an important role. They do not exist as vectors in Hilbert space, but rather as *distribution,* or *generalized vectors.*

Let $\pi$ be a continuous representation of $G$ in a Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is called *smooth* if the map $G \to \mathcal{H}$, $x \mapsto \pi(x)v$ is $C^\infty$. The space of smooth vectors is denoted by $\mathcal{H}^\infty$. It is a natural representation space for $G$ and $\mathfrak{g}$, hence for $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. The space $\mathcal{H}^\infty$ is equipped with a Fréchet topology by means of the seminorms

$$\|\cdot\|_U : v \mapsto \|Uv\|_\mathcal{H}\,, \quad (U \in U(\mathfrak{g}))\,.$$

The continuous linear dual of the conjugate Fréchet space $\overline{\mathcal{H}^\infty}$ is denoted by $\mathcal{H}^{-\infty}$. This space, called the space of *generalized vectors* of $\mathcal{H}$, is equipped with the strong dual topology. It naturally carries the structure of a $G$- and a $\mathfrak{g}$-module. Let $\langle\cdot\,,\,\cdot\rangle$ denote the inner product of $\mathcal{H}$. Then via the map $v \mapsto \langle v\,,\,\cdot\rangle|_{\mathcal{H}^\infty}$ we obtain a continuous linear embedding

$$\mathcal{H} \hookrightarrow \mathcal{H}^{-\infty},$$

via which we shall identify elements. If $\pi$ is unitary, this embedding is equivariant. Finally, for $v \in \mathcal{H}^\infty$ and $\xi \in \mathcal{H}^{-\infty}$ we agree to write

$$\langle \xi\,,\, v\rangle := \xi(v) \quad \text{and} \quad \langle v\,,\,\xi\rangle = \overline{\langle \xi\,,\,v\rangle}.$$

Then $\langle\cdot\,,\,\cdot\rangle : \mathcal{H}^{-\infty}\times\mathcal{H}^\infty \to \mathbb{C}$ is a continuous sesquilinear pairing which is antilinear in the second variable.

Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. We denote by $(\mathcal{H}^{-\infty})^H$ the space of $H$-fixed generalized vectors for $\pi$. Given such a vector $\eta$ we define the map $\mathbf{m}_\eta : \mathcal{H}^\infty \to C^\infty(G/H)$ by

$$\mathbf{m}_\eta(v)(x) = \langle v\,,\,\pi(x)\eta\rangle,$$

for $v \in \mathcal{H}^\infty$ and $x \in G/H$. The map $\mathbf{m}_\eta$ belongs to the space $\mathrm{Hom}_G(\mathcal{H}^\infty, C^\infty(\mathrm{X}))$ of $G$-*equivariant* continuous linear maps $\mathcal{H}^\infty \to C^\infty(\mathrm{X})$. If $\pi$ is irreducible and $\eta \neq 0$, then $\mathbf{m}_\eta$ is an embedding.

**Lemma 2.1** *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of $G$. Then the map $\eta \mapsto \mathbf{m}_\eta$ defines a linear isomorphism*

$$(\overline{\mathcal{H}^{-\infty}})^H \xrightarrow{\simeq} \mathrm{Hom}_G(\mathcal{H}^\infty, C^\infty(\mathrm{X})).$$

*Proof.* If $T \in \mathrm{Hom}_G(\mathcal{H}^\infty, C^\infty(\mathrm{X}))$, we define $\eta_T \in \mathcal{H}^{-\infty}$ by $\eta_T(v) = \overline{(Tv)(e)}$. Then by equivariance of $T$ it follows that, for $h \in H$,

$$(\pi(h)\eta_T)(v) = \eta_T(\pi(h)^{-1}v) = \overline{T(\pi(h)^{-1}v)(e)} = \overline{(Tv)(hH)} = \eta_T(v).$$

Hence, $\eta_T$ is $H$-invariant and we see that $T \mapsto \eta_T$ is a linear map from the space $\mathrm{Hom}_G(V^\infty, C^\infty(\mathrm{X}))$ to $(\overline{\mathcal{H}^{-\infty}})^H$. We will show that this map is a two-sided inverse for the map $\eta \mapsto \mathbf{m}_\eta$. If $T = \mathbf{m}_\eta$, one readily verifies that $\eta = \eta_T$. Conversely, let $\eta = \eta_T$. Then by equivariance of $T$ we find, for $v \in \mathcal{H}^\infty$ and $g \in G$, that

$$\mathbf{m}_\eta(v)(gH) = \langle v\,,\,\pi(g)\eta\rangle = \langle \pi(g)^{-1}v\,,\,\eta\rangle = T(\pi(g)^{-1}v)(e) = T(v)(gH),$$

whence $\mathbf{m}_\eta = T$. □

Let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations of $G$. For each $\pi \in \widehat{G}$ we assume $\mathcal{H}_\pi$ to be a Hilbert space in which $\pi$ is unitarily realized.

**Lemma 2.2** *Let $\pi \in \widehat{G}$. Then* $\dim_{\mathbb{C}}(\mathcal{H}_\pi^{-\infty})^H < \infty$.

*Proof.* See [3], Lemma 3.3. The idea is to select a nontrivial vector $v \in \mathcal{H}_\pi$ that behaves finitely under the action of a suitable maximal compact subgroup. The map $\eta \mapsto \mathbf{m}_\eta(v)$ maps the conjugate space of $(\mathcal{H}_\pi^{-\infty})^H$ injectively and linearly into a space of functions on X that satisfy a certain system of differential equations. The solution space of the system is seen to be finite-dimensional by a method that goes back to Harish-Chandra. □

In view of the two preceding lemmas, it is reasonable to define, for $\pi \in \widehat{G}$, a space of smooth functions on X by

$$C^\infty(\mathrm{X})_\pi := M_\pi(\mathcal{H}_\pi^\infty \otimes (\overline{\mathcal{H}_\pi^{-\infty}})^H), \tag{2.14}$$

where $M_\pi$ is the matrix coefficient map determined by

$$M_\pi(v \otimes \eta)(x) := \mathbf{m}_\eta(v)(x) = \langle v\,,\, \pi(x)\eta \rangle.$$

The space $C^\infty(\mathrm{X})_\pi$ is called the space of smooth functions of type $\pi$. It is the appropriate generalization of the space $L^2(\mathrm{X})_\delta$ in (2.8).

We can now describe our goal of obtaining a Plancherel decomposition for $G/H$. Let

$$\widehat{G}_H := \{\pi \in \widehat{G} \mid \overline{(\mathcal{H}_\pi^{-\infty})}^H \neq 0\}.$$

We wish to specify a locally compact Hausdorff topology on (a subset of) $\widehat{G}_H$, and a Radon measure $d\mu$ on (that part of) $\widehat{G}_H$ together with continuous $G$-equivariant linear operators $C_c^\infty(G/H) \to C^\infty(G/H)_\pi$, $f \mapsto f_\pi$, for $\pi \in \widehat{G}_H$, such that

$$f = \int_{\widehat{G}_H} f_\pi \, d\mu(\pi). \tag{2.15}$$

The integral should converge as an integral with values in the Fréchet space $C^\infty(\mathrm{X})$. It amounts to the decomposition part of the Plancherel theorem. To formulate the unitary nature of the decomposition, we define the Fourier transform $(\hat{f}(\pi) \mid \pi \in \widehat{G}_H)$ of $f$ by

$$\hat{f}(\pi) \in \mathcal{H}_\pi^\infty \otimes \overline{(\mathcal{H}_\pi^{-\infty})}^H, \qquad M_\pi(\hat{f}(\pi)) = f_\pi, \tag{2.16}$$

for $\pi \in \widehat{G}_H$. Since $M_\pi$ is a $G$-equivariant embedding, the map $f \mapsto \hat{f}(\pi)$ intertwines the $G$-representations $L$ and $\pi \otimes 1$.

In addition to (2.15) we now require that $f \mapsto \hat{f}$ be an isometry in the following sense. For each $\pi \in \widehat{G}_H$ there should exist a linear subspace $\mathcal{V}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$, equipped with a positive definite inner product, such that $\hat{f}(\pi) \in \mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi$ for all $f \in C_c^\infty(\mathrm{X})$ and $\pi \in \widehat{G}_H$, and such that

$$\|f\|^2_{L^2(\mathrm{X})} \;=\; \int_{\widehat{G}_H} \|\hat{f}(\pi)\|^2 \, d\mu(\pi). \tag{2.17}$$

Finally, the image of $C_c^\infty(\mathrm{X})$ under $f \mapsto \hat{f}$ should be a dense subspace $\mathfrak{H}_0$ of the Hilbert space $\mathfrak{H}$ consisting of all families $(T_\pi \in \mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi \mid \pi \in \widehat{G}_H)$ that are measurable in a suitable sense and satisfy $\int_{\widehat{G}_H} \|T_\pi\|^2 \, d\mu(\pi) < \infty$.

By (2.15) and (2.16), the inverse operator $\mathcal{J} : \mathfrak{H} \to L^2(\mathrm{X})$ is given by

$$\mathcal{J}T \;=\; \int_{\widehat{G}_H} M_\pi(T_\pi) \, d\mu(\pi)$$

for $T \in \mathfrak{H}_0$. Moreover, by unitarity of the Fourier transform it must be the adjoint of $f \mapsto \hat{f}$. Thus, for $f \in C_c^\infty(\mathrm{X})$ we should have

$$\int_{\widehat{G}_H} \langle f\,,\, M_\pi(T_\pi)\rangle_{L^2(\mathrm{X})} \, d\mu(\pi) \;=\; \int_{\widehat{G}_H} \langle \hat{f}(\pi)\,,\, T_\pi \rangle \, d\mu(\pi).$$

This leads to the insight that the Fourier transform should be given by

$$\langle \hat{f}(\pi)\,,\, T_\pi \rangle = \langle f\,,\, M_\pi(T_\pi)\rangle_{L^2(\mathrm{X})}\,,$$

for a given $\pi \in \widehat{G}_H$ and all $T_\pi \in \mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi$. If $T_\pi = v \otimes \eta$, then the right-hand side becomes

$$\int_{\mathrm{X}} f(x)\, \overline{\langle v\,,\, \pi(x)\eta\rangle}\; dx = \langle \pi(f)\eta\,,\, v\rangle\,,$$

where we have used the notation

$$\pi(f)\eta = \int_{G/H} f(x)\pi(x)\eta\; dx,$$

for $\eta \in (\mathcal{H}_\pi^{-\infty})^H$ and $f \in C_c^\infty(G/H)$, although strictly speaking this notation is in conflict with (2.3). Note that $\pi(f)\eta$ is a smooth vector in $\mathcal{H}_\pi$.

In view of the identification $\mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi \simeq \mathrm{Hom}\,(\mathcal{V}_\pi, \mathcal{H}_\pi)$, it follows from the above that the Fourier transform $\hat{f}(\pi)$ of a function $f \in C_c^\infty(\mathrm{X})$ is given by

$$\hat{f}(\pi) = \pi(f)|_{\mathcal{V}_\pi} = \int_{G/H} f(x)\pi(x)|_{\mathcal{V}_\pi}\; dx\,.$$

**The discrete series** An irreducible unitary representation $\pi$ of $G$ is said to belong to the *discrete series* of $\mathrm{X} = G/H$ if it can be realized on a closed subspace of $L^2(\mathrm{X})$, i.e., if

$$\mathrm{Hom}_{\,G}(\mathcal{H}_\pi, L^2(\mathrm{X})) \neq 0\,. \tag{2.18}$$

By equivariance, an element $T$ from the space on the left-hand side of the above inequality restricts to a continuous linear $G$-equivariant map from $\mathcal{H}_\pi^\infty$ to $L^2(\mathrm{X})^\infty$. By the local Sobolev inequalities it follows that the latter space is contained in $C^\infty(\mathrm{X})$. By density of $\mathcal{H}_\pi^\infty$ in $\mathcal{H}_\pi$ it thus follows that restriction to the space of smooth vectors induces an embedding from the space $\mathrm{Hom}_{\,G}(\mathcal{H}_\pi, L^2(\mathrm{X}))$ onto a subspace of $\mathrm{Hom}_{\,G}(\mathcal{H}_\pi^\infty, C^\infty(\mathrm{X}))$. Via the isomorphism of Lemma 2.1 the latter subspace corresponds to a subspace

$$(\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}} \subset (\mathcal{H}_\pi^{-\infty})^H\,. \tag{2.19}$$

The collection of (equivalence classes of) discrete series representations of X is denoted by $\mathrm{X}^\wedge_{\mathrm{ds}}$. It is at most countable, since $L^2(\mathrm{X})$ is separable.

It follows from these definitions that the restriction of the map $M_\pi$ to $\mathcal{H}_\pi^\infty \otimes \overline{(\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}}}$ has a unique extension to a continuous linear map $\mathcal{H}_\pi \otimes \overline{(\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}}} \to L^2(\mathrm{X})$. In accordance with (2.14), we define, for $\pi \in \mathrm{X}^\wedge_{\mathrm{ds}}$,

$$L^2(G/H)_\pi := M_\pi\left(\mathcal{H}_\pi \otimes \overline{(\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}}}\right)\,.$$

In view of Lemma 2.2 this space equals a finite direct sum of copies of $\pi$, hence is closed. Its elements are called the square integrable functions of discrete series type $\pi$. Alternatively, such a function can be characterized by the condition that its closed $G$-span in $L^2(\mathrm{X})$ is a finite direct sum of copies of $\pi$.

Let $P_\pi$ denote the orthogonal projection $L^2(G/H) \to L^2(G/H)_\pi$. If $\pi'$ is a second representation of the discrete series, not equivalent to $\pi$, then the the restriction of $P_\pi$ to $L^2(G/H)_{\pi'}$ is a continuous linear intertwining operator from a finite multiple of $\pi'$ to a finite multiple of $\pi$, hence must be zero. It follows that

$$\pi \not\sim \pi' \Rightarrow L^2(G/H)_\pi \perp L^2(G/H)_{\pi'}.$$

The discrete part of $L^2(G/H)$ is defined to be the closed $G$-invariant subspace

$$L^2_{\mathrm{d}}(G/H) := \mathrm{cl}\left(\oplus_{\pi \in \mathrm{X}^\wedge_{\mathrm{ds}}} \; L^2(G/H)_\pi\right). \tag{2.20}$$

In the complementary part $L^2_{\mathrm{d}}(G/H)^\perp$, the discrete series will occur with $d\mu$-measure 0 so we may take

$$\mathcal{V}_\pi := (\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}}\,. \tag{2.21}$$

The map

$$M_\pi : \mathcal{H}_\pi \otimes \overline{\mathcal{V}}_\pi \to L^2(G/H)_\pi, \tag{2.22}$$

is a continuous linear bijection, intertwining the representations $\pi \otimes 1$ and $L|_{L^2(G/H)_\pi}$. It is readily seen that $\mathcal{V}_\pi$ carries a unique finite-dimensional Hilbert structure such that $M_\pi$ is an isometry.

**Invariant differential operators** In the process of finding the Plancherel formula, the interaction with invariant differential operators on $\mathrm{X} = G/H$ will play an essential role.

**Definition 2.3** An *invariant differential operator* on X is a linear partial differential operator $D$ with $C^\infty$-coefficients that commutes with the left action of $G$ on $C^\infty(\mathrm{X})$, i.e.,

$$L_g Df = DL_g f,$$

for all $f \in C^\infty(\mathrm{X})$ and $g \in G$. The algebra of these operators is denoted by $\mathbb{D}(G/H)$ or $\mathbb{D}(\mathrm{X})$.

If $D \in \mathbb{D}(\mathrm{X})$, we define its formal adjoint to be the operator $D^* \in \mathbb{D}(\mathrm{X})$ given by the formula

$$\int_{\mathrm{X}} D^* f(x)\overline{g(x)}\, dx = \int_{\mathrm{X}} f(x)\overline{Dg(x)}\, dx\,, \tag{2.23}$$

for $f, g \in C_c^\infty(\mathrm{X})$. Moreover, the conjugate of $D$ is defined by the formula $\bar{D}f = \overline{D\bar{f}}$ and the transpose by $D^t = \bar{D}^*$.

An operator $D \in \mathbb{D}(\mathrm{X})$ with $D = D^*$ is called formally selfadjoint. The following result is due to [3].

**Theorem 2.4** *Let $D \in \mathbb{D}(\mathrm{X})$ be formally selfadjoint. Then $D$, viewed as an operator in $L^2(\mathrm{X})$ with domain $C_c^\infty(\mathrm{X})$, is essentially selfadjoint, i.e., it has a symmetric closure.*

It follows from the above theorem that every formally selfadjoint operator $D \in \mathbb{D}(\mathrm{X})$ allows a spectral decomposition that commutes with the unitary action of $G$ on $L^2(\mathrm{X})$. Let $U_D$ be the unitary group with infinitesimal generator $iD$; then $G$ and $U_D$ commute. Applying the general representation theory of locally compact groups to $G \times U_D$ one can show that there must be a disintegration of $L$ over which the action of $D$ diagonalizes.

Let us see what this means in terms of the decomposition (2.15). If $u \in U(\mathfrak{g})^H$, then $R_u : C^\infty(G) \to C^\infty(G)$ leaves the subspace $C^\infty(G)^H$ of right-$H$-invariant functions invariant. Via the identification $C^\infty(G)^H \simeq C^\infty(\mathrm{X})$, we may view $R_u$ as a smooth differential operator on X which obviously commutes with the $G$-action. Hence $u \mapsto R_u$ defines an algebra homomorphism $U(\mathfrak{g})^H \to \mathbb{D}(\mathrm{X})$.

**Lemma 2.5** *The map $u \mapsto R_u$, $U(\mathfrak{g})^H \to \mathbb{D}(\mathrm{X})$ is a surjective homomorphism of algebras. Its kernel equals $U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$.*

*Proof.* See [82], Prop. 4.1. □

We denote the induced isomorphism by

$$r : U(\mathfrak{g})^H / U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h} \xrightarrow{\simeq} \mathbb{D}(\mathrm{X})\,. \tag{2.24}$$

In the next section we will use this isomorphism to show that $\mathbb{D}(\mathrm{X})$ is a polynomial algebra, as in the Riemannian case. In particular, $\mathbb{D}(\mathrm{X})$ is commutative.

Let $\pi$ be a unitary representation of $G$. Then the action of $U(\mathfrak{g})$ on $\mathcal{H}_\pi^\infty$ naturally extends to an action on $\mathcal{H}_\pi^{-\infty}$. Moreover, $U(\mathfrak{g})^H$ preserves the subspace $(\mathcal{H}_\pi^{-\infty})^H$. This induces the structure of a $U(\mathfrak{g})^H/U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$-module on $(\mathcal{H}_\pi^{-\infty})^H$. Via the isomorphism $r$ we may thus view $(\mathcal{H}_\pi^{-\infty})^H$ as a $\mathbb{D}(\mathrm{X})$-module. Finally, the conjugate space $\overline{(\mathcal{H}_\pi^{-\infty})^H}$ is a $\mathbb{D}(\mathrm{X})$-module for the multiplication map $(D, \eta) \mapsto \bar{D}\eta$.

The following result follows from the definitions given.

**Lemma 2.6** *Let $\pi \in \widehat{G}_H$. Then, for all $D \in \mathbb{D}(\mathrm{X})$,*

$$D \circ M_\pi = M_\pi \circ (I \otimes \bar{D}). \tag{2.25}$$

By the discussion leading up to (2.16), we expect the subspaces $\mathcal{V}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$ to be $\mathbb{D}(\mathrm{X})$-invariant. Moreover, by Theorem 2.4 and commutativity of $\mathbb{D}(\mathrm{X})$, we expect the action of $\mathbb{D}(\mathrm{X})$ on $\mathcal{V}_\pi$ to allow a simultaneous diagonalization.

**Example 2.7** Let $X = {}^{\backprime}G$ with ${}^{\backprime}G$ a Lie group, viewed as a symmetric space for the left times right action of $G = {}^{\backprime}G \times {}^{\backprime}G$. Then $\mathbb{D}(\mathrm{X})$ equals the algebra of bi-invariant differential operators on ${}^{\backprime}G$. Using the canonical identification of $U({}^{\backprime}\mathfrak{g})$ with the left-invariant differential operators on ${}^{\backprime}G$, we see that $\mathbb{D}(\mathrm{X}) \simeq U(\mathfrak{g})^{{}^{\backprime}G}$. If $G$ is a real reductive group of the Harish-Chandra class, then the latter algebra equals the center of the universal algebra. We recall that in this setting $\widehat{G}_H$ consists of the representations of the form $\pi \otimes \pi^*$, with $\pi \in {}^{\backprime}\widehat{G}$. These representations are naturally realized in $E_\pi = \mathrm{End}\,(\mathcal{H}_\pi)_{\mathrm{HS}}$. Moreover, $(E_\pi^{-\infty})^H = \mathbb{C} I_{\mathcal{H}_\pi}$ and the action of $\mathbb{D}(\mathrm{X})$ on this space is given by the infinitesimal character of $\pi$.

For a representation $\pi$ from the discrete series of X it can be shown a priori that the algebra $\mathbb{D}(\mathrm{X})$ has a simultaneous diagonalization on the subspace $\mathcal{V}_\pi$ given by (2.21). In fact, from Theorem 2.4 it can be deduced that each formally selfadjoint operator from $\mathbb{D}(\mathrm{X})$ leaves the space $L^2(\mathrm{X})_\pi$ invariant and admits a simultaeous diagonalization on it; see [3] for details. Since $\mathbb{D}(\mathrm{X})$ is a commutative algebra, spanned by its formally selfadjoint operators, it follows that $\mathbb{D}(\mathrm{X})$ leaves $L^2(\mathrm{X})_\pi$ invariant and admits a simultaneous diagonalization on it. Using (2.22) and (2.25) we can now deduce that $\mathbb{D}(\mathrm{X})$ leaves the subspace $\mathcal{V}_\pi := (\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}}$ of $(\mathcal{H}_\pi^{-\infty})^H$ invariant. Moreover, the following result is valid.

**Lemma 2.8** *Let $\pi$ be a representation from the discrete series of* X. *Then the action of* $\mathbb{D}(\mathrm{X})$ *on* $\mathcal{V}_\pi$ *admits a simultaneous diagonalization.*

# 3 Basic structure theory

**A suitable Cartan involution** From now on we will always assume that $G$ is a real reductive group of the Harish-Chandra class, see Section 15. Moreover, we assume that $\sigma$ is an involution of $G$ and that $H$ is an open subgroup of $G^\sigma$; thus,

$$(G^\sigma)_e < H < G^\sigma. \tag{3.1}$$

**Lemma 3.1** *There exists a Cartan involution $\theta$ of $G$ that commutes with $\sigma$, i.e.,*

$$\sigma \circ \theta = \theta \circ \sigma. \tag{3.2}$$

*Proof.* For $G$ connected semisimple this result can be found in M. Berger's paper [24], where also the classification of all semisimple symmetric spaces is obtained. We refer to [83], Prop. 7.1.1, for details. For $G$ of the Harish-Chandra class one may proceed as follows. We refer to the appendix for unspecified notation. Being an involution, $\sigma$ leaves the semisimple part $\mathfrak{g}_1$ and the center $\mathfrak{c}$ of $\mathfrak{g}$ invariant. On the level of the group, $\sigma$ preserves the maximal compact subgroup $T$ of $C_e$. Hence $\sigma$ preserves $\mathfrak{t}$ and since $\sigma^2 = I$ we may select a $\sigma$-invariant complementary subspace $\mathfrak{v}$ of $\mathfrak{t}$ in $\mathfrak{c}$. It follows that $\sigma(V) = V$. By the result for the semisimple case, $G_1$ has a Cartan involution $\theta_1$ commuting with $\sigma|_{G_1}$. We may extend $\theta_1$ to a Cartan involution $\theta$ of $G$ in the manner explained in the appendix. It is readily verified that $\theta$ commutes with $\sigma$. $\square$

From now on we assume $\theta$ to be as in (3.2). Then the associated maximal compact subgroup $K = G^\theta$ of $G$ is $\sigma$-stable. The involution $\theta$ determines the Cartan decomposition

$$G = K \exp \mathfrak{p}\,. \tag{3.3}$$

Here $\mathfrak{p}$ is the $-1$ eigenspace of $\theta$ in $\mathfrak{g}$ and the map $(k, \mathrm{X}) \mapsto k \exp \mathrm{X}$ is an analytic diffeomorphism from $K \times \mathfrak{p}$ onto $G$. By (3.2), both $K$ and $\mathfrak{p}$ are invariant under $\sigma$, hence from the uniqueness of the Cartan decomposition it follows that

$$G^\sigma = (K \cap G^\sigma) \exp(\mathfrak{p} \cap \mathfrak{g}^\sigma)\,.$$

This in turn implies that $(G^\sigma)_e = (K \cap G^\sigma)_e \exp(\mathfrak{p} \cap \mathfrak{g}^\sigma)$. By looking at tangent spaces, we see that $(K \cap (G^\sigma)_e) \exp(\mathfrak{p} \cap \mathfrak{g}^\sigma)$ is an open subgroup of $G^\sigma$. Hence, $(K \cap G^\sigma)_e = K \cap (G^\sigma)_e$. In view of (3.1) we may now conclude that

$$H = (H \cap K) \exp(\mathfrak{h} \cap \mathfrak{p}) . \tag{3.4}$$

In particular, it follows that $H$ is $\theta$-stable.

Since $\sigma$ and $\theta$ commute, it follows that $\mathfrak{g}$ admits the following joint eigenspace decomposition for $\sigma$ and $\theta$:

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{q} \;\oplus\; \mathfrak{k} \cap \mathfrak{h} \;\oplus\; \mathfrak{p} \cap \mathfrak{q} \;\oplus\; \mathfrak{p} \cap \mathfrak{h} . \tag{3.5}$$

From the fact that $\sigma$ and $\theta$ commute, it also follows that the composition $\sigma\theta$ is an involution of $G$, which commutes with $\theta$. Let

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \tag{3.6}$$

be the associated decomposition of $\mathfrak{g}$ in $+1$ and a $-1$ eigenspaces, respectively. One readily sees that $\mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{g}_- = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}$. Since $K \cap H$ normalizes $\mathfrak{p} \cap \mathfrak{q}$ it follows by application of the Cartan decomposition that

$$G_+ := (K \cap H) \exp(\mathfrak{p} \cap \mathfrak{q}), \tag{3.7}$$

is an open subgroup of $G^{\sigma\theta}$, hence a reductive group with the Cartan decomposition given by (3.7).

**Lemma 3.2** *The map $(k, \mathrm{X}, Y) \mapsto k \exp \mathrm{X} \exp Y$ is a diffeomorphism from $K \times (\mathfrak{p} \cap \mathfrak{q}) \times (\mathfrak{p} \cap \mathfrak{h})$ onto $G$. Accordingly,*

$$G = K \exp(\mathfrak{p} \cap \mathfrak{q}) \exp(\mathfrak{p} \cap \mathfrak{h}).$$

*Proof.* This result is due to G. Mostow, [73]. For details we refer the reader to [44], Thm. 4.1, [83], Prop. 7.1.2, or [82], Prop. 2.2. □

For any given result for reductive symmetric spaces, it is good practice to check what it means for the Riemannian case, which arises for $\sigma = \theta$. In that case the above lemma gives the usual Cartan decomposition, since $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{h} = 0$.

The above lemma has the following immediate corollary.

**Corollary 3.3** *The map $(k, X) \mapsto k \exp X\, H$ is a submersion from $K \times (\mathfrak{p} \cap \mathfrak{q})$ onto $G/H$. It factors to a diffeomorphism*

$$K \times_{K \cap H} (\mathfrak{p} \cap \mathfrak{q}) \simeq G/H , \tag{3.8}$$

*exhibiting $G/H$ as a $K$-homogeneous vector bundle over $K/K \cap H$, with fiber $\mathfrak{p} \cap \mathfrak{q}$.*

**Example 3.4** (The real hyperbolic space) Here $\mathrm{X} = \mathrm{X}_{p,q} \simeq G/H$, with $G = \mathrm{SO}_e(p,q)$ and $H = \mathrm{SO}_e(p-1,q)$; see Example 1.3. The Cartan involution $\theta : A \mapsto (A^t)^{-1}$ commutes with $\sigma$. Thus, $K = \mathrm{SO}\,(p) \times \mathrm{SO}\,(q)$.

In the notation of Example 1.3, let $J : \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ be defined by $J(x', x'') = (-x', x'')$. Then the inner product $\beta$ is given in terms of the standard inner product of $\mathbb{R}^n$ by the formula $\beta(x,y) = (Jx, y)$. From this it follows that the Lie algebra $\mathfrak{so}(p,q)$ consists of the real $n \times n$ matrices $A$ satisfying $A^t = -JAJ$. Thus, $\mathfrak{so}(p,q)$ consists of matrices of the form

$$\begin{pmatrix} B & C^t \\ C & D \end{pmatrix},$$

with $B$ an antisymmetric real $p \times p$ matrix, $D$ an antisymmetric real $q \times q$ matrix, and with $C$ a real $p \times q$ matrix. Moreover, the involution $\sigma$ of $\mathfrak{so}(p,q)$ is given by $A \mapsto SAS$, and a commuting Cartan involution is given by $A \mapsto -A^t$. It follows that the decomposition (3.5) is indicated by the following scheme:

$$\begin{array}{cc} & \begin{array}{ccc} 1 & p-1 & q \end{array} \\ \begin{array}{c} 1 \\ p-1 \\ q \end{array} & \begin{pmatrix} 0 & \mathfrak{k}\cap\mathfrak{q} & \mathfrak{p}\cap\mathfrak{q} \\ \mathfrak{k}\cap\mathfrak{q} & \mathfrak{k}\cap\mathfrak{h} & \mathfrak{p}\cap\mathfrak{h} \\ \mathfrak{p}\cap\mathfrak{q} & \mathfrak{p}\cap\mathfrak{h} & \mathfrak{k}\cap\mathfrak{h} \end{pmatrix} \end{array},$$

which shows where the nonzero entries of the matrices in the mentioned intersections are located.

In the geometric realization of $\mathrm{X}_{p,q}$ the map (3.8) means the following:

$$K/K \cap H \simeq K \cdot e_1 = S^{p-1} \times \{0\},$$

where $S^{p-1}$ is the unit sphere in $\mathbb{R}^p$. The fiber of the vector bundle (3.8) over $e_1$ corresponds to $\exp(\mathfrak{p} \cap \mathfrak{q}) \cdot e_1$; it is given by the equations

$$x_2 = \cdots = x_p = 0, \quad x_1 = \sqrt{1 + x_{p+1}^2 + \cdots + x_n^2}\,.$$

The projection $p : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ restricts to a diffeomorphism of this fiber onto $\mathbb{R}^q$. The other fibers of the vector bundle are readily obtained by applying the action of $\mathrm{SO}\,(p) \times \{I\}$, since $K = \mathrm{SO}\,(p) \times \mathrm{SO}\,(q)$ and $\{I\} \times \mathrm{SO}\,(q)$ stabilizes the fiber over $e_1$.

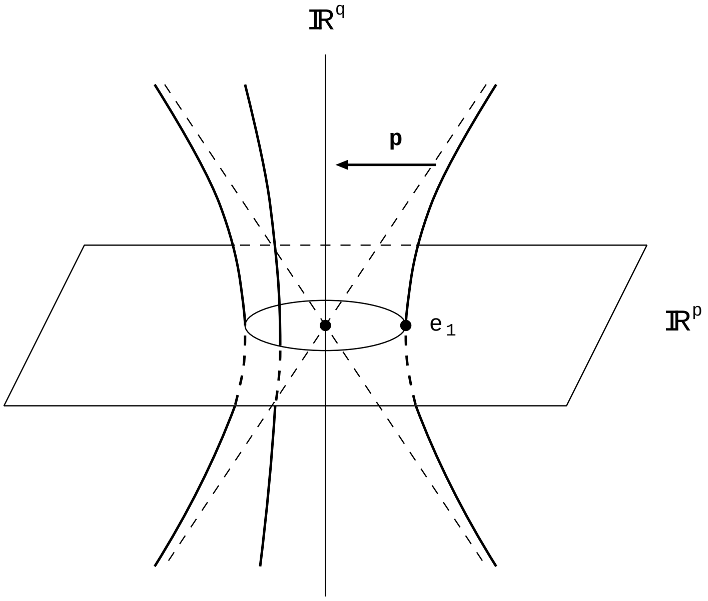


**Figure 3.** $X_{p,q}$ as an $\mathbb{R}^q$-bundle over $S^{p-1}$

**The polar decomposition** We fix a maximal abelian subspace $\mathfrak{a}_q$ of $\mathfrak{p} \cap \mathfrak{q}$. From (3.7) we see that any other choice of $\mathfrak{a}_q$ is conjugate to the present one by an element of $(K \cap H)_e$. The dimension of $\mathfrak{a}_q$ is called the *$\sigma$-split rank* of $G$, or the *split rank* of the symmetric space X. The following lemma specializes to a well-known result in the Riemannian case with $\sigma = \theta$, where $\mathfrak{a}_q$ is a maximal abelian subspace of $\mathfrak{p}$.

**Lemma 3.5** *The nonzero weights of $\mathfrak{a}_q$ in $\mathfrak{g}$ form a possibly nonreduced root system, denoted $\Sigma(\mathfrak{g}, \mathfrak{a}_q) = \Sigma$.*

*The natural map $N_K(\mathfrak{a}_q) \to GL(\mathfrak{a}_q)$ factors to an isomorphism from the quotient group $N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$ onto the reflection group $W$ of $\Sigma$.*

*Proof.* The assertion that $\Sigma$ is a root system is due to [80]. For the remaining assertions, details can be found in [5], Lemma 1.2. □

Note that $\Sigma$ need not span the dual of $\mathfrak{a}_q$, since $\mathfrak{g}$ may have a center. In fact, the intersection $\mathfrak{a}_{\Sigma q}$ of all root hyperplanes in $\mathfrak{a}_q$ is easily seen to be equal to center$(\mathfrak{g}) \cap \mathfrak{p} \cap \mathfrak{q}$.

We define the subgroup $W_{K\cap H}$ of $W$ to be the natural image of the subgroup $N_{K\cap H}(\mathfrak{a}_q)$ of $N_K(\mathfrak{a}_q)$.

From the Cartan decomposition (3.3) it follows that exp is a diffeomorphism from $\mathfrak{a}_q$ onto a closed abelian subgroup $A_q$ of $G$. Via this diffeomorphism $W$ acts on $A_q$. In the Riemannian case ($\sigma = \theta$) we have the $G = KA_qK$ decomposition, where the $A_q$-part is uniquely determined modulo $W$. The generalization of this result to the present context is as follows.

We define $A_q^{\mathrm{reg}} = \exp(\mathfrak{a}_q^{\mathrm{reg}})$, where $\mathfrak{a}_q^{\mathrm{reg}}$ is the complement of the union of the root hyperplanes $\ker \alpha$, $\alpha \in \Sigma$. Alternatively, $A_q^{\mathrm{reg}}$ is the subset of points in $A_q$ not fixed by any element from $W \setminus \{1\}$. The following result can be found in [44], Thm. 4.1.

**Lemma 3.6** (Polar decomposition). *The group $G$ decomposes as $G = KA_{\mathrm{q}}H$. If $x \in G$, then $x \in KaH$ for an element $a \in A_{\mathrm{q}}$ that is uniquely determined modulo $W_{K\cap H}$. Finally,*

$$\mathrm{X}_+ := KA_{\mathrm{q}}^{\mathrm{reg}}H\,,$$

*viewed as a subset of* X, *is open dense.*

*Proof.* We consider the reductive group $G_+$ defined by (3.7). Now $\mathfrak{a}_{\mathrm{q}}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$, and since (3.7) is a Cartan decomposition it follows that

$$G_+ = (K\cap H)A_{\mathrm{q}}(K\cap H)\,,$$

where the $A_{\mathrm{q}}$-part is unique modulo $W_{K\cap H}$. We finish the proof by combining this with the decomposition of Lemma 3.2 and (3.4). ☐

In the following we will always assume that $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathrm{q}})$ is a set of representatives for $W/W_{K\cap H}$. By this we mean that the natural map

$$\mathcal{W} \xrightarrow{\simeq} W/W_{K\cap H} \tag{3.9}$$

is a bijection.

**Corollary 3.7** *Let $A_{\mathrm{q}}^+$ be a chamber in $A_{\mathrm{q}}^{\mathrm{reg}}$. Then*

$$\mathrm{X}_+ = \cup_{v\in\mathcal{W}}\ KA_{\mathrm{q}}^+vH \qquad \text{(disjoint union)}. \tag{3.10}$$

*Moreover, if $x \in \mathrm{X}_+$, then $x \in KavH$ for uniquely determined $v \in \mathcal{W}$ and $a \in A_{\mathrm{q}}^+$.*

*Proof.* Since $\mathcal{W}(K\cap H)$ contains a full set of representatives for $W$, the equality (3.10) follows from the polar decomposition of Lemma 3.6.

To establish uniqueness, let $v_1, v_2 \in \mathcal{W}$ and assume that $Ka_1v_1H = Ka_2v_2H$ for $a_1, a_2 \in A_{\mathrm{q}}^+$. Then $Kv_1^{-1}a_1v_1H = Kv_2^{-1}a_2v_2H$, hence $v_1^{-1}a_1v_1$ and $v_2^{-1}a_2v_2$ are $W_{K\cap H}$-conjugate by Lemma 3.6. This implies that $v_1$ and $v_2$ determine the same coset in $W/W_{K\cap H}$, hence are equal. ☐

**Example 3.8** Let $\mathrm{X} = \mathrm{X}_{p,q}$ be a real hyperbolic space. Let

$$Y = \begin{pmatrix} 0 & \cdots & & 0 & 1 \\ 0 & & & & 0 \\ \vdots & & 0 & & \vdots \\ 0 & & & & \\ 1 & 0 & \cdots & & 0 \end{pmatrix}.$$

Then $\mathfrak{a}_{\mathrm{q}} = \mathbb{R}Y$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$. One readily checks that

$$a_t = \exp tY = \begin{pmatrix} \cosh t & 0 & \cdots & 0 & \sinh t \\ 0 & & & & 0 \\ \vdots & & \mathrm{I} & & \vdots \\ 0 & & & & 0 \\ \sinh t & 0 & \cdots & 0 & \cosh t \end{pmatrix}$$

from which it follows that

$$\begin{aligned} A_{\mathrm{q}}e_1 &= \{\cosh t\, e_1 + \sinh t\, e_n |\, t \in \mathbb{R}\} \\ &= \left\{x \in \mathbb{R}^n |\, x_i = 0\,,\ 1 < i < n\,,\ x_1^2 - x_n^2 = 1\right\}\,. \end{aligned}$$

It is now readily seen that $KA_{\mathrm{q}}e_1 = \mathrm{X}_{p,q}$, and we conclude that $G = \mathrm{SO}_e(p,q)$ acts transitively on $\mathrm{X}_{p,q}$. We also see that $G = KA_{\mathrm{q}}H$. It is now straightforward to verify the statements of Example 3.4.

In the present situation the root system is given by $\Sigma = \{\pm\,\alpha\}$, where $\alpha(Y) = 1$. Thus, $W = \{\pm\, I\}$. Moreover, there is a significant difference between the cases $q = 1$ and $q > 1$. If $q = 1$, then $W_{K\cap H} = \{I\}$, but if $q > 1$, then $W_{K\cap H} = W$. See [82], Example 2.2, for details. The difference is reflected by the fact that $\mathrm{X}_+ = \mathrm{X}_{p,q} \setminus (\mathbb{R}^p \times \{0\})$ consists of two connected components for $q = 1$ and of one connected component for $q > 1$.

The decomposition (3.10) gives rise to an integral decomposition for X. If $\alpha \in \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$, let $\mathfrak{g}_\alpha$ be the associated root space in $\mathfrak{g}$. Since $\sigma\circ\theta = I$ on $\mathfrak{a}_{\mathrm{q}}$, the involution $\sigma\circ\theta$ leaves each root space $\mathfrak{g}_\alpha$, for $\alpha \in \Sigma$, invariant. It follows that the root space decomposes compatibly with (3.6),

$$\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{g}_+) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{g}_-)\,.$$

Accordingly, $m_\alpha := \dim \mathfrak{g}_\alpha = m_\alpha^+ + m_\alpha^-$, where

$$m_\alpha^\pm = \dim(\mathfrak{g}_\alpha \cap \mathfrak{g}_\pm)\,.$$

We will also need the following notation. Recall that $\exp : \mathfrak{a}_{\mathrm{q}} \to A_{\mathrm{q}}$ is a diffeomorphism. We denote its inverse by log. If $\mu \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, we put

$$a^\mu = e^{\mu(\log a)} \qquad (a \in A_{\mathrm{q}})\,. \tag{3.11}$$

In other words, $(\exp X)^\mu = e^{\mu(X)}$, for $X \in \mathfrak{a}_{\mathrm{q}}$.

**Theorem 3.9** *Let $dx$ be a choice of invariant measure on X and let $dk$ be normalized Haar measure. There exists a unique choice of Haar measure $da$ on $A_{\mathrm{q}}$ such that, for $f \in L^1(\mathrm{X})$,*

$$\int_{\mathrm{X}} f(x)dx = \sum_{v\in\mathcal{W}} \int_K \int_{A_{\mathrm{q}}^+} f(kavH)J(a)\, da\, dk\,.$$

*Here $J(a) = \prod_{\alpha\in\Sigma^+}(a^\alpha - a^{-\alpha})^{m_\alpha^+}(a^\alpha + a^{-\alpha})^{m_\alpha^-}$ with $\Sigma^+$ the positive system determined by $\mathfrak{a}_{\mathrm{q}}^+$.*

The computation of the Jacobian is due to M. Flensted-Jensen, [45], Thm. 2.6., Eq. (2.14). Note that in the above formula, $A_{\mathrm{q}}^+$ is a chamber for $\Sigma$, whereas in the mentioned result of [45], the integration is reduced to a bigger chamber in $A_{\mathrm{q}}$ for the smaller root system $\Sigma(\mathfrak{g}_+, \mathfrak{a}_{\mathrm{q}})$. In particular, no summation over $\mathcal{W}$ is needed. The computation of the Jacobian can also be found in [83], proof of Thm. 8.1.1.

**Example 3.10** For the example $\mathrm{X}_{p,q}$ the above result is treated in detail in [82], Example 2.3.

# 4 Invariant differential operators

**A dual Riemannian space** In this section we will explain the structure of the algebra $\mathbb{D}(\mathrm{X})$ of invariant differential operators on X. The key idea is to relate this algebra to the algebra of invariant differential operators on a suitable dual Riemannian symmetric space. The structure of the latter algebra is well known.

If $\mathfrak{v}$ is a finite-dimensional real linear space, we agree to denote the symmetric algebra of its complexification by $S(\mathfrak{v})$. If $\mathfrak{l}$ is a Lie algebra, we denote by $U(\mathfrak{l})$ the universal enveloping algebra of its complexification.

Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ be as in (3.6). Then

$$\mathfrak{g}_+ = \mathfrak{k}\cap\mathfrak{h} \;\oplus\; \mathfrak{p}\cap\mathfrak{q} \qquad \text{and} \qquad \mathfrak{g}_- = \mathfrak{k}\cap\mathfrak{q} \;\oplus\; \mathfrak{p}\cap\mathfrak{h}\,.$$

One readily checks that $\mathfrak{g}_+$ is a subalgebra and that $[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-$, hence

$$\mathfrak{g}^d := \mathfrak{g}_+ \oplus i\mathfrak{g}_-$$

is a real form of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. A nice feature of this dual real form is that the roles of $\theta$ and $\sigma$ become interchanged. More precisely, let $\theta_\mathbb{C}$ and $\sigma_\mathbb{C}$ be the complex linear extensions of $\theta$ and $\sigma$ to $\mathfrak{g}_\mathbb{C}$, and define

$$\theta^d = \sigma_\mathbb{C}|_{\mathfrak{g}^d}\,, \quad \sigma^d = \theta_\mathbb{C}|_{\mathfrak{g}^d}\,.$$

Then $\ker(\theta^d - I) = (\mathfrak{k}\cap\mathfrak{h}) \oplus i(\mathfrak{p}\cap\mathfrak{h})$. It is well known that $\mathfrak{k}\oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_\mathbb{C}$. Hence, $\theta^d$ is a Cartan involution for the dual real form $\mathfrak{g}^d$, which explains the notation. Clearly, $\sigma^d$ is an involution of $\mathfrak{g}^d$ that commutes with $\theta^d$.

Fix a complex linear algebraic group $G_\mathbb{C}$ with algebra $\mathfrak{g}_\mathbb{C}$ and let $G^d$, $K^d$ be the analytic subgroups with Lie algebras $\mathfrak{g}^d$ and

$$\mathfrak{k}^d := \ker(\theta^d - I) = \mathfrak{h}_\mathbb{C} \cap \mathfrak{g}^d\,.$$

Let $H_\mathbb{C}$ be the analytic subgroup with algebra $\mathfrak{h}_\mathbb{C}$. Then $\mathrm{X}^d = G^d/K^d$ is a Riemannian real form of the complex symmetric space $G_\mathbb{C}/H_\mathbb{C}$.

**Example 4.1** Let $\mathrm{X} = \mathrm{X}_{p,q} = \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$. As a complexification of X we may take $\mathrm{SO}\,(n)_\mathbb{C}/\mathrm{SO}\,(n-1)_\mathbb{C}$, where $n = p+q$. Moreover, the dual Riemannian form becomes $\mathrm{X}^d = \mathrm{SO}_e(n-1,1)/\mathrm{SO}_e(n-2,1)$.

**Lemma 4.2** *There is a natural isomorphism*

$$\mathbb{D}(\mathrm{X}) \;\simeq\; \mathbb{D}(\mathrm{X}^d). \tag{4.1}$$

*Proof.* If $H$ is connected, the proof is straightforward, involving the isomorphism (2.24) for both X and $\mathrm{X}^d$,

$$\begin{aligned} \mathbb{D}(\mathrm{X}) &\simeq U(\mathfrak{g})^H/U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h} \\ &= U(\mathfrak{g})\mathfrak{h}/U(\mathfrak{g})^\mathfrak{h} \cap U(\mathfrak{g})\mathfrak{h} \\ &= U(\mathfrak{g}^d)^{\mathfrak{k}^d}/U(\mathfrak{g}^d)^{\mathfrak{k}^d} \cap U(\mathfrak{g}^d)\mathfrak{k}^d \simeq \mathbb{D}(\mathrm{X}^d)\,. \end{aligned}$$

If $H$ is not connected, the second identity is not completely obvious, but can be proved, using information on the structure of $H/H_e$. See [6], Lemma 2.1, for details. □

**Example 4.3** We consider the real hyperbolic space $\mathrm{X} = \mathrm{X}_{p,q}$. In Example 4.1 we saw that the dual space $\mathrm{X}^d$ equals the Riemannian hyperbolic space $\mathrm{SO}_e(n-1,1)/\mathrm{SO}_e(n-2,1)$, where $n = p+q$. Now $\mathbb{D}(\mathrm{X}^d)$ is the polynomial algebra generated by the Laplace-Beltrami operator $\Box$ on $\mathrm{X}^d$. Under the isomorphism (4.1), $\Box$ corresponds to the pseudo-Laplacian $\Box_{p,q}$ on $\mathrm{X}_{p,q}$. Consequently, $\mathbb{D}(\mathrm{X})$ is the polynomial algebra generated by $\Box_{p,q}$.

From the theory of Riemannian symmetric spaces, we recall the existence of a canonical isomorphism

$$\gamma^d : \mathbb{D}(\mathrm{X}^d) \xrightarrow{\simeq} I(\mathfrak{a}_{\mathfrak{p}}^d),$$

where $\mathfrak{a}_{\mathfrak{p}}^d$ is maximal abelian in $\mathfrak{p}$, and where $I(\mathfrak{a}_{\mathfrak{p}}^d)$ is the collection of invariants in $S(\mathfrak{a}_{\mathfrak{p}}^d)$ for the action of the reflection group $W(\mathfrak{g}^d, \mathfrak{a}_{\mathfrak{p}}^d)$ of the root system of $\mathfrak{a}_{\mathfrak{p}}^d$ in $\mathfrak{g}^d$. In particular, it follows that $\mathbb{D}(\mathrm{X}^d)$ is a polynomial algebra of rank $\dim \mathfrak{a}_{\mathfrak{p}}^d$. Combined with Lemma 4.2, this leads to the following.

**Corollary 4.4** *$\mathbb{D}(\mathrm{X})$ is a polynomial algebra of rank $\dim \mathfrak{a}_{\mathfrak{p}}^d$. In particular it is commutative.*

**Corollary 4.5** *The characters of the algebra $\mathbb{D}(\mathrm{X})$ are of the form $\chi_\lambda : D \mapsto \gamma(D, \lambda)$, with $\lambda \in \mathfrak{a}_{\mathfrak{p}\mathbb{C}}^{d*}$. Two characters $\chi_\lambda$ and $\chi_\mu$ are equal if and only if $\lambda$ and $\mu$ are conjugate under $W(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}}^d)$.*

**Cartan subspaces** We shall now discuss, for the symmetric pair $(\mathfrak{g}, \mathfrak{h})$, the analogue of the notion of a Cartan subalgebra for a real reductive Lie algebra.

**Definition 4.6** By a *Cartan subspace* of $\mathfrak{q}$ we mean a subspace $\mathfrak{b} \subset \mathfrak{q}$ that is maximal subject to the following two conditions:

(a) $\mathfrak{b}$ is abelian;

(b) $\mathfrak{b}$ consists of semisimple elements.

By using the method of complexification of the previous subsection, it can be shown that $\dim \mathfrak{b}$ is independent of $\mathfrak{b}$, though in general there are several, but finitely many, $H$-conjugacy classes of Cartan subspaces. The number $\dim \mathfrak{b}$ is called *the rank of* X. It can be shown that every Cartan subspace is $H_e$-conjugate to one that is $\theta$-stable, i.e., invariant under the involution $\theta$.

**Example 4.7** In the case of the group, see Example 1.2,

$$\mathfrak{q} = \{(X, -X) \mid X \in {}^{\backprime}\mathfrak{g}\}.$$

For each Cartan subalgebra ${}^{\backprime}\mathfrak{j} \subset {}^{\backprime}\mathfrak{g}$, the space $\mathfrak{b}_{{}^{\backprime}\mathfrak{j}} := \{(X, -X) \mid X \in {}^{\backprime}\mathfrak{j}\}$ is a Cartan subspace of $\mathfrak{q}$. Moreover, the map ${}^{\backprime}\mathfrak{j} \mapsto \mathfrak{b}_{{}^{\backprime}\mathfrak{j}}$ establishes a bijection between the collection of all Cartan subalgebras of ${}^{\backprime}\mathfrak{g}$ onto the collection of Cartan subspaces of $\mathfrak{q}$; it induces a bijection from the finite set of ${}^{\backprime}G$-conjugacy classes of Cartan subalgebras of ${}^{\backprime}\mathfrak{g}$ onto the set of $H$-conjugacy classes of Cartan subspaces of $\mathfrak{q}$.

A particular Cartan subspace of $\mathfrak{q}$ is obtained as follows. Let

$$\mathfrak{a}_{\mathrm{q}} \subset \mathfrak{p} \cap \mathfrak{q} \tag{4.2}$$

be a maximal abelian subspace. Being a subset of $\mathfrak{p}$, the space $\mathfrak{a}_{\mathrm{q}}$ consists of semisimple elements. Let $\mathfrak{m}_1$ be the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$; then $\mathfrak{m}_1 \cap \mathfrak{q} \subset (\mathfrak{m}_1 \cap \mathfrak{k} \cap \mathfrak{q}) \oplus \mathfrak{a}_{\mathrm{q}}$. Let $\mathfrak{t} \subset \mathfrak{m}_1 \cap \mathfrak{k} \cap \mathfrak{q}$ be maximal abelian. Then $\mathfrak{t}$ consists of semisimple elements, hence so does

$$\mathfrak{b} := \mathfrak{t} \oplus \mathfrak{a}_{\mathrm{q}}.$$

Clearly, $\mathfrak{b}$ is a $\theta$-stable Cartan subspace of $\mathfrak{q}$. We call it *maximally split*, since $\mathfrak{g}$ splits maximally for the action of $\mathfrak{b}$ by $\mathrm{ad}$. Note that the number $\dim \mathfrak{a}_{\mathrm{q}}$ (the rank of the Riemannian pair $(\mathfrak{g}_+, \mathfrak{p} \cap \mathfrak{q})$) is independent of the particular choice of $\mathfrak{a}_{\mathrm{q}}$. This number is called the *split rank of* X or the $\sigma$-split rank of $G$. It can be shown that every maximally split $\theta$-stable subspace of $\mathfrak{q}$ is $K \cap H_e$-conjugate to $\mathfrak{b}$.

To the $\theta$-stable Cartan subspace $\mathfrak{b}$ we associate $\mathfrak{a}_{\mathfrak{p}}^d := \mathfrak{b} \cap \mathfrak{p} \oplus i(\mathfrak{b} \cap \mathfrak{k})$ which is maximal abelian in $\mathfrak{p}^d$. Let $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b})$ be the root system of $\mathfrak{b}$ in $\mathfrak{g}_{\mathbb{C}}$, $W(\mathfrak{b})$ the associated Weyl group and $I(\mathfrak{b})$ the associated collection of $W(\mathfrak{b})$-invariants in $S(\mathfrak{b})$. Then obviously $I(\mathfrak{b}) = I(\mathfrak{a}_{\mathfrak{p}}^d)$. Let $\gamma : \mathbb{D}(\mathrm{X}) \to I(\mathfrak{b})$ be the map which makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{D}(\mathrm{X}) & \xrightarrow{\gamma} & I(\mathfrak{b}) \\ \simeq \downarrow & \circlearrowright & \downarrow = \\ \mathbb{D}(\mathrm{X}^d) & \xrightarrow{\gamma^d} & I(\mathfrak{a}_{\mathfrak{p}}^d)\ . \end{array}$$

The vertical isomorphism on the left side of the diagram is the natural isomorphism indicated in the proof of Lemma 4.2. Since $\gamma^d$ is an isomorphism of algebras, it follows that $\gamma$ is an isomorphism as well. The latter is called the *Harish-Chandra isomorphism* for $\mathbb{D}(\mathrm{X})$ and $\mathfrak{b}$. The well-known description of $\gamma^d$ in terms of the universal enveloping algebra, see, e.g., [63], Ch. II, Thm. 5.17, leads to the following similar description of $\gamma$. The reader may keep in mind that in the Riemannian case, which arises for $\sigma = \theta$, the algebra $\mathfrak{g}$ coincides with its dual form $\mathfrak{g}^d$ and, accordingly, the description of $\gamma$ given below coincides with that of $\gamma^d$.

Let $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b})$ be a choice of positive roots, and let $\mathfrak{g}_{\mathbb{C}}^+$ be the associated sum of positive root spaces. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^+$ complexifies the Iwasawa decomposition $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{a}_{\mathfrak{p}}^d \oplus (\mathfrak{g}_{\mathbb{C}}^+ \cap \mathfrak{g}^d)$ for $\mathfrak{g}^d$. By application of the Poincaré–Birkhoff–Witt (or PBW) theorem, we see that the decomposition induces the following decomposition of the universal enveloping algebra:

$$U(\mathfrak{g}) = [\, \mathfrak{g}_{\mathbb{C}}^+ U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{h}_{\mathbb{C}} \,] \oplus U(\mathfrak{b}).$$

Let $D \in \mathbb{D}(\mathrm{X})$. Then $D = R_u$ for a $u \in U(\mathfrak{g})^H$. There is a unique $u_0 \in U(\mathfrak{b})$, only depending on $u$ through its image $D$, such that

$$u - u_0 \in \mathfrak{g}_{\mathbb{C}}^+ U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{h}_{\mathbb{C}}\,.$$

The element $\gamma(D) \in I(\mathfrak{b})$ is now given by $\gamma(D) = T_{\rho_{\mathfrak{b}}} u_0$, where $\rho_{\mathfrak{b}} = \frac{1}{2} \mathrm{tr}_{\mathbb{C}}\, \mathrm{ad}\,(\cdot)|\mathfrak{g}_{\mathbb{C}}^+$ and where $T_{\rho_{\mathfrak{b}}}$ denotes the automorphism of $S(\mathfrak{b})$ induced by the map $x \mapsto x + \rho_{\mathfrak{b}}(x)$, $\mathfrak{b} \to S(\mathfrak{b})$.

## 5 The discrete series

**Flensted-Jensen's duality** The idea of passing to the dual Riemannian form $\mathrm{X}^d$ plays an important role in the theory of the discrete series of X. We shall restrict ourselves to giving a short account of some of the main ideas involved. For simplicity of the exposition we make the mild assumption in this section that $G$ is the analytic subgroup with Lie algebra $\mathfrak{g}$ of a complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $G^d, K^d$ and $H^d$ be the analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}^d, \mathfrak{k}^d$ and $\mathfrak{h}^d$, respectively. We put $X^d = G^d/K^d$ and agree to write $C^\infty(\mathrm{X}^d)_{H^d}$ for the space of smooth $H^d$-finite functions on $\mathrm{X}^d$. The following result, due to Flensted-Jensen [45], establishes an important duality between functions on X and on $\mathrm{X}^d$. We observe that $A_{\mathrm{q}}$ naturally embeds into each of the spaces X and $\mathrm{X}^d$. Moreover, by Lemma 3.6,

$$\mathrm{X} = KA_{\mathrm{q}} \quad \text{and} \quad \mathrm{X}^d = H^d A_{\mathrm{q}}. \tag{5.1}$$

The isomorphism $\mathbb{D}(\mathrm{X}) \to \mathbb{D}(\mathrm{X}^d)$ of Lemma 4.2 is denoted by $D \mapsto D^d$.

**Theorem 5.1** *There exists a unique linear map $f \mapsto f^d$ from $C^\infty(\mathrm{X})_K$ to $C^\infty(\mathrm{X}^d)_{H^d}$ with the property that, for all $f$, and all $X \in U(\mathfrak{k}) = U(\mathfrak{h}^d)$,*

$$L_X f|_{A_{\mathrm{q}}} = L_X f^d|_{A_{\mathrm{q}}},$$

*where $L$ denotes the infinitesimal left regular representation.*

*The map $f \mapsto f^d$ is injective. Moreover, for every $f \in C^\infty(\mathrm{X})_K$,*

$$(Df)^d = D^d f^d, \qquad (D \in \mathbb{D}(\mathrm{X})).$$

*Proof.* It suffices to prove the result for functions with a fixed $K$-type. One then combines the decompositions (5.1) with the fact that each finit-dimensional representation of $K$ extends to a holomorphic representation of the analytic group $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_{\mathbb{C}}$, which has $K$ as a compact real form, and $H^d$ as another real form. □

We now turn to the application of the above idea to the study of the collection $\mathrm{X}^\wedge_{\mathrm{ds}}$ of discrete series representations for X. Given a continuous representation of $G$ in a locally convex space $V$, we denote by $V_K$ the set of $K$-finite vectors in $V$.

Let $\pi \in \mathrm{X}^\wedge_{\mathrm{ds}}$. Then it follows from Lemma 2.8, combined with the fact that the map $M_\pi$ in (2.22) is bijective, that every function in the space $L^2(\mathrm{X})_{\pi,K}$ decomposes inside the mentioned space as a finite sum of simultaneous eigenfunctions for $\mathbb{D}(\mathrm{X})$. Thus, a first step towards the classification of the discrete series is the determination of all eigenfunctions in $C^\infty(\mathrm{X}) \cap L^2(\mathrm{X})_K$ for $\mathbb{D}(\mathrm{X})$. It can be shown that such functions automatically belong to $L^2_{\mathrm{d}}(\mathrm{X})$.

Let a function $f$ of the mentioned type be given. Thus, $f \in C^\infty(\mathrm{X}) \cap L^2(\mathrm{X})_K$ and in the notation of Section 4, there exists a $\Lambda \in \mathfrak{b}^*_{\mathbb{C}}$ such that

$$Df = \gamma(D, \Lambda)\, f \qquad (D \in \mathbb{D}(\mathrm{X})). \tag{5.2}$$

Applying Theorem 5.1 we see that the associated dual function $f^d \in C^\infty \mathrm{X}^d)_{H^d}$ satisfies the system of differential equations

$$Df^d = \gamma^d(D, \Lambda)\, f^d \qquad (D \in \mathbb{D}(\mathrm{X}^d)). \tag{5.3}$$

Moreover, the $L^2$-behavior of $f$ can be formulated in terms of growth conditions of $f^d$ at infinity. Attached to the system (5.3) is a certain Poisson transform $\mathcal{P}^d$, which we shall briefly describe. Let $G^d = K^d A^d_{\mathfrak{p}} N^d$ be an Iwasawa decomposition, let $M^d$ be the centralizer of $A^d_{\mathfrak{p}}$ in $K^d$ and let $P^d = M^d A^d_{\mathfrak{p}} N^d$ be the corresponding minimal parabolic subgroup of $G^d$. Let $\rho^d \in (\mathfrak{a}^d_{\mathfrak{p}})^*$ be defined by

$$\rho^d(\,\cdot\,) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}\,(\,\cdot\,)|_{\mathfrak{n}^d}).$$

In other words, $\rho^d = \frac{1}{2}\Sigma_\alpha \dim(\mathfrak{g}^d_\alpha)\alpha$, where the summation extends over the $\mathfrak{a}^d_{\mathfrak{p}}$-roots $\alpha$ in $\mathfrak{n}^d$.

Let $\mathbb{C}_\Lambda$ denote $\mathbb{C}$ equipped with the $A^d_{\mathfrak{p}}$-action determined by $-\Lambda + \rho^d$. Thus, $a \in A^d_{\mathfrak{p}}$ acts on $\mathbb{C}_\Lambda$ by the scalar $a^{-\Lambda+\rho^d}$. The action of $A^d_{\mathfrak{p}}$ is extended to an action of $P^d$ on $\mathbb{C}_\Lambda$, by letting $M^d$ and $N^d$ act trivially. We now define the $G$-equivariant line bundle $\mathcal{L}_\Lambda$ on the flag manifold $G^d/P^d$ by

$$\mathcal{L}_\Lambda := G^d \times_{P^d} \mathbb{C}_\Lambda.$$

The space $\Gamma(\mathcal{L}_\Lambda)$ of continuous sections of this bundle is naturally identified with the space of continuous functions $f : G \to \mathbb{C}$ transforming according to the rule

$$f(xman) = a^{\Lambda - \rho^d} f(x),$$

for all $x \in G^d$ and $(m, a, n) \in M^d \times A^d_{\mathfrak{p}} \times N^d$. The natural action of $G^d$ on sections defines a continuous representation $\pi_\Lambda$ of $G^d$ in $\Gamma(\mathcal{L}_\Lambda)$. Let $\mathcal{B}(\mathcal{L}_\Lambda)$ denote the space of hyperfunction sections of the line bundle $\mathcal{L}_\Lambda$. This space may be identified with the dual of the locally convex space of analytic sections of the bundle $\mathcal{L}_{-\Lambda}$, and thus is a locally convex space. We define the Poisson transform $\mathcal{P}_\Lambda : \mathcal{B}(\mathcal{L}_\Lambda) \to C^\infty(\mathrm{X}^d)$ by the formula

$$\mathcal{P}_\Lambda \varphi(\bar{x}) = \int_{K^d} \varphi(xk)\, dk, \qquad (x \in G^d),$$

where $dk$ denotes normalized Haar measure of $K^d$. From the definition it readily follows that the Poisson transform intertwines the representation $\pi_\Lambda$ with the left regular representation $L$. Moreover, it maps into the space $\mathcal{E}_\Lambda(\mathrm{X}^d)$ of smooth functions $f \in C^\infty(\mathrm{X}^d)$ satisfying the system (5.3).

Let $\mathcal{E}^*_\Lambda(\mathrm{X}^d)$ denote the space of functions $f \in \mathcal{E}_\Lambda(\mathrm{X}^d)$ for which there exist constants $r \in \mathbb{R}$ and $C > 0$ such that

$$\|f\| := |f(x)| \leq Ce^{r\mathrm{dist}(x, \bar{e})} \qquad (x \in \mathrm{X}^d), \tag{5.4}$$

where $\mathrm{dist}(x, \bar{e})$ denotes the Riemannian distance in $\mathrm{X}^d$ between $x$ and the origin $\bar{e} = eK^d$. Note that $\mathrm{dist}(x, \bar{e}) = |X|$ for $X \in \mathfrak{p}^d$ and $x = \exp XK^d$.

It is not hard to show that $\mathcal{P}_\Lambda$ maps the space $\mathcal{D}'(\mathcal{L}_\Lambda)$ of distribution sections of $\mathcal{L}_\Lambda$ continuously into the space $\mathcal{E}^*_\Lambda(\mathrm{X}^d)$, equipped with the locally convex topology suggested by the estimates (5.4).

**Theorem 5.2** *Let* $\mathrm{Re}\,(\Lambda)$ *be dominant with respect to the roots of* $\mathfrak{a}_{\mathfrak{p}}^d$ *in* $\mathfrak{n}^d$. *Then the Poisson transform* $\mathcal{P}_\Lambda$

(a) *is a topological linear isomorphism* $\mathcal{B}(\mathcal{L}_\Lambda) \overset{\simeq}{\to} \mathcal{E}_\Lambda(\mathrm{X}^d)$, *and*

(b) *restricts to a topological linear isomorphism* $\mathcal{D}'(\mathcal{L}_\Lambda) \overset{\simeq}{\to} \mathcal{E}_\Lambda^*(\mathrm{X}^d)$.

For $\mathrm{X}^d$ of rank one, part (a) of the theorem is due to Helgason [61]. In [62] he conjectured part (a) to be true in general, and established it on the level of $K^d$-finite functions. Part (a) was established in generality by M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and T. Tanaka, [65], by means of the microlocal machinary developed by the school around M. Sato. In particular, this machinary allowed one to define a boundary value inverting the Poisson transform, generalizing the classical boundary value for harmonic functions on the disk. Part (b) is due to Oshima and Sekiguchi [78]. Later, Wallach gave a different proof of (b), [87], based on the theory of asymptotic behavior of matrix coefficients. Inspired by this work, van den Ban and Schlichtkrull, [12], gave a proof of (b) via a theory of asymptotic expansions with distribution coefficients, which allowed them to define a distributional boundary value inverting the restricted Poisson transform of (b).

The system (5.2) remains unchanged if $\Lambda$ is replaced by a conjugate under the Weyl group $W(\mathfrak{g}^d, \mathfrak{a}_{\mathfrak{p}}^d)$. Without loss of generality, $\mathrm{Re}\,\Lambda$ may therefore be assumed to be dominant. It can be shown that for a function $f \in C^\infty(\mathrm{X}) \cap L^2(\mathrm{X})_K$ satisfying the system (5.2), the function $f^d$ satisfies a growth condition which in particular implies (5.4). Therefore, for such a function $f$, the dual function $f^d$ may be realized as a Poisson transform of a unique distribution section $\varphi \in \mathcal{D}'(\mathcal{L}_\Lambda)$, which by equivariance is $H^d$-finite.

**Theorem 5.3** *Assume that* $\mathrm{rk}\,(G/H) = \mathrm{rk}\,(K/K \cap H)$. *Then there exist infinitely many discrete series representations for* X.

For the case of the group this result is due to Harish-Chandra [53], [54], who also established the necessity of the above rank condition, and gave the full classification of the discrete series via character theory.

The generalization to symmetric spaces is due to Flensted-Jensen [45]. The idea of his proof is to construct, for infinitely many dominant values of $\Lambda \in (\mathfrak{a}_{\mathfrak{p}}^d)^*$, a nontrivial function $f \in C^\infty(\mathrm{X}) \cap L^2(\mathrm{X})_K$ satisfying (5.2), via its dual $f^d$. The dual is obtained as a Poisson transform $f^d = \mathcal{P}_\Lambda(\varphi)$, with $\varphi$ a distribution section of $\mathcal{L}_\Lambda$, with support contained in a closed $H^d$-orbit on $G^d/P^d$.

**The classification of Oshima and Matsuki** In [77], Oshima and Matsuki established the necessity of the rank condition of Theorem 5.3 for the existence of discrete series. They used the mentioned theory of boundary values to show that the growth condition on $f^d$ can be translated into a condition on the support of $\mathcal{P}_\Lambda^{-1}(f^d)$, which by equivariance is a union of $H^d$ orbits on $G^d/P^d$. Moreover, this condition can only be met if the rank condition is fulfilled. This leads to the following.

**Theorem 5.4** $\mathrm{X}_{\mathrm{ds}}^\wedge \neq \varnothing \iff \mathrm{rk}\,(G/H) = \mathrm{rk}\,(K/K \cap H)$.

**Example 5.5** If $\mathrm{rk}\,(G/H) = 1$, then clearly the theorem implies that $\mathrm{X}^{\wedge}_{\mathrm{ds}} \neq \varnothing$. It is readily seen that the hyperbolic spaces $\mathrm{X}_{p,q}$, see Example 3.4, have rank 1; therefore, each of these has infinitely many representations in the discrete series.

In addition, in [77], Oshima and Matsuki proved, under the rank condition, that a function $f \in C^\infty(\mathrm{X})$ is in $L^2(\mathrm{X})$ if and only if its dual $f^d$ is the Poisson transform of a distribution section of $\mathcal{L}_\Lambda$ with support contained in a union of closed $H^d$-orbits. Moreover, they obtained the following information about the infinitesimal character $\Lambda$.

**Theorem 5.6** ([77]). *Let $\pi \in \mathrm{X}^{\wedge}_{\mathrm{ds}}$ and let $\mathfrak{b} \subset \mathfrak{q}$ be a $\theta$-stable Cartan subspace. Then the eigenvalues of the $\mathbb{D}(\mathrm{X})$-module $(\mathcal{H}_\pi^{-\infty})^H_{\mathrm{ds}}$ are all of the form*

$$D \mapsto \gamma(D, \Lambda)\,,$$

*with $\Lambda \in \mathfrak{b}^*_{\mathbb{C}}$ real and regular, i.e.,*

$$\langle \Lambda\,,\, \alpha\rangle \in \mathbb{R} \setminus \{0\}, \quad \text{for all} \quad \alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{b}).$$

In addition to this, for $G$ connected semisimple, Oshima and Matsuki gave a list of representations spanning $L^2_{\mathrm{d}}(\mathrm{X})$. For a few of these it remained an open problem whether they are nonzero or irreducible (a priori they are finite sums of irreducibles). The irreducibility was settled by D. Vogan, [85]. In [70] Matsuki gave necessary conditions for nontriviality, which he announced to be sufficient. The final problem is whether the list contains double occurrences. The answer is believed to be no under the mentioned assumption on $G$, implying that for a representation $\pi$ from the discrete series, $\dim(\mathcal{H}_\pi^{-\infty})_{\mathrm{ds}} = 1$, or equivalently, $\pi$ occurs with multiplicity one in the Plancherel formula. This fact has been established by F. Bien [26], except for spaces that have as a factor one of four exceptional symmetric spaces.

In the proof of the Plancherel theorem that we will describe, we do not need the full description of $\mathrm{X}^{\wedge}_{\mathrm{ds}}$. The formulation of the Plancherel theorem is put in a form that avoids precise description of the discrete series. As a consequence, Theorems 5.4 and 5.6 are sufficient for the proof. At the same time, it should be emphasized that the mentioned theorems are absolutely indispensable for the proof. This is also true for Delorme's proof in [40].

# 6 Parabolic subgroups

**The Coxeter complex** In the proof of the Plancherel formula, the asymptotic behavior of $K$-finite eigenfunctions of $\mathbb{D}(\mathrm{X})$ plays a crucial role. In view of the polar decomposition of Lemma 3.6, for a proper description of this behavior it is necessary to use an appropriate description of asymptotic directions to infinity in $A_{\mathrm{q}} \simeq \mathfrak{a}_{\mathrm{q}}$. The description of these directions relies on the following notion of a facet for the root system $\Sigma$. The collection of facets will turn out to be in one-to-one correspondence with the collection of $\sigma\theta$-stable parabolic subgroups of $G$ containing $A_{\mathrm{q}}$.

**Definition 6.1** A *facet* of $(\mathfrak{a}_\mathrm{q}, \Sigma)$ is defined to be an equivalence class for the equivalence relation $\sim$ on $\mathfrak{a}_\mathrm{q}$ defined by

$$\mathrm{X} \sim Y \Leftrightarrow \{\alpha \in \Sigma \mid \alpha(\mathrm{X}) > 0\} = \{\alpha \in \Sigma \mid \alpha(Y) > 0\}.$$

The dimension of a facet is defined to be the dimension of its linear span.

**Example 6.2** Consider the root system $A_2$ in $\mathbb{R}^2$, consisting of the roots $\pm\alpha, \pm\beta$ and $\pm(\alpha+\beta)$, where $\{\alpha, \beta\}$ is a fundamental system. The 6 open Weyl chambers are the facets of dimension 2, the 6 open ended halflines that border them (the 'walls') are those of dimension 1. Finally, $\{0\}$ is the unique facet of dimension 0.

The collection of facets, also called the *Coxeter complex* of $\Sigma$, is denoted by $\mathcal{P}(\Sigma)$. It is equipped with a natural action by the Weyl group $W$ of $\Sigma$. If $\mathrm{X} \in \mathfrak{a}_\mathrm{q}$, we denote its class by $C_\mathrm{X} \in \mathcal{P}(\Sigma)$. For $C \in \mathcal{P}(\Sigma)$, we put

$$\begin{aligned} \Sigma(C) &= \{\alpha \in \Sigma \mid \alpha > 0 \text{ on } C\}, \\ \Sigma_C &= \{\alpha \in \Sigma \mid \alpha = 0 \text{ on } C\}. \end{aligned}$$

Then

$$\Sigma = -\Sigma(C) \cup \Sigma_C \cup \Sigma(C) \quad \text{(disjoint union).} \tag{6.1}$$

Let $S$ be the intersection of the root hyperplanes $\ker \alpha$, $\alpha \in \Sigma_C$. Then, clearly, the set $D = \{\mathrm{X} \in S \mid \forall_{\alpha \in \Sigma(C)} \;\; \alpha(\mathrm{X}) > 0\}$ contains $C$, hence is a nonempty open subset of $S$ and therefore spans $S$. On the other hand, from (6.1) it follows that $D \in \mathcal{P}(\Sigma)$. Hence $C = D$ and we conclude that the linear span of the facet $C$ is given by

$$\mathrm{span}(C) = \cap_{\alpha \in \Sigma_C} \;\; \ker \alpha\,.$$

We now fix a closed Weyl chamber for $\Sigma$, which we call positive and denote by $\bar{\mathfrak{a}}_\mathrm{q}^+$. The following result is well known, see, e.g., [29].

**Lemma 6.3** *Let $C \in \mathcal{P}(\Sigma)$. Then there exists a unique $D \in \mathcal{P}(\Sigma)$ such that*

(a) *$C$ is $W$-conjugate to $D$,*

(b) *$D \subset \bar{\mathfrak{a}}_\mathrm{q}^+$.*

Let $\Sigma^+$ be a positive system and let $\Delta$ be the collection of simple roots in $\Sigma$ associated with the chamber $\bar{\mathfrak{a}}_\mathrm{q}^+$. If $F \subset \Delta$, we put

$$\begin{aligned} \mathfrak{a}_{F\mathrm{q}} &= \cap_{\alpha \in F} \;\ker \alpha\,, \\ \mathfrak{a}_{F\mathrm{q}}^+ &= \{\mathrm{X} \in \mathfrak{a}_{F\mathrm{q}} \mid \forall \alpha \in \Sigma^+ \quad \alpha(\mathrm{X}) \neq 0 \Rightarrow \alpha(\mathrm{X}) > 0\}. \end{aligned}$$

Then $\mathfrak{a}_{F\mathrm{q}}^+ \in \mathcal{P}(\Sigma)$. Moreover, the following result is well known, see, e.g., [29].

**Lemma 6.4** *$F \mapsto \mathfrak{a}_{F\mathrm{q}}^+$ is a bijection from the collection of all subsets of $\Delta$ onto the collection of all facets $C \in \mathcal{P}(\Sigma)$ contained in $\bar{\mathfrak{a}}_\mathrm{q}^+$. Finally, $\bar{\mathfrak{a}}_\mathrm{q}^+$ is the disjoint union of the sets $\mathfrak{a}_{F\mathrm{q}}^+$, for $F \subset \Delta$.*

**Definition 6.5** A *standard facet* (relative to $\Sigma^+$) is a facet $C$ satisfying one of the following equivalent conditions:

(a) $C \subset \overline{\mathfrak{a}}_{\mathrm{q}}^+$;

(b) $C = \mathfrak{a}_{F\mathrm{q}}^+$ for some $F \subset \Delta$.

To each facet $C \in \mathcal{P}(\Sigma)$ we associate the following subalgebra of $\mathfrak{g}$:

$$\mathfrak{p}_C := \mathfrak{g}_0 \oplus \bigoplus_{\substack{\alpha \in \Sigma \\ \alpha|_C \geq 0}} \mathfrak{g}_\alpha\,, \tag{6.2}$$

where $\mathfrak{g}_0$ denotes the centralizer of $\mathfrak{a}_{\mathrm{q}}$. The algebra $\mathfrak{p}_C$ is readily checked to be a *parabolic subalgebra* of $\mathfrak{g}$, i.e., it is a subalgebra that is its own normalizer in $\mathfrak{g}$. The fact that $\sigma\theta = I$ on $\mathfrak{a}_{\mathrm{q}}$ implies that the parabolic subalgebra $\mathfrak{p}_C$ is $\sigma\theta$-stable.

**Lemma 6.6** *The map $C \mapsto \mathfrak{p}_C$ is a bijective correspondence between $\mathcal{P}(\Sigma)$ and the set of $\sigma\theta$-stable parabolic subalgebras containing $\mathfrak{a}_{\mathrm{q}}$.*

*Proof.* The proof is not difficult, see [20], Sect. 2. □

**Remark 6.7** If $\sigma = \theta$, then $\mathfrak{a}_{\mathrm{q}}$ is maximal abelian in $\mathfrak{p}$. We write $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}_{\mathrm{q}}$ in this case, and $A_{\mathfrak{p}} = \exp \mathfrak{a}_{\mathfrak{p}}$. In the present setting the above result amounts to the well-known fact that the map $C \mapsto \mathfrak{p}_C$ is a bijective correspondence between the Coxeter complex $\mathcal{P}(\Sigma)$ of $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ and the collection of all parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{a}_{\mathfrak{p}}$.

If $\mathfrak{r}$ is a parabolic subalgebra of $\mathfrak{g}$, then its normalizer $R = N_G(\mathfrak{r})$ in $G$ is a closed subgroup with algebra $\mathfrak{r}$. Thus, $R$ is a subgroup of $G$ that equals the normalizer of its Lie algebra. A subgroup with this property is called a *parabolic subgroup* of $G$. Clearly, the map $\mathfrak{r} \mapsto N_G(\mathfrak{r})$ defines a bijective correspondence between the collection of all parabolic subalgebras of $\mathfrak{g}$ and the collection of all parabolic subgroups of $G$. If $C \in \mathcal{P}(\Sigma)$, we put $P_C = N_G(\mathfrak{p}_C)$.

**Corollary 6.8** *The map $C \mapsto P_C$ is a bijective correspondence between $\mathcal{P}(\Sigma)$ and the collection $\mathcal{P}_\sigma$ of all parabolic subgroups of $G$ that are $\sigma\theta$-stable and contain $A_{\mathrm{q}}$.*

**Langlands decomposition** If $P \in \mathcal{P}_\sigma$, let $C = C_P \in \mathcal{P}(\Sigma)$ be the unique facet with $P_C = P$. We agree to write

$$\mathfrak{a}_{P\mathrm{q}}^+ := C\,, \quad \Sigma_P = \Sigma_C\,, \quad \Sigma(P) = \Sigma(C)\,. \tag{6.3}$$

We also agree to write $\mathfrak{a}_{P\mathrm{q}} := \mathrm{span}(\mathfrak{a}_{P\mathrm{q}}^+)$; then

$$\begin{aligned} \mathfrak{a}_{P\mathrm{q}} &= \cap_{\alpha \in \Sigma_P} \ker \alpha\,, \\ \mathfrak{a}_{P\mathrm{q}}^+ &= \{\mathrm{X} \in \mathfrak{a}_{P\mathrm{q}} \mid \forall \beta \in \Sigma(P): \quad \beta(\mathrm{X}) > 0\}\,. \end{aligned}$$

Let $\mathfrak{m}_{1P}$ denote the centralizer of $\mathfrak{a}_{P\mathrm{q}}$ in $\mathfrak{g}$. Then

$$\mathfrak{m}_{1P} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma_P} \mathfrak{g}_\alpha \,. \tag{6.4}$$

Moreover, it follows from (6.2) that

$$\mathrm{Lie}(P) = \mathfrak{m}_{1P} \oplus \mathfrak{n}_P \,, \qquad \text{where} \quad \mathfrak{n}_P = \oplus_{\alpha \in \Sigma(P)} \; \mathfrak{g}_\alpha \,. \tag{6.5}$$

It is readily checked that $\mathfrak{m}_P$ is a reductive Lie algebra and that $\mathfrak{n}_P$ is the nilpotent radical of Lie$(P)$; therefore, the decomposition in (6.5) is a Levi decomposition. In fact, it is the unique Levi decomposition with a $\theta$-stable Levi component, cf. [84], Sect. II.6.

Let $C \in \mathcal{P}(\Sigma)$, then $-C \in \mathcal{P}(\Sigma)$ as well. Accordingly, for $P \in \mathcal{P}_\sigma$ we define the *opposite* parabolic subgroup $\bar{P}$ by requiring that $\mathfrak{a}_{\bar{P}\mathrm{q}}^+ = -\mathfrak{a}_{P\mathrm{q}}^+$. The following lemma is a straightforward consequence of the definitions given above.

**Lemma 6.9** *Let $P \in \mathcal{P}_\sigma$. Then*

(a) $\overline{P} = \theta(P) = \sigma(P)$;

(b) $\mathfrak{n}_{\bar{P}} = \theta\mathfrak{n}_P = \sigma\mathfrak{n}_P$;

(c) $\mathfrak{m}_{1\bar{P}} = \theta\mathfrak{m}_{1P} = \sigma\mathfrak{m}_{1P} = \mathfrak{m}_{1P}$,

(d) $\mathfrak{g} = \mathfrak{n}_{\bar{P}} \oplus \mathfrak{m}_{1P} \oplus \mathfrak{n}_P$.

We define the following subgroups of $G$:

$$M_{1P} := Z_G(\mathfrak{a}_{P\mathrm{q}}) \quad \text{and} \quad N_P = \exp \mathfrak{n}_P \,.$$

**Proposition 6.10** *Let $P \in \mathcal{P}_\sigma$.*

(a) *$N_P$ is a closed subgroup of $G$.*

(b) *$M_{1P}$ is a group of the Harish-Chandra class.*

(c) *$P = M_{1P}N_P$; the multiplication map is a diffeomorphism from $M_{1P} \times N_P$ onto $P$.*

*Proof.* See [84], Sect. II.6. □

As also mentioned in the appendix, assertion (b) of the above proposition is of particular importance, since it allows induction with respect to dimension within Harish-Chandra's class of real reductive groups.

**Remark 6.11** If $\sigma = \theta$, then $\mathfrak{a}_\mathrm{q}$ is maximal abelian in $\mathfrak{p}$. In the notation of Remark 6.7, let $\mathfrak{p}_0$ denote the parabolic subalgebra determined by $C := \mathfrak{a}_\mathfrak{p}^+$, let $P_0 = P_C$ denote its normalizer in $G$ and let $N_0 := N_C$. We recall that $G$ admits the Iwasawa decomposition $G = N_0 A_\mathfrak{p} K$, where the natural multiplication map $N_0 \times A_\mathfrak{p} \times K \to G$ is an analytic diffeomorphism. In particular, it follows that $G = P_0 K$.

**Lemma 6.12** *Let $P \in \mathcal{P}_\sigma$. Then $G = PK$. Moreover, the multiplication map induces a diffeomorphim $P \times_{K \cap M_{1P}} K \to G$.*

*Proof.* This is a rather straightforward consequence of the Iwasawa decomposition described in Remark 6.11. See [84], Sect. II.6, for details. □

We can now describe the so-called *Langlands decomposition* of a parabolic subgroup $P \in \mathcal{P}_\sigma$. Let us first do this on the level of Lie algebras. Let $P \in \mathcal{P}_\sigma$. Then $\mathfrak{m}_{1P}$ is $\theta$-invariant, by Lemma 6.9, hence $\mathfrak{m}_{1P} = (\mathfrak{m}_{1P} \cap \mathfrak{k}) \oplus (\mathfrak{m}_{1P} \cap \mathfrak{p})$. We define

$$\mathfrak{a}_P = \mathrm{center}(\mathfrak{m}_{1P}) \cap \mathfrak{p}\,.$$

Clearly, $\mathfrak{a}_{P\mathrm{q}}$ is contained in this space. On the other hand, if $X \in \mathfrak{p} \cap \mathfrak{q}$ centralizes $\mathfrak{m}_{1P}$, then $X$ centralizes the maximal abelian subspace $\mathfrak{a}_\mathrm{q}$ of $\mathfrak{p} \cap \mathfrak{q}$, hence belongs to it. Moreover, in view of (6.4), $\alpha(X) = 0$ for all $\alpha \in \Sigma_P$, from which we deduce that $X \in \mathfrak{a}_{P\mathrm{q}}$. Thus,

$$\mathfrak{a}_{P\mathrm{q}} = \mathfrak{a}_P \cap \mathfrak{q}\,.$$

This justifies the notation with subscript q in hindsight. The group

$$A_P := \exp \mathfrak{a}_P$$

is called the *split component* of $P$, and $A_{P\mathrm{q}} := \exp \mathfrak{a}_{P\mathrm{q}}$ the *$\sigma$-split component.*

Define $\mathfrak{m}_P := (\mathfrak{m}_{1P} \cap \mathfrak{k}) \oplus ([\mathfrak{m}_{1P}, \mathfrak{m}_{1P}] \cap \mathfrak{p})$. Then $\mathfrak{m}_P$ is a reductive Lie algebra with $\mathrm{center}(\mathfrak{m}_P) \cap \mathfrak{p} = 0$. Moreover,

$$\mathfrak{m}_{1P} = \mathfrak{m}_P \oplus \mathfrak{a}_P\,.$$

It follows that

$$\mathrm{Lie}(P) = \mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P.$$

This is called the infinitesimal Langlands decomposition. Define $M_P = (M_{1P} \cap K)\exp(\mathfrak{m}_P \cap \mathfrak{p})$. Then the following result describes the Langlands decomposition of the parabolic subgroup $P$.

**Lemma 6.13** (Langlands decomposition). *Let $P \in \mathcal{P}_\sigma$. Then $M_P$ is a group of the Harish-Chandra class. Moreover,*

$$M_{1P} = M_P A_P\,, \quad P = M_P A_P N_P.$$

*The multiplication maps induce diffeomorphisms $M_P \times A_P \to M_{1P}$ and $M_P \times A_P \times N_P \to P$.*

*Proof.* For a proof the reader is referred to [84], Sect. II.6. □

**Remark 6.14** In his work on the Plancherel decomposition, P. Delorme reserves the above notation for the so-called *$\sigma$-Langlands decomposition.* More precisely, let $A_{P\mathrm{h}} = A_P \cap H$; then $A_P = A_{P\mathrm{h}} A_{P\mathrm{q}}$. Put $M_{P\sigma} = M_P A_{P\mathrm{h}}$. Then

$$P = M_{P\sigma} A_{P\mathrm{q}} N_P$$

is called the $\sigma$-Langlands decomposition of the parabolic subgroup. Delorme uses the notation $M_P$ instead of $M_{P\sigma}$ and $A_P$ instead of $A_{P\mathrm{q}}$.

# 7 Parabolically induced representations

**Induced representations** In this section we assume that $G$ is a real reductive group of the Harish-Chandra class, and that $P \in \mathcal{P}_\sigma$. We shall describe the process of inducing representations from $P$ to $G$, and its relation with function theory on $G/H$. It is a good idea to keep in mind that, in particular, the Riemannian case with $\sigma = \theta$ is covered. In this case $\sigma\theta = I$, so that $\mathcal{P}_\sigma$ consists of *all* parabolic subgroups containing $A_\mathfrak{p} = A_\mathrm{q}$, see Remark 6.7.

Let $\xi \in \widehat{M}_P$ (the unitary dual of $M_P$) and let $\mathcal{H}_\xi$ be a Hilbert space in which $\xi$ is unitarily realized. Let $\lambda$ belong to $\mathfrak{a}_{P\mathbb{C}}^* := \mathrm{Hom}_{\,\mathbb{R}}(\mathfrak{a}_P, \mathbb{C})$, the complexified linear dual of $\mathfrak{a}_P$. We define the representation $\xi \otimes \lambda \otimes 1$ of $P = M_P A_P N_P$ in $\mathcal{H}_\xi$ by

$$(\xi \otimes \lambda \otimes 1)(man) = a^\lambda \xi(m)\,,$$

for $m \in M_P$, $a \in A_p$, $n \in N_P$. This indeed defines a representation of $P$, since $M_P$ centralizes $A_P$, and since $M_{1P} = M_P A_P$ normalizes $N_P$.

We shall now proceed to define the parabolically induced representation

$$\pi_{P,\xi,\lambda} := \mathrm{ind}_P^G(\xi \otimes (\lambda + \rho_P) \otimes 1)\,. \tag{7.1}$$

Here $\rho_P \in \mathfrak{a}_P^*$ is defined by

$$\begin{aligned}\rho_P(x) &= \frac{1}{2}\mathrm{tr}\,\left[\mathrm{ad}\,(x)|_{\mathfrak{n}_P}\right] \\ &= \frac{1}{2}\sum_{\alpha\in\Sigma(P)} \dim(\mathfrak{g}_\alpha)\,\alpha\,.\end{aligned}$$

The translation over $\rho_P$ will turn out to be needed to ensure that the representation $\pi_{P,\xi,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}_{P\mathrm{q}}^*$. To describe the representation space for $\pi_{P,\xi,\lambda}$ we first define

$$C(P : \xi : \lambda)$$

to be the space of continuous functions $f : G \to \mathcal{H}_\xi$ transforming according to the rule

$$f(man\, x) = a^{\lambda+\rho_P}\xi(m)f(x)\,, \tag{7.2}$$

for $x \in G$, $m \in M_P$, $a \in A_P$, $n \in N_P$.

In $C(P : \xi : \lambda)$, the representation $\pi_{P,\xi,\lambda}$ is defined by restricting the right regular representation, i.e., if $f \in C(P : \xi : \lambda)$ and $x \in G$, then

$$\pi_{P,\xi,\lambda}(x)f(y) = f(yx)\,, \qquad (y \in G)\,.$$

Our next goal is to extend $\pi_{P,\xi,\lambda}$ to a suitable Hilbert space completion of $C(P : \xi : \lambda)$. It follows from Lemma 6.12 that a function $f$ in $C(P : \xi : \lambda)$ is completely determined by its restriction $f|_K$ to $K$. Let $C(K : \xi)$ denote the space of continuous functions $\varphi : K \to \mathcal{H}_\xi$ transforming according to the rule

$$\varphi(mk) = \xi(m)\varphi(k)\,, \tag{7.3}$$

for $k \in K$ and $m \in K \cap P = K \cap M_P$.

**Lemma 7.1** *The map $f \mapsto f|_K$ defines a topological linear isomorphism from $C(P:\xi:\lambda)$ onto $C(K:\xi)$.*

*Proof.* This follows by application of Lemma 6.12. $\square$

Via the above isomorphism, $\pi_{P,\xi,\lambda}$ may be viewed as a ($\lambda$-dependent) representation of $G$ on the ($\lambda$-independent) space $C(K:\xi)$. This realization of $\pi_{P,\xi,\lambda}$ is sometimes called the *compact picture* of the induced representation.

According to the above, we may equip $C(P:\xi:\lambda)$ with the pre-Hilbert structure defined by

$$\begin{aligned}\langle f\,,\,g\rangle &= \langle f|_K\,,\,g|_K\rangle_{L^2(K,\mathcal{H}_\xi)}\\ &= \int_K \langle f(k)\,,\,g(k)\rangle_{\mathcal{H}_\xi}\,dk\,,\end{aligned} \tag{7.4}$$

where $dk$ denotes normalized Haar measure on $K$. The Hilbert completion of $C(P:\xi:\lambda)$ for this structure is denoted by $\mathcal{H}_{P,\xi,\lambda}$. It can be shown that $\pi_{P,\xi,\lambda}$ extends uniquely to a continuous representation of $G$ in $\mathcal{H}_{P,\xi,\lambda}$.

Alternatively, the Hilbert space $\mathcal{H}_{P,\xi,\lambda}$ may also be characterized as the space of measurable functions $f: G \to \mathcal{H}_\xi$ that transform according to the rule (7.2) and satisfy $f|_K \in L^2(K,\mathcal{H}_\xi)$, equipped with the inner product given by (7.4).

**Generalized vectors** We now come to the result that motivated the introduction of the shift by $\rho_P$ in (7.1).

**Proposition 7.2** *Let $\xi \in \widehat{M_P}$ and $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$. Then the sesquilinear pairing $\mathcal{H}_{P,\xi,\lambda} \times \mathcal{H}_{P,\xi,-\bar\lambda} \to \mathbb{C}$ defined by*

$$\langle f\,,\,g\rangle := \int_K \langle f(k)\,,\,g(k)\rangle\rangle_{\mathcal{H}_\xi}\,dk \tag{7.5}$$

*is $G$-equivariant. In particular, the representation $\pi_{P,\xi,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$.*

*Proof.* It suffices to prove the equivariance for $f$ and $g$ smooth. In that case the function $\langle f\,,\,g\rangle_\xi : x \mapsto \langle f(x)\,,\,g(x)\rangle_{\mathcal{H}_\xi}$ belongs to $C^\infty(P:1:1)$, which may be identified with the space of smooth sections of the density bundle over $P\backslash G$. Its naturally defined integral over $P\backslash G$ is readily shown to equal the integral on the right-hand side of (7.5). To see that the pairing is equivariant, we note that, for $x \in G$, $\langle \pi_{\xi,\lambda}(x)f\,,\,\pi_{\xi,-\bar\lambda}(x)g\rangle_\xi$ equals the pullback of $\langle f\,,\,g\rangle_\xi$ under the diffeomorphism $Pg \mapsto Pgx$. The integration of densities is invariant under diffeomorphisms. For more details concerning this proof in terms of densities, we refer the reader to [7], Lemma 2.1. $\square$

The space of smooth vectors for $\pi_{P,\xi,\lambda}$ equals the space

$$C^\infty(P:\xi:\lambda)$$

of smooth functions $G \to \mathcal{H}_\xi^\infty$ transforming according to the rule (7.2), see [28], Sect. III.7. for details. The sesquilinear pairing of the above proposition induces a $G$-equivariant linear embedding

$$\mathcal{H}_{P,\xi,-\bar\lambda} \hookrightarrow (\overline{C^\infty(P:\xi:\lambda)})' = \mathcal{H}^{-\infty}_{P,\xi,\lambda}\,.$$

This provides motivation for us to use the notation

$$C^{-\infty}(P:\xi:-\overline{\lambda}) := (\overline{C^\infty(P:\xi:\lambda)})'.$$

The sesquilinear pairing of Proposition 7.2 then naturally extends to a sesquilinear pairing

$$C^\infty(P:\xi:\lambda)\times C^{-\infty}(P:\xi:-\overline{\lambda})\to\mathbb{C},$$

also denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$.

Similarly, we define $C^\infty(K:\xi)$ to be the space of smooth functions $K\to\mathcal{H}_\xi^\infty$ transforming according to the rule (7.3) and $C^{-\infty}(K:\xi)$ for its continuous antilinear dual. Then the restriction map $f\mapsto f|_K$ induces topological linear isomorphisms $C^{\pm\infty}(P:\xi:\lambda)\simeq C^{\pm\infty}(K:\xi)$. Accordingly, the representations $\pi^{\pm\infty}_{P,\xi,\lambda}$ may then be realized in the $\lambda$-independent spaces $C^{\pm\infty}(K:\xi)$.

The Plancherel formula will essentially be built from the representations of $\mathrm{X}^\wedge_{\mathrm{ds}}$ and from the induced representations $\pi_{P,\xi,\lambda}$, where $P\in\mathcal{P}_\sigma$, $P\neq G$, and where $\xi$ belongs to the discrete series of $M_P/M_P\cap vHv^{-1}$, for some $v\in\mathcal{W}$, and $\lambda\in i\mathfrak{a}^*_{P\mathrm{q}}$.

# 8 H–fixed generalized vectors

**Orbit structure** We assume that $P\in\mathcal{P}_\sigma$, $\xi\in\widehat{M}_P$ and $\lambda\in\mathfrak{a}^*_{P\mathbb{C}}$ and will try to describe sufficiently many $H$-fixed elements in $C^{-\infty}(P:\xi:\lambda)$. With this in mind it is important to have knowledge of the $H$-orbits on $P\backslash G$. The following result is a direct consequence of results of Matsuki, [69], and, independently, Rossmann, [80]; see also [5], App. B.

We agree to write $W_P$ for the centralizer of $\mathfrak{a}_{P\mathrm{q}}$ in $W$. Equivalently, $W_P$ is the subgroup of $W$ generated by the reflections in the roots of $\Sigma_P$. We fix a collection ${}^P\mathcal{W}$ of representatives for $W_P\backslash W/W_{K\cap H}$, contained in $N_K(\mathfrak{a}_\mathrm{q})$. We denote by $P\backslash G/H$ the collection of $H$-orbits on $P\backslash G$ and by $(P\backslash G/H)_{\mathrm{open}}$ the subset of open orbits.

**Proposition 8.1** *The set $P\backslash G/H$ is finite. Moreover, the map $v\mapsto PvH$ is a bijection from ${}^P\mathcal{W}$ onto $(P\backslash G/H)_{\mathrm{open}}$. In particular, the union $\cup_{v\in{}^P\mathcal{W}}PvH$ is an open dense subset of $P\backslash G$.*

On the open $H$-orbits one expects the elements of $C^{-\infty}(P:\xi:\lambda)^H$ to be just functions, which may be evaluated in points. Let $\varphi\in C^{-\infty}(P:\xi:\lambda)^H$ and let $v\in^P\mathcal{W}$. Then one expects that $\varphi(v)$ is a vector in $\mathcal{H}_\xi^{-\infty}$ which is fixed for $\xi\otimes(\lambda+\rho_P)\otimes 1|_{P\cap vHv^{-1}}$, because of the formal identity, for $p\in P\cap vHv^{-1}$,

$$\begin{aligned}[\xi\otimes(\lambda+\rho_p)\otimes 1](p)\,\varphi(v) &= \varphi(pv)=\varphi(vv^{-1}pv)\\ &= [\pi_{\xi,\lambda}(v^{-1}pv)\varphi](v)\\ &= \varphi(v),\end{aligned}$$

since $v^{-1}pv\in H$. This implies that $\varphi(v)\in(\mathcal{H}_\xi^{-\infty})^{M_P\cap vHv^{-1}}$ and $(\lambda+\rho_P)|_{\mathfrak{a}_P\cap\mathfrak{h}}=0$. The latter condition is equivalent to $\lambda|_{\mathfrak{a}_P\cap\mathfrak{h}}=0$, in view of the following lemma.

**Lemma 8.2** *$\rho_P$ vanishes on $\mathfrak{a}_P \cap \mathfrak{h}$.*

*Proof.* Since $\theta\sigma(\mathfrak{n}_P) = \mathfrak{n}_P$, by Lemma 6.9 (b), it follows that $\rho_P(\theta\sigma X) = \rho_P(X)$ for all $X \in \mathfrak{a}_P$. Hence $\rho_P = -\rho_P$ on $\mathfrak{a}_P \cap \mathfrak{h}$. $\square$

Writing $\mathfrak{a}_{P\mathrm{h}} := \mathfrak{a}_P \cap \mathfrak{h}$, we have the direct sum decomposition

$$\mathfrak{a}_P = \mathfrak{a}_{P\mathrm{h}} \oplus \mathfrak{a}_{P\mathrm{q}}$$

via which we may identify $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with the subspace of $\mathfrak{a}^*_{P\mathbb{C}}$ consisting of elements that vanish on $\mathfrak{a}_P \cap \mathfrak{h}$. The heuristic argument given above suggests that it is reasonable to expect that the induction parameter $\lambda$ should be restricted to the subspace $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ of $\mathfrak{a}^*_{P\mathbb{C}}$.

We note that, for $v \in {}^P\mathcal{W}$, the space

$$\mathrm{X}_{P,v} := M_P / M_P \cap vHv^{-1} \tag{8.1}$$

is reductive symmetric in the class under consideration. Indeed, $M_P$ is of the Harish-Chandra class by Lemma 6.13. Moreover, as $\mathrm{Ad}\,(v) \circ \sigma \circ \mathrm{Ad}\,(v^{-1}) = -I$ on $\mathfrak{a}_{P\mathrm{q}}$, the map $\sigma^v : x \mapsto v\sigma(v^{-1}xv)v^{-1}$ leaves the group $M_P$ invariant and defines an involution on it, having $vG^\sigma v^{-1} \cap M_P$ as its set of fixed points. The involution $\sigma^v$ commutes with $\theta$. For later purposes we observe that the space

$${}^*\mathfrak{a}_{P\mathrm{q}} := \mathfrak{m}_P \cap \mathfrak{a}_\mathrm{q} \tag{8.2}$$

equals the orthocomplement of $\mathfrak{a}_{P\mathrm{q}}$ in $\mathfrak{a}_\mathrm{q}$ with respect to any $W$-invariant inner product. Moreover, ${}^*A_{P\mathrm{q}} = \exp({}^*\mathfrak{a}_{P\mathrm{q}})$ is the analogue of $A_\mathrm{q}$ for each of the spaces (8.1).

We now agree to define the finite-dimensional Hilbert space $V(P, \xi, v)$, for $\xi \in \widehat{M_P}$ and $v \in {}^P\mathcal{W}$, by

$$\begin{aligned} V(P,\xi,v) &= (\mathcal{H}_\xi^{-\infty})_{\mathrm{ds}}^{M_P \cap vHv^{-1}} && \text{if } \xi \in \mathrm{X}^\wedge_{P,v,\mathrm{ds}} \\ &= 0 && \text{otherwise}\,. \end{aligned}$$

(See (2.19) for the notation used.)

**Definition 8.3** Let $\xi \in \widehat{M_P}$. We define $V(P, \xi)$ to be the formal direct Hilbert sum

$$V(P,\xi) = \bigoplus_{v \in {}^P\mathcal{W}} V(P,\xi,v)\,. \tag{8.3}$$

If $\eta \in V(P, \xi)$, then $\eta_v$ denotes its component in $V(P, \xi, v)$.

The idea now is to invert the map $\varphi \mapsto (\varphi(v))_{v \in {}^P\mathcal{W}}$ described above. An element $\mu \in \mathfrak{a}^*_{P\mathrm{q}}$ is called strictly $P$-dominant if

$$\langle \mu \,,\, \alpha \rangle > 0\,, \quad \text{for all } \alpha \in \Sigma(P)\,.$$

**Definition 8.4** Let $\eta \in V(P,\xi)$. For $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with $-(\mathrm{Re}\,\lambda + \rho_P)$ strictly $P$-dominant we define the function $j(P:\xi:\lambda:\eta): G \to \mathcal{H}_\xi^{-\infty}$ by

$$j(P:\xi:\lambda:\eta)(man\,vh) = a^{\lambda+\rho_P}\xi(m)\eta_v \tag{8.4}$$

for $v \in {}^P\mathcal{W}$, $man \in P$, $h \in H$ and by 0 outside $\cup_{v\in {}^P\mathcal{W}}\, PvH$ (the union of the open $P \times H$-orbits).

**Theorem 8.5** *Let $\xi \in \widehat{M_P}$ and let $\eta \in V(P,\xi)$. For every $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with $-(\mathrm{Re}\,\lambda + \rho_P)$ strictly $P$-dominant, the function $j(P:\xi:\lambda:\eta)$ defines an element of $C^{-\infty}(P:\xi:\lambda)^H$.*

*Moreover, $\lambda \mapsto j(P:\xi:\lambda:\eta)$ extends meromorphically to $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ as a function with values in $C^{-\infty}(K:\xi)$. The singular locus of this extended function is the union of a locally finite collection of hyperplanes of the form $\langle\lambda\,,\,\alpha\rangle = c$, with $\alpha \in \Sigma(P)$ and $c \in \mathbb{C}$.*

*Finally, if $\lambda$ is a regular value, then*

$$j(P:\xi:\lambda:\eta) \in C^{-\infty}(P:\xi:\lambda)^H\,.$$

**Remark 8.6** For the case of minimal $P \in \mathcal{P}_\sigma$, Theorem 8.5 is due to [5], where a proof based on the meromorphic continuation of intertwining operators is given and to [74], where a proof based on Bernstein's result on the meromorphic continuation of a complex power of a polynomial is given. In the same setting of a minimal $\sigma$-parabolic subgroup, in [6], Sect. 9, it is shown that $j(P:\xi:\lambda)$ satisfies a functional equation that allows for translation in the parameter $\lambda$.

For general $P \in \mathcal{P}_\sigma$, Theorem 8.5 is due to [31]. Later, in [34] a proof based on a generalization of the mentioned functional equation was given.

The meromorphic continuation is absolutely crucial for the development of the theory, since the set $i\mathfrak{a}^*_{P\mathrm{q}}$ (where the $\pi_{P,\xi,\lambda}$ are unitary) is not contained in the region $\langle\mathrm{Re}\,\lambda + \rho_P\,,\,\alpha\rangle < 0$ ($\alpha \in \Sigma(P)$).

By meromorphic continuation one still has that $j(P:\xi:\lambda:\eta)(v) = \eta_v$, showing that $j(P:\xi:\lambda) = j(P:\xi:\lambda:\cdot)$ defines an injective homomorphism

$$V(P,\xi) \hookrightarrow C^{-\infty}(P:\xi:\lambda)^H\,,$$

for regular $\lambda$. Thus $V(P,\xi)$ becomes a model for the space $\mathcal{V}_{\pi_{P,\xi,\lambda}}$ defined in the text above (2.17). The inner product of $V(P,\xi)$ may be transferred to an inner product on $\mathcal{V}_{\pi_{P,\xi,\lambda}}$. However, it is more convenient to keep working with $V(P,\xi)$, since this space is independent of $\lambda$.

**Definition 8.7** We define $\mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ to be the set of $\xi \in \widehat{M_P}$ for which the space $V(P,\xi)$, defined in (8.3), is nonzero.

**Remark 8.8** The above condition on $\xi \in \widehat{M_P}$ is equivalent to

$$\exists v \in {}^P\mathcal{W}: \;\; \xi \in \mathrm{X}^\wedge_{P,v,\mathrm{ds}}\,.$$

**Remark 8.9** Let $P$ be a minimal element of $\mathcal{P}_\sigma$ (with respect to inclusion). Then $\mathfrak{a}_{P\mathrm{q}}^+$ is maximal among the facets of $\Sigma$, hence an open Weyl chamber. Thus, $\mathfrak{a}_{P\mathrm{q}} = \mathfrak{a}_\mathrm{q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. From this one can derive that $M_P/M_P \cap vHv^{-1}$ is compact, for $v \in^P \mathcal{W} = \mathcal{W}$. It follows that $\mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ consists of finite-dimensional unitary representations of $M$. This makes the nature of the functional analysis involved in Theorem 5 considerably simpler. Under the assumption $[W : W_{K\cap H}] = 1$ this case is discussed in [82].

**Example 8.10** (Riemannian case). Let $\sigma = \theta$ and let $P \in \mathcal{P}_\sigma$ be minimal. Then $M_P \subset K$, hence $\mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ consists of the trivial representation 1. Moreover, one may take ${}^P\mathcal{W} = \{e\}$ and $V(P, 1) = \mathbb{C}$, equipped with the standard inner product. Then

$$j(P : 1 : \lambda : 1)(nak) = a^{\lambda+\rho_P}.$$

Thus, $j(P : 1 : \lambda : 1)$ equals $\mathbf{1}_\lambda$, the unique $K$-fixed vector in $C(P : 1 : \lambda)$ determined by $\mathbf{1}_\lambda(e) = 1$.

**Definition 8.11** Let $P \in \mathcal{P}_\sigma$. The series of unitary representations $\pi_{P,\xi,\lambda}$, for $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ and $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$, is called the *generalized $\sigma$-principal series* associated with $P$.

The Plancherel measure will turn out to be supported by the the generalized $\sigma$-principal series associated with the finite set of parabolic subgroups $P \in \mathcal{P}_\sigma$.

In order to work in a uniform way, we understand the above definitions to include the discrete series for $G/H$. More precisely, assume that $G/H$ satisfies the rank condition of Theorem 5.4, so that $\mathrm{center}(\mathfrak{g}) \cap \mathfrak{p} \cap \mathfrak{q} = \{0\}$. We consider the parabolic subgroup $P = G$. Then $M_P/M_P \cap H \simeq G/H$ and accordingly one may identify $\mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ with $\mathrm{X}^\wedge_{\mathrm{ds}}$. Moreover, $\mathfrak{a}_{P\mathrm{q}} = \{0\}$. For $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ we now have $\mathcal{H}_{P,\xi,0} \simeq \mathcal{H}_\xi$ and $\pi_{P,\xi,0} \sim \xi$. Thus, the discrete series may be thought of as the generalized $\sigma$-principal series associated with the $\sigma$-parabolic subgroup $G$.

# 9 The action of invariant differential operators

**A canonical homomorphism** In this section we describe the action of the algebra $U(\mathfrak{g})^H$ of invariant differential operators on the $H$-fixed generalized vectors introduced in the previous section. This action factors to an action of the algebra $\mathbb{D}(\mathrm{X})$ of invariant differential operators on these vectors.

Let $P \in \mathcal{P}_\sigma$. Then from Lemma 6.9 we infer that the map $\bar{\mathfrak{n}}_P \to \mathfrak{h}$ given by $X \mapsto X + \sigma(X)$ induces a linear isomorphism from $\bar{\mathfrak{n}}_P$ onto $\mathfrak{h}/\mathfrak{h} \cap \mathfrak{m}_{1P}$. It follows that $\mathfrak{g} = \mathfrak{n}_P \oplus [\mathfrak{m}_{1P} + \mathfrak{h}]$. By the Poincaré–Birkhoff–Witt theorem this implies that

$$U(\mathfrak{g}) = \mathfrak{n}_P U(\mathfrak{g}) \oplus [U(\mathfrak{m}_{1P}) \otimes_{U(\mathfrak{m}_{1P}\cap\mathfrak{h})} U(\mathfrak{h})]. \tag{9.1}$$

Moreover, this decomposition is stable under the adjoint action by $H_P := M_{1P} \cap H$. Accordingly, for $D \in U(\mathfrak{g})^H$, we define the element ${}^\backprime\mu_P(D)$ in the space $U(\mathfrak{m}_{1P})^{H_P}/U(\mathfrak{m}_{1P})^{H_P} \cap U(\mathfrak{m}_{1P})\mathfrak{h}_P$ by

$$D - {}^\backprime\mu_P(D) \in [\mathfrak{n}_P U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{h}],$$

where we have abused language in an obvious way. In view of Lemma 2.5, applied to X and $\mathrm{X}_{1P} = M_{1P}/M_{1P} \cap H$, it is now readily seen that the map ${}^{`}\mu_P$ factors to an algebra homomorphism from $\mathbb{D}(\mathrm{X})$ to $\mathbb{D}(M_{1P}/M_{1P} \cap H)$. We define the character $d_P : M_{1P} \to ]\,0, \infty\,[$ by

$$d_P(m) = |\det(\mathrm{Ad}\,(m)|_{\mathfrak{n}_P})|^{1/2}.$$

Using the duality of Section 4 it is seen that the function $d_P$ is right $H_P$-invariant. Moreover, $d_P = 1$ on $M_P$ and $d_P = e^{\rho_P}$ on $A_P$. Multiplication by the function $d_P$ induces a topological linear isomorphism from $C^\infty(\mathrm{X}_{1P})$ onto itself; moreover, if $m \in M_{1P}$, then $L_m^{-1} \circ d_P \circ L_m = d_P(m)\, d_P$. It follows that conjugation by $d_P$ induces a linear automorphism of $\mathbb{D}(\mathrm{X}_{1P})$. Accordingly, for $D \in \mathbb{D}(\mathrm{X})$ we define the differential operator

$$\mu_P(D) := d_P^{-1} \circ {}^{`}\mu_P(D) \circ d_P \in \mathbb{D}(\mathrm{X}_{1P}).$$

Let $\mathfrak{b}$ be a $\theta$-stable maximal abelian subalgebra of $\mathfrak{q}$, containing $\mathfrak{a}_\mathrm{q}$. Let $\gamma : \mathbb{D}(\mathrm{X}) \to I(\mathfrak{b})$ be the Harish-Chandra isomorphism introduced in Section 4 and let $\gamma^{\mathrm{X}_{1P}} : \mathbb{D}(\mathrm{X}_{1P}) \to I_P(\mathfrak{b})$ be the similar isomorphism for the space $\mathrm{X}_{1P}$; here $I_P(\mathfrak{b})$ denotes the subalgebra of $W(\mathfrak{m}_{1P\mathbb{C}}, \mathfrak{b})$-invariants in $S(\mathfrak{b})$. Since $\mathrm{X}_{1P}$ depends on $P$ through its $\sigma$-split component $\mathfrak{a}_{P\mathrm{q}}$, the same holds for the isomorphism $\gamma^{\mathrm{X}_{1P}}$.

**Lemma 9.1** *The map $\mu_P$ is an injective algebra isomorphism from $\mathbb{D}(\mathrm{X})$ into $\mathbb{D}(\mathrm{X}_{1P})$ which depends on $P$ through its split component $\mathfrak{a}_{P\mathrm{q}}$. Moreover,*

$$\gamma^{\mathrm{X}_{1P}} \circ \mu_P = \gamma. \tag{9.2}$$

Equation (9.2) is proved in the same fashion as the analogous result for Riemannian symmetric spaces. See [6], Sect. 2, for details. The remaining assertions readily follow.

The multiplication map $M_{P\sigma} \times A_{P\mathrm{q}} \to M_{1P}$ induces a diffeomorphism $\mathrm{X}_P \times A_{P\mathrm{q}} \to \mathrm{X}_{1P}$, which in turn induces an algebra isomorphism

$$\mathbb{D}(\mathrm{X}_{1P}) \simeq \mathbb{D}(\mathrm{X}_P) \otimes U(\mathfrak{a}_{P\mathrm{q}}).$$

Moreover, since $\mathfrak{a}_{P\mathrm{q}}$ is abelian, the universal enveloping algebra $U(\mathfrak{a}_{P\mathrm{q}})$ of its complexification is naturally isomorphic with the symmetric algebra $S(\mathfrak{a}_{P\mathrm{q}})$, which in turn is naturally isomorphic with the algebra $P(\mathfrak{a}_{P\mathrm{q}}^*)$ of complex polynomial functions $\mathfrak{a}_{P\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$. Accordingly, for every $D \in \mathbb{D}(\mathrm{X})$, the associated element $\mu_P(D) \in \mathbb{D}(\mathrm{X}_{1P})$ may be viewed as a $\mathbb{D}(\mathrm{X}_P)$-valued polynomial function on $\mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$; this polynomial function is denoted by

$$\lambda \mapsto \mu_P(D : \lambda).$$

If $v \in N_K(\mathfrak{a}_\mathrm{q})$, we denote the analogue of $\mu_P$ for the symmetric pair $(G, vHv^{-1})$ by $\mu_P^v$. Thus, $\mu_P^v$ is an algebra homomorphism $\mathbb{D}(G/vHv^{-1}) \to \mathbb{D}(\mathrm{X}_{1P,v})$. Conjugation by $v$ in $U(\mathfrak{g})$ naturally induces an algebra isomorphism

$\mathbb{D}(\mathrm{X}) \to \mathbb{D}(G/vHv^{-1})$, which we denote by $\mathrm{Ad}\,(v)$. We define an algebra homomorphism $\mu_{P,v} : \mathbb{D}(\mathrm{X}) \to \mathbb{D}(\mathrm{X}_{1P,v})$ by

$$\mu_{P,v} = \mu_P^v \circ \mathrm{Ad}\,(v).$$

Let $D \in \mathbb{D}(\mathrm{X})$; then by the natural isomorphism $\mathbb{D}(\mathrm{X}_{1P,v}) \simeq \mathbb{D}(\mathrm{X}_{P,v}) \otimes P(\mathfrak{a}_{P\mathrm{q}}^*)$, the operator $\mu_{P,v}(D)$ may be viewed as a $\mathbb{D}(\mathrm{X}_{P,v})$-valued polynomial function on $\mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$. As such it is denoted by $\lambda \mapsto \mu_{P,v}(D : \lambda)$.

We now recall from the text preceding Lemma 2.8 that for $\xi \in \mathrm{X}_{P,v,\mathrm{ds}}^\wedge$ the finite-dimensional space

$$\mathcal{V}_\xi = (\mathcal{H}_\xi^{-\infty})_{\mathrm{ds}}^{M_P \cap vHv^{-1}}$$

has a natural structure of $\mathbb{D}(\mathrm{X}_{P,v})$-module. The endomorphism by which the operator $\mu_{P,v}(D : \lambda)$ acts on this module is denoted by $\mu_{P,v}(D : \xi : \lambda)$. Finally, the direct sum of these endomorphisms, for $v \in {}^P\mathcal{W}$, is an endomorphism of $V(P, \xi)$, denoted by

$$\mu_P(D : \xi : \lambda) := \oplus_{v \in {}^P\mathcal{W}} \quad \mu_{P,v}(D : \xi : \lambda).$$

The following result is a straightforward consequence of Lemma 2.8.

**Lemma 9.2** *The space $V(P, \xi)$ has a basis, subordinate to the decomposition (8.3), with respect to which every endomorphism $\mu_P(D : \xi : \lambda)$ diagonalizes, for $D \in \mathbb{D}(\mathrm{X})$ and $\lambda \in \mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$.*

**Action on generalized vectors** We can now finally describe the action of the algebra of invariant differential operators on the $H$-fixed generalized vectors introduced in the previous section.

**Lemma 9.3** *Let $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}_{P,v,\mathrm{ds}}^\wedge$. Then for all $\eta \in V(P, \xi)$ and $D \in U(\mathfrak{g})^H$,*

$$\pi_{P,\xi,\lambda}(D) j(P : \xi : \lambda)\eta = j(P : \xi : \lambda)\mu_P(D : \xi : \lambda)\eta, \tag{9.3}$$

*as a meromorphic $C^{-\infty}(K : \xi)$-valued identity in the variable $\lambda$.*

We give a sketch of the proof. For the case that $P$ is minimal, details can be found in [5] and [6], Section 4. For general $P$, details can be found in [38], Proof of Prop. 3.

By a technique going back to F. Bruhat, [30], see also [34], it can be shown that for $\lambda$ in an open dense subset $\Omega(P, \xi)$ of $\mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$, no element $j$ of $C^{-\infty}(P : \xi : \lambda)^H$ is supported by lower dimensional $P \times H$ double cosets. Therefore, such an element is completely determined by its restriction to the open $P \times H$ double cosets. According to Proposition 8.1, the latter are parametrized by ${}^P\mathcal{W}$. On every open orbit, $j$ has equivariance properties for the transitive action by $P \times H$ and is therefore a genuine function with values in $\mathcal{H}_\xi^{-\infty}$; in particular it may be evaluated at the points of ${}^P\mathcal{W}$. We conclude that, for $\lambda \in \Omega(P, \xi)$, any element $j \in C^{-\infty}(P : \xi : \lambda)^H$ is completely determined by the values $\mathrm{ev}_v j := j(v)$, for $v \in {}^P\mathcal{W}$. Thus, by meromorphy, it suffices to check the identity (9.3) when evaluated in $v \in {}^P\mathcal{W}$, for generic $\lambda \in \Omega(P, \xi)$. It follows from (8.4) and Theorem 8.5 that $\mathrm{ev}_v j(P : \xi : \lambda)\eta = \eta_v$. Hence, evaluation in $v$ of the right-hand side of (9.3) yields $\mu_{P,v}(D : \xi : \lambda)\eta_v$.

On the other hand, combining the equivariance properties of $j(P : \xi : \lambda)$ with the definition of $\mu_{P,v}(D : \xi : \lambda)$ given earlier in this section, we infer, writing $\pi_\lambda := \pi_{P,\xi,\lambda}$, that

$$\begin{aligned}
\mathrm{ev}_v\, \pi_\lambda(D)\, j(P : \xi : \lambda)\eta &= \mathrm{ev}_e\, \pi_\lambda(\mathrm{Ad}\, v(D))\, \pi_\lambda(v)\, j(P : \xi : \lambda)\eta \\
&= \mathrm{ev}_e\, \pi_\lambda({}^{`}\mu_P^v(\mathrm{Ad}\,(v)D))\, \pi_\lambda(v)\, j(P : \xi : \lambda)\eta \\
&= \mu_{P,v}(D : \xi : \lambda)\, \mathrm{ev}_e\, \pi_\lambda(v)\, j(P : \xi : \lambda)\eta \\
&= \mu_{P,v}(D : \xi : \lambda)\eta_v.
\end{aligned}$$

□

**Example 9.4** (Riemannian case) In the notation of Example 8.10, the above formula (9.3) becomes $\pi_{P,1,\lambda}(D)\mathbf{1}_\lambda = \gamma(D : \lambda)\mathbf{1}_\lambda$, for $D \in \mathbb{D}(G/K)$.

# 10 The Plancherel theorem

**Normalization of measures** In this section we will formulate the Plancherel theorem for the symmetric space $\mathrm{X} = G/H$ in the sense of representation theory. We first need to describe the precise relations between the normalizations of the measures that come into play.

First, we equip $i\mathfrak{a}_\mathrm{q}^*$ with a $W$-invariant positive definite inner product so that it becomes a Euclidean space. For each $P \in \mathcal{P}_\sigma$ this inner product restricts to a positive definite inner product on $i\mathfrak{a}_{P\mathrm{q}}^*$. The associated Euclidean Lebesgue measure is denoted by $d\mu_P$. Similarly, the orthocomplement $i^*\mathfrak{a}_{P\mathrm{q}}^*$, see also (8.2), is equipped with the Euclidean Lebesgue measure $d\lambda_P$. Accordingly, the product measure $d\lambda_P d\mu_P$ equals the Euclidean Lebesgue measure on $i\mathfrak{a}_\mathrm{q}^*$.

On each group ${}^*A_{P\mathrm{q}}$, let $dm_P$ denote a choice of Haar measure. In terms of this Haar measure we may define an associated Euclidean Fourier transform by

$$\hat{f}(\lambda) = \int_{{}^*A_{P\mathrm{q}}} f(a)\, a^{-\lambda}\, dm_P(a).$$

We fix $dm_P$ uniquely by the requirement that the associated Fourier transform extends to an isometry from $L^2({}^*A_{P\mathrm{q}}, dm_P(a))$ onto $L^2(i^*\mathfrak{a}_{P\mathrm{q}}^*, |W_P| d\lambda_P)$.

We recall from the text following (8.2) that ${}^*A_{P\mathrm{q}}$ is the analogue of $A_\mathrm{q}$ for the symmetric spaces $\mathrm{X}_{P,v} = M_P/M_P \cap vHv^{-1}$, for $v \in {}^P\mathcal{W}$. Finally, we agree to fix the normalization of the invariant measure $dx_{P,v}$ on $\mathrm{X}_{P,v}$ so that $dm_P$ is the invariant measure of ${}^*A_{P\mathrm{q}}$ specified in Theorem 3.9 applied to the data $\mathrm{X}_{P,v}$, $K \cap M_P$ and ${}^*A_{P\mathrm{q}}$.

**Fourier transform** The first step towards the Plancherel decomposition is the introduction of a suitable Fourier transform. This is done in terms of the $H$-fixed generalized vectors introduced in Section 8.

**Definition 10.1** The (unnormalized) *Fourier transform* ${}^u\hat{f}$ of a function $f \in C_c^\infty(\mathrm{X})$ is defined by

$$
{}^u\hat{f}(P:\xi:\lambda) = \int_{G/H} f(x)\, \pi_{P,\xi,\lambda}(x) j(P:\xi:\lambda)\, dx \tag{10.1}
$$

for $P \in \mathcal{P}_\sigma$, $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ and generic $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$.

We note that the Fourier transform in (10.1) belongs to the Hilbert space $\mathrm{Hom}\,(V(P,\xi), \mathcal{H}_{P,\xi,\lambda})$ which is canonically identified with $\overline{V(P,\xi)} \otimes \mathcal{H}_{P,\xi,\lambda}$. The tensor product of the trivial and the principal series representation on these spaces, respectively, is denoted by $1 \otimes \pi_{P,\xi,\lambda}$.

**Example 10.2** (Riemannian case) In the notation of Example 8.10, we find that

$$
{}^u\hat{f}(P:1:\lambda)(k) = \int_{G/K} f(x) \mathbf{1}_\lambda(kx)\, dx, \quad (k \in K),
$$

the usual formula for the Fourier transform of $G/K$.

The following result is an immediate consequence of Definition 10.1.

**Lemma 10.3** *Let $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$. For generic $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$, the map*

$$
f \mapsto {}^u\hat{f}(P:\xi:\lambda)
$$

*intertwines the regular representation $L$ with the representation $1 \otimes \pi_{P,\xi,\lambda}$.*

The next result is a straightforward consequence of Definition 10.1 combined with Lemma 9.3.

**Lemma 10.4** *Let $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$. Then for all $D \in \mathbb{D}(\mathrm{X})$ and all $f \in C_c^\infty(\mathrm{X})$,*

$$
{}^u\widehat{(Df)}(P:\xi:\lambda) = {}^u\hat{f}(P:\xi:\lambda) \circ \mu_P(D^t:\xi:\lambda)
$$

*for generic $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.*

It follows from the above lemma combined with Lemma 9.2 that the action of the algebra $\mathbb{D}(\mathrm{X})$ of invariant differential operators allows for a simultaneous diagonalization on the Fourier transform side.

The Fourier transform defined above is one of the ingredients of the Plancherel decomposition. To define the Plancherel measure, we need to introduce the so-called standard intertwining operators.

Let $P, Q \in \mathcal{P}_\sigma$ and assume that $A_{P\mathrm{q}} = A_{Q\mathrm{q}}$. Then also $M_P = M_Q$ and $A_P = A_Q$. Let $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}} = \mathrm{X}^\wedge_{Q,*,\mathrm{ds}}$. Write

$$
\Sigma(Q:P) = \{\alpha \in \Sigma \mid \mathfrak{g}_\alpha \subset \bar{\mathfrak{n}}_Q \cap \mathfrak{n}_P\}\,.
$$

Then for $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with $\langle \mathrm{Re}\,\lambda\,,\,\alpha\rangle$ sufficiently large for each $\alpha \in \Sigma(Q:P)$, the following integral converges absolutely, for $f \in C^\infty(P:\xi:\lambda)$ and $x \in G$,

$$A(Q:P:\xi:\lambda)f(x) = \int_{N_Q \cap \overline{N}_P} f(nx)dn\,. \tag{10.2}$$

Here $dn$ is suitably normalized Haar measure of $N_Q \cap \overline{N}_P$. Moreover, it can be shown that $A(Q:P:\xi:\lambda)$ defined above is a continuous linear operator $C^\infty(P:\xi:\lambda) \to C^\infty(Q:\xi:\lambda)$, intertwining the representations $\pi_{P,\xi,\lambda}$ and $\pi_{Q,\xi,\lambda}$. Finally, $A(Q:P:\xi:\lambda)$ can be meromorphically extended in the parameter $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$. For details, we refer the reader to [66] and [86].

For every $P \in \mathcal{P}_\sigma$ we put

$$\mathfrak{a}^{*\mathrm{reg}}_{P\mathrm{q}} = \{\lambda \in \mathfrak{a}^*_{P\mathrm{q}} \mid \langle \lambda\,,\,\alpha\rangle \neq 0\;\forall \alpha \in \Sigma(P)\}\,.$$

**Theorem 10.5** *Let $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$. Then for every $\lambda \in i\mathfrak{a}^{*\mathrm{reg}}_{P\mathrm{q}}$, the representation $\pi_{P,\xi,\lambda}$ is irreducible (unitary).*

*Proof.* If $P$ is a minimal element of $\mathcal{P}_\sigma$, this follows from a result of Bruhat, [30]. For $P$ nonminimal it follows from a result of Harish-Chandra, see [67]. The application of Harish-Chandra's result requires the information on $\xi$ provided by Theorems 5.4 and 5.6, see [22], Thm. 10.7. $\square$

We retain the notation introduced in the text preceding Theorem 10.5. The standard intertwining operator has an adjoint

$$A(Q:P:\xi:-\overline{\lambda})^* : C^{-\infty}(Q:\xi:\lambda) \to C^{-\infty}(P:\xi:\lambda)$$

which depends meromorphically on $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ and is $G$-equivariant. This adjoint equals the continuous linear extension of $A(P:Q:\xi:\lambda)$ and will therefore be denoted by $A(P:Q:\xi:\lambda)$ as well. The operator

$$A(\overline{P}:P:\xi:-\bar{\lambda})^* \circ A(\overline{P}:P:\xi:\lambda) \tag{10.3}$$

is a $G$-intertwining operator from $C^\infty(P:\xi:\lambda)$ to $C^\infty(P:\xi:\lambda)$ for generic $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$, hence a scalar by the above theorem. By meromorphy it follows that (10.3) equals

$$\eta(P:\xi:\lambda)\,I \tag{10.4}$$

with $\eta(P:\xi:\cdot\,) : \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}} \to \mathbb{C}$ a meromorphic function. From the fact that (10.4) equals the composed map in (10.3) it follows that $\eta \geq 0$ on $i\mathfrak{a}^*_{P\mathrm{q}}$. Hence $\eta(P:\xi:\cdot\,)^{-1}$ defines a measurable function on $i\mathfrak{a}^*_{P\mathrm{q}}$ with values in $[0,\infty[$. At this point the function $\eta(P:\xi)$ is only defined up to a positive scalar, due to the fact that no precise normalization of the Haar measure $d\bar{n}_P$ of $\bar{N}_P$ has been specified. In what follows the precise normalization will be of importance. Let $\varphi_P : G \to ]\,0,\infty\,[$ be the function defined by $\varphi_P(namk) = a^{2\rho_P}$, for $nam \in P$ and $k \in K$. Thus, $\varphi_P \in C^\infty(P:1:\rho_P)$. We fix the normalization of our measure by requiring that

$$[A(\bar{P}:P:1:\rho_P)\varphi_P](1) = \int_{\bar{N}_P} \varphi_P(\bar{n})\,d\bar{n}_P = 1,$$

where the integral is known to converge absolutely. We now define the measure $d\mu_{P,\xi}$ on $i\mathfrak{a}_{P\mathrm{q}}^*$ by

$$d\mu_{P,\xi}(\lambda) := \frac{1}{\eta(P:\xi:\lambda)}\, d\mu_P(\lambda)\,, \tag{10.5}$$

where $d\mu_P$ is Lebesgue measure on $i\mathfrak{a}_{P\mathrm{q}}^*$, normalized as described in the first paragraph of the present section.

**Example 10.6** (Riemannian case) We use the notation of Example 8.10 and recall from the theory of Riemannian symmetric spaces that

$$A(\bar{P}:P:1:\lambda)\mathbf{1}_\lambda = c(\lambda)\ \mathbf{1}_\lambda, \tag{10.6}$$

with $c(\lambda)$ the well-known scalar $c$-function for $G/K$. In view of the definition of $\eta$, this leads to

$$\eta(P:1:\lambda) = |c(\lambda)|^2 \qquad (\lambda \in i\mathfrak{a}_\mathfrak{p}^*),$$

so that the measure $d\mu_{P,1}(\lambda)$ takes the familiar form $|c(\lambda)|^{-2}$ times Lebesgue measure.

The normalizer of $\mathfrak{a}_{P\mathrm{q}}$ in the Weyl group $W$ is denoted by $W_P^*$; recall that the centralizer in $W$ of the same set is denoted by $W_P$. We define the group

$$W(\mathfrak{a}_{P\mathrm{q}}) = W_P^*/W_P\,.$$

Restriction to $\mathfrak{a}_{P\mathrm{q}}$ induces a natural isomorphism from this group onto a subgroup of $\mathrm{GL}(\mathfrak{a}_{P\mathrm{q}})$.

**Definition 10.7** Two parabolic subgroups $P, Q \in \mathcal{P}_\sigma$ are said to be ($\sigma$–) *associated* if their $\sigma$-split components $\mathfrak{a}_{P\mathrm{q}}$ and $\mathfrak{a}_{Q\mathrm{q}}$ are conjugate under the Weyl group $W$. The equivalence relation of being associated is denoted by $\sim$.

Let $\mathbf{P}_\sigma$ be a set of representatives in $\mathcal{P}_\sigma$ for the classes of $\sim$. Thus, $\mathbf{P}_\sigma$ is in one-to-one correspondence with $\mathcal{P}_\sigma/\sim$. The following result is a first version of the Plancherel theorem.

**Theorem 10.8** *Let $f \in C_c^\infty(\mathrm{X})$. Then*

$$\|f\|_{L^2(\mathrm{X})}^2 = \sum_{P\in\mathbf{P}_\sigma} [W:W_P^*] \sum_{\xi\in\mathrm{X}^\wedge_{P,*,\mathrm{ds}}} \int_{i\mathfrak{a}_{P\mathrm{q}}^*} \|{}^u\hat{f}(P:\xi:\lambda)\|^2\, d\mu_{P,\xi}(\lambda)\,. \tag{10.7}$$

**Example 10.9** (Riemannian case) We use the notation of Example 8.10. We now need to use the information that noncompact Riemannian symmetric spaces have no discrete series, which follows from Harish-Chandra's work on the discrete series, but also from Theorem 5.4. This allows us to conclude that $\mathrm{X}^\wedge_{P,*,\mathrm{ds}} = \emptyset$, unless $P$ is a minimal parabolic subgroup containing $A_\mathfrak{p} = A_\mathrm{q}$ and $\xi = 1$. Moreover, then $V(P,1)$ equals $\mathbb{C}$, equipped with the standard inner product, and $W = W_P^*$. In view of Example 10.6 we see that (10.7) takes the usual form of the Plancherel formula for $G/K$.

Theorem 10.8 motivates the following definition of a unitary direct integral representation. First, for $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, we define the Hilbert space

$$\mathfrak{H}(P,\xi) := \overline{V(P,\xi)} \otimes L^2(K:\xi),$$

equipped with the tensor product inner product. In addition, we define

$${}^u\mathfrak{L}^2(P,\xi) := L^2(i\mathfrak{a}^*_{P\mathrm{q}},\, \mathfrak{H}(P,\xi),\, [W:W^*_P]\, d\mu_{P,\xi}),$$

the space of square integrable functions $i\mathfrak{a}^*_{P\mathrm{q}} \to \mathfrak{H}(P,\xi)$, equipped with the $L^2$-Hilbert structure associated with the indicated measure. The above space is equipped with the representation $\pi_{P,\xi}$ of $G$ given by

$$[\pi_{P,\xi}(x)\varphi](\lambda) = [1 \otimes \pi_{P,\xi,\lambda}(x)]\varphi(\nu),$$

for $\varphi \in {}^u\mathfrak{L}^2(P,\xi)$ and $x \in G$. It can be shown that $\pi_{P,\xi}$ is a continuous unitary representation of $G$. In fact, it provides a realization of a direct integral,

$$\pi_{P,\xi} \simeq \int^\oplus_{i\mathfrak{a}^*_{P\mathrm{q}}} 1_{\overline{V(P,\xi)}} \otimes \pi_{P,\xi,\lambda}\, [W:W^*_P]\, d\mu_{P,\xi}(\lambda)\,.$$

We define $(\pi_P, {}^u\mathfrak{L}^2(P))$ as the Hilbert direct sum of the unitary representations $(\pi_{P,\xi}, {}^u\mathfrak{L}^2(P,\xi))$, for $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$. Finally, we define $(\pi, {}^u\mathfrak{L}^2)$ as the unitary direct sum of the representations $(\pi_P, {}^u\mathfrak{L}^2(P))$, for $P \in \mathbf{P}_\sigma$. Thus,

$$\pi \simeq \oplus_{P\in\mathbf{P}_\sigma} \widehat{\oplus}_{\xi\in\mathrm{X}^\wedge_{P,*,\mathrm{ds}}}\ \pi_{P,\xi}.$$

The following result is now a straightforward consequence of Theorem 10.8.

**Corollary 10.10** *$f \mapsto {}^u\hat{f}$ extends to an isometry ${}^u\mathfrak{F}$ from $L^2(\mathrm{X})$ into the Hilbert space ${}^u\mathfrak{L}^2$, intertwining the representation $L$ with the representation $\pi$.*

**Remark 10.11** The reason for the summation over $\mathbf{P}_\sigma$ rather than $\mathcal{P}_\sigma$ is that the principal series for associated $P, Q \in \mathcal{P}_\sigma$ are related by intertwining operators, as we shall now explain.

First, assume that $\mathfrak{a}_{P\mathrm{q}} = \mathfrak{a}_{Q\mathrm{q}}$. Then the standard intertwining operator $A(Q:P:\xi:\lambda)$ intertwines the representations $\pi_{P,\xi,\lambda}$ and $\pi_{Q,\xi,\lambda}$ for $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}} = \mathrm{X}^\wedge_{Q,*,\mathrm{ds}}$ and generic $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$.

If $Q = wPw^{-1}$ for some Weyl group element $w \in W$, then there also exists an intertwining operator between principal series representations for $P$ and $Q$. It is defined as follows. We observe that $wM_Pw^{-1} = M_Q$. Hence if $\xi \in \widehat{M_P}$, then the representation $w\cdot\xi$ defined by $w\xi(m) = \xi(w^{-1}mw)$ belongs to $\widehat{M_Q}$ (here we have abused notation, $w$ should be replaced by a representative in $N_K(\mathfrak{a}_\mathrm{q})$). Now the map $L(w)$ given by

$$L(w)\varphi(x) = \varphi(w^{-1}x)$$

defines an intertwining operator from $\mathcal{H}_{P,\xi,\lambda}$ to $\mathcal{H}_{Q,w\xi,w\lambda}$ which is obviously unitary.

Finally, if ${}^P\mathcal{W}$ is a set of representatives for $W_P\backslash W/W_{K\cap H}$ in $N_K(\mathfrak{a}_{\mathrm{q}})$, then ${}^Q\mathcal{W} = w^P\mathcal{W}$ is a set of representatives for $W_Q\backslash W/W_{K\cap H}$. This implies that $\xi \mapsto w\xi$ is a bijection from $\mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ onto $\mathrm{X}^\wedge_{Q,*,\mathrm{ds}}$.

In general, if $Q \sim P$, then there exists a $P' \in \mathcal{P}_\sigma$ with $\mathfrak{a}_{P\mathrm{q}} = \mathfrak{a}_{P'\mathrm{q}}$ and $Q = wP'w$ for some $w \in W$. From the above two cases we see that the principal series for $P$ and for $Q$ are related by intertwining operators.

We have called Theorem 10.8 a preliminary version of the Plancherel theorem, since it does not yet describe the image of ${}^u\mathfrak{F}$. In fact, ${}^u\mathfrak{F}$ is not onto ${}^u\mathfrak{L}^2$, due to the presence of intertwining operators. These intertwining operators are also the cause of double occurrences of irreducible representations in the direct integral representation $\pi$.

To be more precise, let $w \in W_P^*$. Then the operator $L(w)$, introduced in Remark 10.11, intertwines $\pi_{P,\xi,\lambda}$ with $\pi_{wPw^{-1},w\xi,w\lambda}$. Since

$$w(\mathfrak{a}_{P\mathrm{q}}) = \mathfrak{a}_{wPw^{-1}\mathrm{q}}\,,$$

the latter representation is intertwined with $\pi_{P,w\xi,w\lambda}$ by the standard intertwining operator $A(P : wPw^{-1} : w\xi : w\lambda)$, see Remark 10.11. The following result implies that more than this cannot happen.

**Proposition 10.12** *For $j = 1, 2$, let $P_j \in \mathbf{P}_\sigma$, $\xi_j \in \mathrm{X}^\wedge_{P_j,*,ds}$ and $\lambda_j \in i\mathfrak{a}^{*\mathrm{reg}}_{P_j\mathrm{q}}$. Then the representations $\pi_{P_1,\xi_1,\lambda_1}$ and $\pi_{P_2,\xi_2,\lambda_2}$ are equivalent if and only if $P_1 = P_2$ and if there exists a $w \in W(\mathfrak{a}_{P_1\mathrm{q}})$ such that $\xi_2 = w\xi_1$ and $\lambda_2 = w\lambda_1$.*

*Proof.* This result, which is closely related to Theorem 10.5, is due to Harish-Chandra, see [67] and [22], Prop. 10.8, for details. □

The next result describes the effect on the Fourier transform of the intertwining operators mentioned above.

**Proposition 10.13** *Let $P \in \mathcal{P}_\sigma$, $w \in W_P^*$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$. Then $w\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, and $d\mu_{P,w\xi}(w\lambda) = d\mu_{P,\xi}(\lambda)$. Moreover, there exists a unique unitary isomorphism ${}^u\mathfrak{C}_{P,w}(\xi,\lambda)$ from $\overline{V(P,\xi)} \otimes L^2(K:\xi)$ onto $\overline{V(P,w\xi)} \otimes L^2(K : w\xi)$, depending on $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$ in a measurable way, such that*

$${}^u\hat{f}(P : w\xi : w\lambda) = {}^u\mathfrak{C}_{P,w}(\xi,\lambda)\, {}^u\hat{f}(P : \xi : \lambda)\,.$$

*The operator ${}^u\mathfrak{C}_{P,w}(\xi,\lambda)$ intertwines $1 \otimes \pi_{P,\xi,\lambda}$ with $1 \otimes \pi_{P,w\xi,w\lambda}$. Moreover, if $u, v \in W_P^*$, then*

$${}^u\mathfrak{C}_{P,uv}(\xi,\lambda) = {}^u\mathfrak{C}_{P,u}(v\xi, v\lambda){}^u\mathfrak{C}_{P,v}(\xi,\lambda),$$

*for almost every $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$.*

It follows from the above proposition that for $P, w, \xi$ as above we may define a unitary operator ${}^u\mathfrak{C}_{P,w}(\xi) : {}^u\mathfrak{L}^2(P,\xi) \to {}^u\mathfrak{L}^2(P, w\xi)$ by

$${}^u\mathfrak{C}_{P,w}(\xi)\varphi(\lambda) = {}^u\mathfrak{C}_{P,w}(\xi, w^{-1}\lambda)\varphi(w^{-1}\lambda).$$

The direct sum of these operators, for $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, defines a unitary operator ${}^u\mathfrak{C}_{P,w}$ from

$$ {}^u\mathfrak{L}^2(P) := \widehat{\oplus}_{\xi\in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}}\ {}^u\mathfrak{L}^2(P,\xi) $$

onto itself, intertwining the representation $\pi_P := \widehat{\oplus}_\xi\, \pi_{P,\xi}$ with itself. Moreover, $w \mapsto {}^u\mathfrak{C}_{P,w}$ defines a unitary representation of $W_P^*$ in ${}^u\mathfrak{L}^2(P)$. The direct sum of these representations, for $P \in \mathbf{P}_\sigma$, defines a unitary representation of $W_P^*$ in ${}^u\mathfrak{L}^2$, commuting with the representation $\pi$ of $G$. Accordingly, the space of $W_P^*$-invariants is a closed invariant subspace of ${}^u\mathfrak{L}^2$.

**Proposition 10.14** *The image of the map ${}^u\mathfrak{F}$, see Corollary 10.10, is given by*

$$ \mathrm{image}({}^u\mathfrak{F}) = ({}^u\mathfrak{L}^2)^{W_P^*}. $$

The group $W(\mathfrak{a}_{P\mathrm{q}}) = W_P^*/W_P$ acts freely, but in general not transitively, on the components of $\mathfrak{a}_{P\mathrm{q}}^{*\mathrm{reg}}$, which are the facets in $\mathfrak{a}_\mathrm{q}^*$ whose spans equal $\mathfrak{a}_{P\mathrm{q}}^*$.

Let $\Omega_P$ be a fundamental domain for the action of $W(\mathfrak{a}_{P\mathrm{q}})$ on $i\mathfrak{a}_{P\mathrm{q}}^{*\mathrm{reg}}$, consisting of connected components of $i\mathfrak{a}_{P\mathrm{q}}^{*\mathrm{reg}}$. Then, for each $P \in \mathcal{P}_\sigma$ and every $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, we denote by ${}^u\mathfrak{L}^2_{\Omega_P}(P,\xi)$ the closed $G$-invariant subspace of functions in ${}^u\mathfrak{L}^2(P,\xi)$ that vanish almost everywhere outside $\Omega_P$. Finally, we define the following closed $G$-invariant subspace of ${}^u\mathfrak{L}^2$,

$$ {}^u\mathfrak{L}^2_0 := \oplus_{P\in\mathbf{P}_\sigma} \widehat{\oplus}_{\xi\in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}}\ {}^u\mathfrak{L}^2_{\Omega_P}(P,\xi). $$

The orthogonal projection onto this subspace is denoted by $\varphi \mapsto \varphi_0$. After these preparations we can now describe the Plancherel decomposition induced by the unnormalized Fourier transform.

**Theorem 10.15** (Plancherel theorem) *The map $f \mapsto ({}^u\mathfrak{F}f)_0$ defines an isometry from $L^2(\mathrm{X})$ onto ${}^u\mathfrak{L}^2_0$, intertwining $L$ with $\pi$, establishing the Plancherel decomposition*

$$ L \sim \oplus_{P\in\mathbf{P}_\sigma}\ \widehat{\oplus}_{\xi\in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}} \int_{\Omega_P} 1_{\overline{V(P,\xi)}} \otimes \pi_{P,\xi,\lambda} \quad [W:W_P]\, d\mu_{P,\xi}(\lambda)\,. \qquad (10.8) $$

*In particular, for all $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, the representation $\pi_{P,\xi,\lambda}$ occurs with multiplicity* $\dim V(P,\xi)$, *for almost every $\lambda \in i\mathfrak{a}_{P\mathrm{q}}^*$.*

*Proof.* The fact that ${}^u\mathfrak{F}$ establishes a direct integral decomposition follows from Theorem 10.8 combined with Corollary 10.10 and Proposition 10.13. The occurring representations are almost all irreducible by Theorem 10.5, and almost all inequivalent by Proposition 10.12. □

**Remark 10.16** It follows from Lemma 10.4 and Lemma 9.2 that the action of the algebra $\mathbb{D}(\mathrm{X})$ on $C_c^\infty(\mathrm{X})$ transfers via ${}^u\mathfrak{F}$ to an action on the space on the right-hand side of (10.8), which respects the direct integral decomposition and allows a compatible simultaneous diagonalization.

The fact that the functions $\lambda \mapsto {}^u\hat{f}(P : \xi : \lambda)$ may have singularities on $i\mathfrak{a}_{Pq}^*$, even for $f \in C_c^\infty(\mathrm{X})$, is no problem from the Hilbert space direct integral point of view. However, in the proofs that will be described later, it will turn out to be crucial to have a different normalization of the Fourier transform available. The normalized Fourier transform is going to be defined as in Definition 10.1, but with $j(P : \xi : \lambda)$ replaced by a differently normalized element of $\mathrm{Hom}\,(V(P,\xi), C^{-\infty}(P : \xi : \lambda)^H)$.

**Definition 10.17** Let $P \in \mathcal{P}_\sigma$, $\xi \in \mathrm{X}_{P,*,\mathrm{ds}}^\wedge$. We define

$$j^\circ(P : \xi : \lambda) = A(\bar{P} : P : \xi : \lambda)^{-1} j(\bar{P} : \xi : \lambda)$$

as a meromorphic $\mathrm{Hom}\,(V(P,\xi), C^{-\infty}(K : \xi))$-valued function of $\lambda \in \mathfrak{a}_{Pq\mathbb{C}}^*$.

From the fact that $\mu_P = \mu_{\bar{P}}$, see Lemma 9.1, it is readily seen that Lemma 9.3 is valid with $j^\circ$ in place of $j$.

The normalization of $j$ in Definition 10.17 is motivated by the following remarkable property.

**Theorem 10.18** (Regularity theorem) *The* $\mathrm{Hom}\,(V(P,\xi), C^{-\infty}(K : \xi))$-*valued meromorphic function* $\lambda \mapsto j^\circ(P : \xi : \lambda)$ *is regular on* $i\mathfrak{a}_{Pq}^*$.

The proof Theorem 10.18 is based on a similar result, formulated in the next section, for the so-called normalized Eisenstein integral. For $P$ a minimal $\sigma$-parabolic subgroup the result is due to [15]. For general $P$ it is due to [35].

**Example 10.19** (Riemannian case) We use the notation of Example 8.10. In view of (10.6), it follows that

$$j^\circ(P : \xi : \lambda)(1) = c(\lambda)^{-1}\mathbf{1}_\lambda. \tag{10.9}$$

Thus, in this case the regularity theorem amounts to the well-known fact that the $c$-function has no zeros on $i\mathfrak{a}_\mathfrak{p}^*$.

We now define the normalized Fourier transform $\hat{f}$ of a function $f \in C_c^\infty(\mathrm{X})$ as ${}^u\hat{f}$, but with everywhere $j(P : \xi : \lambda)$ replaced by $j^\circ(P : \xi : \lambda)$. Then Theorem 10.18 has the following immediate consequence.

**Corollary 10.20** *Let* $P \in \mathcal{P}_\sigma$ *and* $\xi \in \mathrm{X}_{P,*,\mathrm{ds}}^\wedge$. *Then for every* $f \in C_c^\infty(\mathrm{X})$, *the function* $\lambda \mapsto \hat{f}(P : \xi : \lambda)$ *is analytic as a* $\overline{V(P,\xi)} \otimes C^\infty(K : \xi)$- *valued function on* $i\mathfrak{a}_{Pq}^*$.

A simple calculation leads to the following relation between ${}^u\hat{f}$ and $\hat{f}$, for $f \in C_c^\infty(\mathrm{X})$:

$$\hat{f}(P : \xi : \lambda) = [I \otimes A(\bar{P} : P : \xi : \lambda)^{-1}]\, {}^u\hat{f}(\bar{P} : \xi : \lambda).$$

From this it readily follows that

$$\|\hat{f}(P : \xi : \lambda)\|^2 = \eta(\bar{P} : \xi : \lambda)^{-1}\, \|{}^u\hat{f}(\bar{P} : \xi : \lambda)\|^2\,. \tag{10.10}$$

This relation has the effect that the normalized Fourier transform induces a Plancherel decomposition with Plancherel measure equal to ordinary Lebesgue measure. Indeed, let $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$; then, for $f \in C_c^\infty(\mathrm{X})$,

$$\|{}^u\hat{f}(P : \xi : \lambda)\|^2 \, d\mu_{P,\xi}(\lambda) = \|\hat{f}(P : \xi : \lambda)\|^2 \, d\mu_P.$$

It follows that Theorem 10.8 is equivalent to a similar result for the normalized Fourier transform $f \mapsto \hat{f}$ with $d\mu_P(\lambda)$ in place of $d\mu_{P,\xi}(\lambda)$.

For $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, let $\mathfrak{L}^2(P,\xi)$ be defined as ${}^u\mathfrak{L}^2(P,\xi)$, but with the Lebesgue measure $d\mu_P(\lambda)$ in place of $d\mu_{P,\xi}(\lambda)$. Thus,

$$\mathfrak{L}^2(P,\xi) := L^2(i\mathfrak{a}^*_{P\mathrm{q}},\, \mathfrak{H}(P,\xi),\, [W : W^*_P]\, d\mu_P). \tag{10.11}$$

Moreover, let $(\pi, \mathfrak{L}^2)$ be defined accordingly. Then Corollary 10.10 is equivalent to the similar result involving the unnormalized Fourier transform and $(\pi, \mathfrak{L}^2)$. We denote the continuous linear extension of the normalized Fourier transform by $\mathfrak{F} : L^2(\mathrm{X}) \to \mathfrak{L}^2$. Proposition 10.13 is now equivalent to a normalized version, with different intertwining operators $\mathfrak{C}_{P,w}(\xi : \lambda) : \mathfrak{H}(P,\xi) \to \mathfrak{H}(P, w\xi)$. In view of (10.10), the connection of these intertwining operators with their unnormalized analogues is given by

$${}^u\mathfrak{C}_{\bar{P},w}(\xi : \lambda) \circ [1 \otimes A(\bar{P} : P : \xi : \lambda)] = [1 \otimes A(\bar{P} : P : w\xi : w\lambda)]\mathfrak{C}_{P,w}(\xi : \lambda).$$

As before we define a unitary representation of $W(\mathfrak{a}_{P\mathrm{q}})$ in $\mathfrak{L}^2$ commuting with $\pi$, so that Proposition 10.14 is equivalent to its normalized analogue. Finally, we fix fundamental domains $\Omega_P \subset i\mathfrak{a}^{*\mathrm{reg}}_{P\mathrm{q}}$ for the action of $W(\mathfrak{a}_{P\mathrm{q}})$, and define $\mathfrak{L}^2_0$ in a similar fashion as ${}^u\mathfrak{L}^2_0$. Then $\mathfrak{L}^2_0$ is a closed $G$-invariant subspace of $\mathfrak{L}^2$. The orthogonal projection onto it is denoted by $\varphi \mapsto \varphi_0$. Theorem 10.15 is now equivalent to the following normalized analogue.

**Theorem 10.21** (Normalized version of the Plancherel theorem) *The map $f \mapsto (\mathfrak{F}f)_0$ defines an isometry from $L^2(\mathrm{X})$ onto $\mathfrak{L}^2_0$, intertwining $L$ with $\pi$, establishing the Plancherel decomposition*

$$L \sim \oplus_{P \in \mathbf{P}_\sigma} \widehat{\oplus}_{\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}} \int_{i\Omega_P} 1_{\overline{V(P,\xi)}} \otimes \pi_{P,\xi,\lambda} \quad [W : W_P]\, d\mu_P(\lambda)\,.$$

**Remark 10.22** Since Lemma 9.3 is valid with $j^\circ$ in place of $j$, it follows that Lemma 10.4 is valid with $f \mapsto \hat{f}$ in place of $f \mapsto {}^u\hat{f}$. Therefore, the obvious analogue of Remark 10.16 is valid for $\mathfrak{F}$.

**Remark 10.23** In Delorme's paper [40] the above formula occurs without the constants $[W : W_P]$. This is due to a different normalization of measures. More precisely, Delorme uses the normalization of measures described in the first paragraph of this section, with the exception of the Lebesgue measure $d\lambda_P$ on $i^*\mathfrak{a}^*_{P\mathrm{q}}$, which he normalizes by the requirement that the Euclidean Fourier transform of ${}^*A_{P\mathrm{q}}$ extend to an isometry $L^2({}^*A_{P\mathrm{q}}, dm_P) \to L^2(i^*\mathfrak{a}^*_{P\mathrm{q}}, d\lambda_P)$.

## 11 The spherical Plancherel theorem

**The Eisenstein integral** A main step towards proving the Plancherel theorem consists of proving the analogue of Theorem 10.8 for the normalized Fourier transform $f \mapsto \hat{f}$. It suffices to do this on the subspace of functions of a specific left $K$-type. To be more precise, let $\delta \in \widehat{K}$ and let $C_c^\infty(\mathrm{X})_\delta$ be the subspace of $C_c^\infty(\mathrm{X})$ consisting of left $K$-finite functions of type $\delta$. Then it suffices to show that $f \mapsto \hat{f}$ is an isometry from $C_c^\infty(\mathrm{X})_\delta$, equipped with the inner product of $L^2(\mathrm{X})$, to the space $\mathfrak{L}^2_\delta$ of $K$-finite elements of type $\delta$ in $\mathfrak{L}^2$. The restriction of the Fourier transform to $C_c^\infty(\mathrm{X})_\delta$ naturally leads to the concept of Eisenstein integral, as we will now explain.

We start by observing that

$$\begin{aligned} L^2(\mathrm{X})_\delta &\simeq \mathrm{Hom}_K(V_\delta, L^2(\mathrm{X})) \otimes V_\delta \\ &\simeq (L^2(\mathrm{X}) \otimes V_\delta^*)^K \otimes V_\delta\,. \end{aligned}$$

Here and in what follows, unspecified isomorphisms are assumed to be the obvious natural ones. Put $\tau = \tau_\delta := \delta^* \otimes 1$ and $V_\tau = V_{\tau_\delta} := V_\delta^* \otimes V_\delta$; then it follows that

$$\begin{aligned} L^2(\mathrm{X})_\delta &\simeq (L^2(\mathrm{X}) \otimes V_\tau)^K \\ &= L^2(\mathrm{X} : \tau)\,, \end{aligned}$$

where the latter space is the space of functions $\varphi$ in $L^2(\mathrm{X}, V_\tau)$ that are $\tau$-spherical, i.e.,

$$\varphi(kx) = \tau(k)\varphi(x)\,, \qquad (x \in \mathrm{X}\,,\, k \in K)\,. \tag{11.1}$$

Similar considerations lead to analogous definitions for spaces of spherical functions associated with $C_c^\infty(\mathrm{X})$, $C^\infty(\mathrm{X})$, $C(\mathrm{X})$. In the $L^2$-context all the natural isomorphisms are isometric if we agree to equip $V_\delta^* \otimes V_\delta \simeq \mathrm{End}\,(V_\delta)$ with $d_\delta^{-1}$ times the Hilbert–Schmid inner product. Note that so far we have used nothing special about X; the whole construction applies to a manifold X equipped with a smooth $K$-action and a $K$-invariant density.

The natural isomorphism $L^2(\mathrm{X})_\delta \to L^2(\mathrm{X} : \tau_\delta)$ will be called sphericalization, and is denoted by

$$f \mapsto f^{\mathrm{sph}}\,. \tag{11.2}$$

We recall from the general considerations in Section 2 that the Fourier transform $f \mapsto \hat{f}(P : \xi : \lambda)$ may also be given by testing with a matrix coefficient. In the present context, let $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$. Then for generic $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$ we define the map

$$\mathrm{M}_{P,\xi,\lambda} :\ \overline{V(P,\xi)} \otimes C^\infty(K : \xi) \to C^\infty(\mathrm{X}) \tag{11.3}$$

by the following formula, for $\eta \otimes \varphi \in \overline{V(P,\xi)} \otimes C^\infty(K : \xi)$ and $x \in \mathrm{X}$,

$$\mathrm{M}_{P,\xi,\lambda}(\eta \otimes \varphi)(x) = \langle \varphi\,,\ \pi_{P,\xi,-\overline{\lambda}}(x)\, j^\circ(P : \xi : -\overline{\lambda})\eta \rangle.$$

Here $j^\circ(P : \xi : -\overline{\lambda})$ is as in Definition 10.17. From the definition of the normalized Fourier transform, we now obtain, for $f \in C_c^\infty(\mathrm{X})$ and $T \in \overline{V(P,\xi)} \otimes C^\infty(K : \xi)$, that

$$\langle \hat{f}(P : \xi : \lambda)\,,\, T\rangle = \langle f\,,\, \mathrm{M}_{P,\xi,-\overline{\lambda}}(T)\rangle\,. \tag{11.4}$$

We observe that $\mathrm{M}_{P,\xi,\lambda}$ intertwines the generalized principal series representation $\pi_{P,\xi,\lambda} \otimes I$ with the left regular representation $L$. In particular, it follows that $\mathrm{M}_{P,\xi,\lambda}$ maps $\overline{V(P,\xi)} \otimes C^\infty(K : \xi)_\delta$ into $C^\infty(\mathrm{X})_\delta \simeq C^\infty(\mathrm{X} : \tau_\delta)$. An Eisenstein integral is essentially an element in the image of $\mathrm{M}_{P,\xi,\lambda}$, viewed as an element of $C^\infty(\mathrm{X} : \tau_\delta)$. It becomes a very practical tool if we realize the parameter space $\overline{V(P,\xi)} \otimes C^\infty(K : \xi)_\delta$ in a different fashion.

Let $P \in \mathcal{P}_\sigma$ and $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ and let $v \in {}^P\mathcal{W}$. If $V(P,\xi,v)$ is nontrivial, then $\xi$ belongs to the discrete series of the space $\mathrm{X}_{P,v}$ defined by (8.1). In any case, we consider the natural matrix coefficient map $\overline{V(P,\xi,v)} \otimes \mathcal{H}_\xi \to L^2(\mathrm{X}_{P,v})$ and denote its image by

$$L^2(\mathrm{X}_{P,v})_\xi.$$

In particular, this space is nontrivial if and only if $\xi \in \mathrm{X}^\wedge_{P,v,\mathrm{ds}}$. We put $K_P := K \cap M_P$ and define $\tau_P := \tau_{\delta,P} = \tau_\delta|_{K_P}$. Then $L^2(\mathrm{X}_{P,v} : \tau_P) = (L^2(\mathrm{X}_{P,v}) \otimes V_\tau)^{K_P}$. Accordingly, we define

$$L^2(\mathrm{X}_{P,v} : \tau_P)_\xi := (L^2(\mathrm{X}_{P,v})_\xi \otimes V_\tau)^{K_P}\,.$$

**Lemma 11.1** *The space $\overline{V(P,\xi)} \otimes C^\infty(K : \xi)_\delta$ is finite-dimensional and equals the Hilbert space $\overline{V(P,\xi)} \otimes L^2(K : \xi)_\delta$. Moreover, there is a natural isometrical isomorphism*

$$\overline{V(P,\xi)} \otimes C^\infty(K : \xi)_\delta \ \xrightarrow{\ \simeq\ }\ \oplus_{v \in {}^P\mathcal{W}}\ L^2(\mathrm{X}_{P,v} : \tau_{\delta,P})_\xi\,,$$

*where $\oplus$ denotes the formal direct sum of Hilbert spaces.*

*Proof.* First, we note that $L^2(K : \xi)$ is the representation space for the induced representation $\mathrm{ind}_{K_P}^K(\xi \mid K_P)$. By Frobenius reciprocity we have

$$\mathrm{Hom}_K(V_\delta, L^2(K : \xi)) \simeq \mathrm{Hom}_{K_P}(V_\delta, \mathcal{H}_\xi)\,. \tag{11.5}$$

Hence,

$$\begin{aligned} L^2(K : \xi)_\delta &\simeq \mathrm{Hom}_{K_P}(V_\delta, \mathcal{H}_\xi) \otimes V_\delta \\ &\simeq (\mathcal{H}_\xi \otimes V_{\tau_\delta})^{K_P}\,. \end{aligned} \tag{11.6}$$

It is a standard fact from representation theory that each $K_P$-type occurs with finite multiplicity in $\xi \in \widehat{M}_P$. Therefore, the space in (11.6) is finite-dimensional. It follows that $L^2(K : \xi)_\delta$ is finite-dimensional, hence equals its dense subspace $C^\infty(K : \xi)_\delta$. This establishes the first two assertions.

From (11.5) it follows, for $v \in {}^P\mathcal{W}$, that

$$\begin{aligned} \overline{V(P,\xi,v)} \otimes L^2(K : \xi)_\delta &\simeq (\overline{V(P,\xi,v)}_{(1)} \otimes \mathcal{H}_\xi \otimes V_{\tau_\delta})^{K_P} \\ &\simeq (L^2(\mathrm{X}_{P,v})_\xi \otimes V_{\tau_\delta})^{K_P} \end{aligned}$$

by the matrix coefficient map of $\xi$. Here, the index (1) on a tensor component indicates that the action of the group $K_P$ is trivial on that component. The argument is completed by taking the direct sum over $v \in {}^P\mathcal{W}$. $\square$

We denote the isomorphism of Lemma 11.1 by $T \mapsto \psi_T$, a notation that is compatible with Harish-Chandra's notation in the case of the group, see [58], §7, Lemma 1.

**Definition 11.2** Let $\psi \in \oplus_{v \in {}^P\mathcal{W}} L^2(\mathrm{X}_{P,v} : \tau_{\delta,P})_\xi$. Then, for $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, the *normalized Eisenstein integral* $E^\circ(P : \psi : \lambda)$ is defined by

$$E^\circ(P : \psi : \lambda) = \mathrm{M}^{\mathrm{sph}}_{P,\xi,-\lambda}(T) \in C^\infty(\mathrm{X} : \tau_\delta)\,,$$

where $T \in \overline{V(P,\xi)} \otimes C^\infty(K : \xi)_\delta$ is such that $\psi = \psi_T$.

**Example 11.3** (Riemannian case) We use the notation of Example 8.10 and consider the case of the trivial $K$-type $\delta = 1$, so that $V_\tau = \mathbb{C}$ and $C^\infty(G/K : \tau)$ equals the space $C^\infty(K\backslash G/K)$ of bi-K-invariant smooth functions on $G$. Then $\overline{V(P,1)} \otimes C^\infty(K : 1)_\delta \simeq \mathbb{C}$. Moreover, ${}^P\mathcal{W} = \{e\}$ and $\psi_1 = 1$. In view of (10.9) it follows that in this setting the normalized Eisenstein integral is given by

$$E^\circ(P : 1 : \lambda)(x) = \int_K \overline{c(\bar\lambda)^{-1})\mathbf{1}_{\bar\lambda}(kx)}\, dk = c(\lambda)^{-1}\varphi_\lambda(x),$$

where $\varphi_\lambda$ is the zonal spherical function in $C^\infty(K\backslash G/K)$, determined by the parameter $\lambda$.

The above definition of the Eisenstein integral can be extended to a bigger $\psi$-space, by collecting all $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$ together. We need some preparation for this.

**Definition 11.4** Let $(\tau, V_\tau)$ be any finite-dimensional unitary representation of $K$. Then by $\mathcal{A}_2(\mathrm{X} : \tau)$ we denote the space of smooth functions $f \in C^\infty(\mathrm{X} : \tau)$ satisfying the following conditions:

(a) $f \in L^2(\mathrm{X} : \tau)$;

(b) $\mathbb{D}(\mathrm{X})f$ is finite-dimensional.

**Theorem 11.5** *The space $\mathcal{A}_2(\mathrm{X} : \tau)$ is finite-dimensional. Moreover, it decomposes as the orthogonal direct sum*

$$\mathcal{A}_2(\mathrm{X} : \tau) = \oplus_{\xi \in \mathrm{X}^\wedge_{\mathrm{ds}}} L^2(\mathrm{X} : \tau)_\xi\,.$$

*In particular, only finitely many summands in the direct sum are nonzero.*

*Proof.* This is a deep result, which is equivalent to the assertion that for a given $\delta \in \widehat{K}$ only finitely many representations from the discrete series of $X$ contain the $K$-type $\delta$. It follows from the classification of the discrete series by Oshima and Matsuki [77]. We will see that it also follows from our proof of the Plancherel formula, if one uses the information on the discrete series given in Theorems 5.4 and 5.6. Of course, the latter results are due to [77] as well. □

We define the finite-dimensional Hilbert space $\mathcal{A}_{2,P} = \mathcal{A}_{2,P}(\tau_\delta)$ by

$$\mathcal{A}_{2,P} := \oplus_{v \in {}^P\mathcal{W}} \mathcal{A}_2(\mathrm{X}_{P,v} : \tau_{\delta,P}) , \tag{11.7}$$

where $\oplus$ denotes the formal orthogonal direct sum of Hilbert spaces. By Theorem 11.5, applied to $\mathrm{X}_{P,v}$ for $v \in {}^P\mathcal{W}$, the space $\mathcal{A}_{2,P}$ decomposes as the orthogonal direct sum, for $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$, of the spaces

$$\mathcal{A}_{2,P,\xi} := \oplus_{v \in {}^P\mathcal{W}} L^2(\mathrm{X}_{P,v} : \tau_{\delta,P})_\xi .$$

Accordingly, given $\psi \in \mathcal{A}_{2,P}$, we write $\psi_\xi$ for the component determined by $\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}$.

**Definition 11.6** For $\psi \in \mathcal{A}_{2,P}$ we define the *normalized Eisenstein integral*L $E^\circ(P : \psi : \lambda) \in C^\infty(\mathrm{X} : \tau)$ by

$$E^\circ(P : \psi : \lambda) = \sum_{\xi \in \mathrm{X}^\wedge_{P,*,\mathrm{ds}}} E^\circ(P : \psi_\xi : \lambda) .$$

**The regularity theorem** We shall now describe the action of invariant differential operators on the Eisenstein integral. Let $D \in \mathbb{D}(\mathrm{X})$. If $v \in {}^P\mathcal{W}$ and $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, then $\mu_{P,v}(D : \lambda)$ is an operator in $\mathbb{D}(\mathrm{X}_{P,v})$, which naturally acts on the space $\mathcal{A}_2(\mathrm{X}_{P,v}, \tau_P)$ by an endomorphism $\underline{\mu}_{P,v}(D, \lambda)$. The direct sum of these endomorphisms, for $v \in \mathcal{W}$, is an endomorphism of $\mathcal{A}_{2,P}$, denoted by

$$\underline{\mu}_P(D : \lambda) := \oplus_{v \in \mathcal{W}} \ \underline{\mu}_{P,v}(D : \lambda). \tag{11.8}$$

**Proposition 11.7** *For every $\psi \in \mathcal{A}_{2,P}$, the Eisenstein integral $E^\circ(P : \psi : \lambda)$ is meromorphic as a function of $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$ with values in $C^\infty(\mathrm{X} : \tau)$. Moreover, it behaves finitely under the action of $\mathbb{D}(\mathrm{X})$, for generic $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$. More precisely, for every $D \in \mathbb{D}(\mathrm{X})$,*

$$DE^\circ(P : \psi : \lambda) = E^\circ(P : \underline{\mu}_P(D : \lambda)\psi : \lambda),$$

*as a meromorphic identity in the variable $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.*

The regularity theorem for $j^\circ$, Theorem 10.18, is essentially equivalent to the following regularity theorem for the Eisenstein integral.

**Theorem 11.8** (Regularity theorem) *The $C^\infty(\mathrm{X} : \tau)$-valued meromorphic function $\lambda \mapsto E^\circ(P : \psi : \lambda)$ is regular on $i\mathfrak{a}^*_{P\mathrm{q}}$, for every $\psi \in \mathcal{A}_{2,P}$.*

This result is proved by a careful asymptotic analysis combined with the Maass–Selberg relations presented in Theorem 11.22 below. For $P$ a minimal $\sigma$-parabolic subgroup it is due to [15]. For general $P$ it is due to [35], which in turn makes use of [10].

The Fourier transform defined in the text below Theorem 10.18 can be expressed in terms of the normalized Eisenstein integral.

**Lemma 11.9** *Let $F \in C_c^\infty(\mathrm{X} : \tau_\delta)$, and let $f$ be the corresponding function in $C_c^\infty(\mathrm{X})_\delta$, i.e., $F = f^{\mathrm{sph}}$. Then*

$$\langle \hat{f}(P : \xi : \lambda)\,,\, T\rangle = \int_{\mathrm{X}} \langle F(x)\,,\, E^\circ(P : \psi_T : -\lambda)(x)\rangle_{V_\tau}\, dx\,,$$

*for every $T \in \overline{V(P,\xi)} \otimes L^2(K : \xi)$ and all $\lambda \in i\mathfrak{a}_{P\mathrm{q}}^*$.*

*Proof.* Let $T$ be as mentioned. Then using (11.4) we find that

$$\begin{aligned}\langle \hat{f}(P : \xi : \lambda)\,,\, T\rangle &= \langle f\,,\, \mathrm{M}_{P,\xi - \overline{\lambda}}(T)\rangle\\ &= \langle F\,,\, \mathrm{M}^{\mathrm{sph}}_{P,\xi,-\overline{\lambda}}(T)\rangle\\ &= \int_{\mathrm{X}} \langle F(x)\,,\, E^\circ(P : \psi_T : \overline{\lambda})(x)\rangle_{V_\tau}\, dx.\end{aligned}$$

□

**The spherical Fourier transform** The relation in Lemma 11.9 provides motivation for the following definition of the spherical Fourier transform. We write $E^\circ(P : \lambda)$ for the $\mathrm{Hom}\,(\mathcal{A}_{2,P}, V_\tau)$-valued function on X given by

$$E^\circ(P : \lambda)(x)\psi = E^\circ(P : \psi : \lambda)(x)\,,$$

$\psi \in \mathcal{A}_{2,P}$, $x \in \mathrm{X}$. Moreover, we define the so-called *dual Eisenstein integral* to be the $\mathrm{Hom}\,(V_\tau, \mathcal{A}_{2,P})$-valued function on X given by

$$E^*(P : \lambda : x) := E(P : -\overline{\lambda} : x)^*\,,$$

for $x \in \mathrm{X}$ and generic $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.

**Definition 11.10** Let $F \in C_c^\infty(\mathrm{X} : \tau)$. The *spherical Fourier transform* $\mathcal{F}_P F : i\mathfrak{a}^*_{P\mathrm{q}} \to \mathcal{A}_{2,P}$ is defined by

$$\mathcal{F}_P F(\lambda) = \int_{\mathrm{X}} E^*(P : \lambda : x) F(x) dx\,. \tag{11.9}$$

**Example 11.11** (Riemannian case) In the setting of Example 11.3 it follows that $\mathcal{F}_P f(\lambda)$ equals $c(-\lambda)^{-1}$ times the usual spherical Fourier transform $\tilde{f}(\lambda)$, i.e.,

$$\mathcal{F}_P(f)(\lambda) = c(-\lambda)^{-1}\tilde{f}(\lambda) = c(-\lambda)^{-1}\int_{G/K} f(x)\varphi_{-\lambda}(x)\, dx,$$

for $f \in C_c^\infty(K\backslash G/K)$ and generic $\lambda \in \mathfrak{a}^*_{\mathfrak{p}\mathbb{C}}$.

It follows from Definition 11.10 that

$$\langle \hat{f}(P : \xi : \lambda)\,,\, T\rangle = \langle \mathcal{F}_P F(-\lambda)\,,\, \psi_T\rangle \tag{11.10}$$

in the notation of Lemma 11.9. The change from $\lambda$ to $-\lambda$ is somewhat awkward, but nevertheless incorporated here to guarantee a traditional form for the asymptotic approximations of the Eisenstein integral as the space variable tends to infinity. Given a finite-dimensional real linear space $V$ we shall use the notation $\mathcal{S}(V)$ for the usual space of rapidly decreasing or Schwartz functions $V \to \mathbb{C}$.

**Proposition 11.12** *Let $P \in \mathcal{P}_\sigma$. The Fourier transform $\mathcal{F}_P$ maps $C_c^\infty(\mathrm{X} : \tau)$ continuous linearly into the Schwartz space $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$.*

This result is a consequence of uniformly tempered estimates for the Eisenstein integral combined with partial integration. More comments on the proof are given in the text following Theorem 11.16.

The operator $\mathcal{F}_P$ has the so-called wave packet operator as its adjoint.

**Definition 11.13** Let $P \in \mathcal{P}_\sigma$. The *wave packet* operator $\mathcal{J}_P$ is the operator from $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$ to $C^\infty(\mathrm{X} : \tau)$, defined by

$$\mathcal{J}_P\varphi(x) = \int_{i\mathfrak{a}_{P\mathrm{q}}^*} E^\circ(P : \lambda : x)\varphi(\lambda)\, d\lambda\,.$$

Here $d\lambda$ is abbreviated notation for the Lebesgue measure $d\mu_P(\lambda)$ on $i\mathfrak{a}_{P\mathrm{q}}^*$, normalized as in the beginning of Section 10.

Theorem 10.8 and its normalized version are now equivalent to the following result, which is the major ingredient in the Plancherel theorem for spherical functions.

**Theorem 11.14** *Let $f \in C_c^\infty(\mathrm{X} : \tau)$. Then*

$$f = \sum_{P \in \mathbf{P}_\sigma} [W : W_P^*]\, \mathcal{J}_P \mathcal{F}_P f\,. \tag{11.11}$$

**Example 11.15** (Riemannian case) In the setting of Example 11.3, we have that

$$\mathcal{J}_P\psi(x) = \int_{i\mathfrak{a}_\mathfrak{p}^*} \psi(\lambda)c(\lambda)^{-1}\varphi_\lambda(x)\, d\lambda.$$

If we combine this with Example 11.11 and use the remarks of Example 10.9, we see that (11.11) takes the form of the inversion formula for Riemannian symmetric spaces,

$$f(x) = \int_{i\mathfrak{a}_\mathfrak{p}^*} \tilde{f}(\lambda)\varphi_\lambda(x)\frac{d\lambda}{|c(\lambda)|^2}.$$

There is a natural notion of *Schwartz function* on X, which generalizes Harish-Chandra's notion of Schwartz function for the group. Let $l_\mathrm{X} : \mathrm{X} \to [0, \infty[$ be defined by $l_\mathrm{X}(kah) = \|\log a\|$. Then the *$L^2$-Schwartz space* of X is defined by

$$\mathcal{C}(\mathrm{X}) = \{f \in C^\infty(\mathrm{X}) \mid (1 + l_\mathrm{X})^n L_u f \in L^2(\mathrm{X}) \quad \forall u \in U(\mathfrak{g}),\, n \in \mathbb{N}\}\,.$$

Here $L_u$ denotes the infinitesimal left regular action of $u$. Alternatively, the Schwartz space can be characterized in terms of sup-norms of derivatives. Let $\Theta : G/H \to\, ]\,0, \infty\,[$ be defined by $\Theta(x) = \Xi(x\sigma(x)^{-1})^{1/2}$, where $\Xi$ denotes Harish-Chandra's elementary spherical function for $G/K$ with spectral parameter 0. Then a function $f \in C^\infty(\mathrm{X})$ belongs to $\mathcal{C}(X)$ if and only if for every $u \in U(\mathfrak{g})$ and all $n \in \mathbb{N}$,

$$\sup_{x \in \mathrm{X}} |(1 + l_\mathrm{X}(x))^n\, \Theta(x)^{-1} L_u f(x)| < \infty.$$

For more details we refer the reader to [6], Sect. 17.

**Theorem 11.16** *Let $P \in \mathcal{P}_\sigma$. Then*

(a) *The Fourier transform $\mathcal{F}_P$ extends to a continuous linear operator from $\mathcal{C}(\mathrm{X} : \tau)$ to $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$.*

(b) *The wave packet operator $\mathcal{J}_P$ is a continuous linear operator from $\mathcal{C}(\mathrm{X} : \tau)$ to $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$.*

The fact that $\mathcal{F}_P$ extends continuous linearly to the Schwartz space is a consequence of uniformly tempered estimates for the Eisenstein integral. These will be formulated at a later stage, see Theorem 14.1. That the extension maps into the Euclidean Schwartz space $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$ is a consequence of the tempered estimates combined with partial integration, involving an application of Proposition 11.7.

Assertion (b) of the theorem is a consequence of the mentioned tempered estimates and an application of the theory of the constant term of [33]. The proof is due to [15] for $P$ minimal and to [10] for $P$ general.

The natural action of the algebra of invariant differential operators on $C^\infty(\mathrm{X} : \tau)$ leaves the subspace $\mathcal{C}(\mathrm{X} : \tau)$ invariant. Moreover, it behaves well with respect to the Fourier and wave packet transforms. For $D \in \mathbb{D}(\mathrm{X})$ we denote by $\underline{\mu}_P(D : \cdot)$ the endomorphism of $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$ given by $[\underline{\mu}_P(D : \cdot)\varphi](\lambda) = \underline{\mu}_P(D : \lambda)\varphi(\lambda)$, see also (11.8).

**Lemma 11.17** *Let $P \in \mathcal{P}_\sigma$ and $D \in \mathbb{D}(\mathrm{X})$. Then*

(a) $\mathcal{F}_P \circ D = \underline{\mu}_P(D : \cdot) \circ \mathcal{F}_P$;

(b) $D \circ \mathcal{J}_P = \mathcal{J}_P \circ \underline{\mu}_P(D : \cdot)$.

*Proof.* Property (a) follows from Proposition 11.7 combined with Definition 11.10 and the relation $\underline{\mu}_P(D^* : -\bar{\lambda})^* = \underline{\mu}_P(D : \lambda)$. For a proof of the latter relation, see [21], Lemma 14.7. Property (b) follows from the mentioned Proposition 11.7 combined with Definition 11.13. $\square$

**$C$-functions and Maass–Selberg relations** For the full Plancherel theorem, we need to give a description of the image of $(\mathcal{F}_P)_{P \in \mathbf{P}_\sigma}$. This description involves so-called $C$-functions, which occur in the asymptotic behavior of the Eisenstein integral.

If $P, Q \in \mathcal{P}_\sigma$ are associated, see Definition 10.7, we define

$$W(\mathfrak{a}_{Q\mathrm{q}} \mid \mathfrak{a}_{P\mathrm{q}}) = \{ s|_{\mathfrak{a}_{P\mathrm{q}}} \mid s \in W,\ s(\mathfrak{a}_{P\mathrm{q}}) \subset \mathfrak{a}_{Q\mathrm{q}} \}\,.$$

**Theorem 11.18** *Let $P, Q \in \mathcal{P}_\sigma$ be associated. There exist uniquely determined* $\mathrm{Hom}\,(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$*-valued meromorphic functions $\lambda \mapsto C^\circ_{Q|P}(s : \lambda)$ on $\mathfrak{a}_{P\mathrm{q}\mathbb{C}}^*$, for $s \in W(\mathfrak{a}_{Qq} \mid \mathfrak{a}_{P\mathrm{q}})$, such that*

$$E^\circ(P : \lambda : mav)\psi \sim \sum_{s \in W(\mathfrak{a}_{Qq} | \mathfrak{a}_{P\mathrm{q}})} a^{s\lambda - \rho_Q} [C^\circ_{Q|P}(s : \lambda)\psi]_v(m)\,,$$

*as $a \to \infty$ in $A^+_{Q\mathrm{q}}$, for all $\lambda \in i\mathfrak{a}_{P\mathrm{q}}^*$, $v \in {}^Q\mathcal{W}$, $m \in \mathrm{X}_{Q,v}$ and $\psi \in \mathcal{A}_{2,P}$; here $s\lambda := \lambda \circ s^{-1}$.*

The motivation for the particular normalization in the definition of $j^\circ$, see Definition 10.17, is ultimately given by the following result.

**Proposition 11.19** $C^\circ_{P|P}(1:\lambda) = I$.

The $C$-functions allow us to formulate the so-called functional equations for the Eisenstein integral.

**Theorem 11.20** *Let $P, Q \in \mathcal{P}_\sigma$ be associated. Then, for all $s \in W(\mathfrak{a}_{Q\mathrm{q}}|\mathfrak{a}_{P\mathrm{q}})$,*

$$E^\circ(P:\lambda:x) = E^\circ(Q:s\lambda:x)\circ C^\circ_{Q|P}(s:\lambda), \tag{11.12}$$

*for every $x \in \mathrm{X}$, as a meromorphic identity in the variable $\lambda \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$.*

This result generalizes Harish-Chandra's functional equations for the case of the group, see [55]. For symmetric spaces, and $P, Q$ minimal, the result is due to [6]. The general result is due to [35]. Later, in [21] a different proof has been given, based on the principle of induction of relations developed in [20]. It involves the idea that the functions on both sides of (11.12) are essentially eigenfunctions depending meromorphically on the parameter $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$. Moreover, they satisfy conditions that allow one to conclude their equality from the equality of the coefficient of $a^{s\lambda-\rho_Q}$ in the expansion along $A^+_{Q\mathrm{q}}v$, for each $v \in {}^Q\mathcal{W}$. The latter equalities amount to

$$C^\circ_{Q|P}(s:\lambda) = C^\circ_{Q|Q}(1:s\lambda)\, C^\circ_{Q|P}(s:\lambda),$$

which is valid in view of Proposition 11.19.

The functional equations for the Eisenstein integral imply transformation formulas for the normalized $C$-functions

**Proposition 11.21** *Let $P, Q, R \in \mathcal{P}_\sigma$ be associated, and let $s \in W(\mathfrak{a}_{Q\mathrm{q}}|\mathfrak{a}_{P\mathrm{q}})$ and $t \in W(\mathfrak{a}_{R\mathrm{q}}|\mathfrak{a}_{Q\mathrm{q}})$. Then*

$$C^\circ_{R|P}(ts:\lambda) = C^\circ_{R|Q}(t:s\lambda)C^\circ_{Q|P}(s:\lambda), \tag{11.13}$$

*as a* $\mathrm{Hom}\,(\mathcal{A}_{2,P}, \mathcal{A}_{2,R})$*-valued identity of meromorphic functions in the variable $\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.*

*Proof.* This follows from (11.12) by comparing the coefficients of $a^{ts\lambda-\rho_R}$ in the asymptotic expansions along $A^+_{R\mathrm{q}}v$, for $v \in {}^R\mathcal{W}$. □

The proof of the regularity theorem, Theorem 11.8, is based on an asymptotic analysis together with the following important fact.

**Theorem 11.22** (Maass–Selberg relations). *Let $P, Q \in \mathcal{P}_\sigma$ be associated and let $s \in W(\mathfrak{a}_{Q\mathrm{q}} \mid \mathfrak{a}_{P\mathrm{q}})$. Then*

$$C^\circ_{Q|P}(s:-\overline{\lambda})^* C^\circ_{Q|P}(s:\lambda) = I\,.$$

*In particular, if $\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}$, then $C^\circ_{Q|P}(s:\lambda)$ is unitary.*

In the case of the group, the above result is due to Harish-Chandra, [58]. He introduced the name Maass–Selberg relations to emphasize remarkable analogies with the theory of automorphic forms.

For symmetric spaces and $P$ minimal, Theorem 11.22 (Maass–Selberg) is due to E.P. van den Ban, [6]. For general $P$ it is due to J. Carmona and P. Delorme, [35]. The latter paper depends in an essential way on Delorme's paper [39].

Later, in [21], van den Ban and Schlichtkrull managed to obtain the Maass–Selberg relations for general $\sigma$-parabolic subgroups from those for the minimal ones. We will give more details at a later stage, see Theorem 14.2.

**Example 11.23** (Riemannian case) In the setting of a Riemannian symmetric space with $P$ minimal, see Example 11.3, we see that $C^\circ_{P|P}(s:\lambda) = c(\lambda)^{-1}c(s\lambda)$. Thus, in this case the Maass–Selberg relations with $Q = P$ and $s \in W$ amount to $|c(\lambda)|^2 = |c(s\lambda)|^2$ for $\lambda \in i\mathfrak{a}^*_{\mathfrak{p}}$ (imaginary).

**The Plancherel theorem for spherical functions** The functional equations for the Eisenstein integral, together with the Maass–Selberg relations, imply transformation formulas for the associated Fourier transforms.

**Proposition 11.24** *Let $P, Q \in \mathcal{P}_\sigma$ be associated. Then, for every $f \in \mathcal{C}(\mathrm{X}:\tau)$ and each $s \in W(\mathfrak{a}_{Q\mathrm{q}}|\mathfrak{a}_{P\mathrm{q}})$,*

$$\mathcal{F}_Q f(s\lambda) = C^\circ_{Q|P}(s:\lambda)\mathcal{F}_P f(\lambda), \qquad (\lambda \in i\mathfrak{a}^*_{P\mathrm{q}}). \tag{11.14}$$

*Proof.* Taking adjoints on both sides of (11.12), with $-\bar{s}^{-1}\lambda$ in place of $\lambda$, and using the Maass–Selberg relations (Theorem 11.22), we obtain the following functional equation for the dual Eisenstein integral

$$C^\circ_{Q|P}(s:\lambda)\circ E^*(P:\lambda:x) = E^*(Q:s\lambda:x). \tag{11.15}$$

The transformation formula for the Fourier transforms is an immediate consequence. □

**Proposition 11.25** *Let $P, Q \in \mathcal{P}_\sigma$ be associated. Then $\mathcal{J}_P \circ \mathcal{F}_P = \mathcal{J}_Q \circ \mathcal{F}_Q$ on $\mathcal{C}(\mathrm{X}:\tau)$.*

*Proof.* Let $f \in \mathcal{C}(\mathrm{X}:\tau)$. From (11.14) it follows that, for $x \in \mathrm{X}$,

$$\begin{aligned}
\mathcal{J}_Q\mathcal{F}_Q f(x) &= \int_{i\mathfrak{a}^*_{P\mathrm{q}}} E^\circ(Q:s\lambda:x)\,\mathcal{F}_Q f(s\lambda)\,d\lambda \\
&= \int_{i\mathfrak{a}^*_{P\mathrm{q}}} E^\circ(Q:s\lambda:x)\,C^\circ_{Q|P}(s:\lambda)\,\mathcal{F}_P f(\lambda)\,d\lambda \\
&= \int_{i\mathfrak{a}^*_{P\mathrm{q}}} E^\circ(P:\lambda:x)\,)\,\mathcal{F}_P f(\lambda)\,d\lambda \\
&= \mathcal{J}_P\mathcal{F}_P f(x).
\end{aligned}$$

The next to last equality follows from Theorem 11.20. □

The above result implies that the summation in Theorem 11.14 essentially ranges over equivalence classes of associated parabolic subgroups.

Motivated by the transformation formula (11.14) with $P = Q$, we define

$$[\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}]^{W(\mathfrak{a}_{P\mathrm{q}})} \tag{11.16}$$

to be the subspace of $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$ consisting of the functions $\varphi$ satisfying

$$\varphi(s\lambda) = C^{\circ}_{P|P}(s : \lambda)\varphi(\lambda)\,,$$

for all $\lambda \in i\mathfrak{a}_{P\mathrm{q}}^*$, $s \in W(\mathfrak{a}_{P\mathrm{q}})$. It follows that $\mathcal{F}_P$ maps $\mathcal{C}(\mathrm{X} : \tau)$ into the space (11.16) introduced above.

Proposition 10.14 and its normalized version are consequences of the following Plancherel theorem for spherical Schwartz functions.

**Theorem 11.26** (The Plancherel formula for spherical functions) *The map* $\mathcal{F} := \oplus_{P\in\mathbf{P}_\sigma} \mathcal{F}_P$ *is a topological linear isomorphism*

$$\mathcal{C}(\mathrm{X} : \tau) \ \xrightarrow{\simeq}\ \oplus_{P\in\mathbf{P}_\sigma}\ \ (\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P})^{W(\mathfrak{a}_{P\mathrm{q}})}\,.$$

*Its inverse is given by*

$$\mathcal{J} := \oplus_{P\in\mathbf{P}_\sigma}\,[W : W_P^*]\,\mathcal{J}_P\,.$$

*Moreover, for every* $f \in C(\mathrm{X} : \tau)$,

$$\|f\|^2_{L^2(\mathrm{X}:\tau)} = \sum_{P\in\mathbf{P}_\sigma} [W : W_P^*]\,\|\mathcal{F}_P f\|^2_{L^2}.$$

The set $\mathbf{P}_\sigma$ contains precisely one minimal $\sigma$-parabolic subgroup of $\mathcal{P}_\sigma$, since all such are associated. Let us denote it by $P_0$, and its Langlands decomposition by

$$P_0 = MAN_0.$$

We denote by $\mathcal{C}_{\mathrm{mc}}(\mathrm{X} : \tau)$ the image of $\mathcal{J}_{P_0}$ and call it the most continuous part of the Schwartz space. The following results were proved in my earlier work with Schlichtkrull on the most continuous part of the Plancherel theorem, [16].

**Theorem 11.27** *There exists a differential operator* $D_\tau \in \mathbb{D}(\mathrm{X})$, *depending on* $\tau = \tau_\delta$, *such that*

(a) $D_\tau$ *is injective on* $C_c^\infty(\mathrm{X} : \tau)$,

(b) $D_\tau \circ \mathcal{J}_{P_0} \circ \mathcal{F}_{P_0} = D_\tau$ *on* $\mathcal{C}(\mathrm{X} : \tau)$.

It follows from the above result combined with the spherical Plancherel theorem that

$$D_\tau \circ \mathcal{J}_P = 0 \quad \text{for all} \quad P \in \mathbf{P}_\sigma \setminus \{P_0\},$$

hence also for all nonminimal $\sigma$-parabolic subgroups $P \in \mathcal{P}_\sigma$. In particular, it follows that $\mathcal{C}_{\mathrm{mc}}(\mathrm{X} : \tau)$ equals the orthocomplement of the kernel of $D_\tau$ in $\mathcal{C}(\mathrm{X} : \tau)$.

In the final sections of our exposition, we shall give a sketch of the proof of Theorem 11.27, and derive the full Plancherel theorem from it by application of a residue calculus.

## 12 The most continuous part

**Expansions for Eisenstein integrals** In this section we shall give a sketch of the proof of Theorem 11.27, which serves as the starting point for our further derivations.

Let $A_{\mathrm{q}}^+$ be the positive Weyl chamber associated with the choice of $P_0$; i.e., $A_{\mathrm{q}}^+ = A_{P_0\mathrm{q}}^+$. In the notation of (6.3), the associated choice of positive roots is given by

$$\Sigma^+ := \Sigma(P_0).$$

In this section we shall briefly write $E^\circ(\lambda : x)$ for the normalized Eisenstein integral $E^\circ(P_0 : \lambda : x)$. The theory of this Eisenstein integral, connected with the minimal principal series for $(G, H)$, is developed in [5], [6] and [15], along the lines sketched in the previous sections. In Delorme's work, partially in collaboration with Carmona, the Eisenstein integrals associated with the non-minimal (or generalized) principal series are introduced in a similar manner. In my work together with H. Schlichtkrull, the more general Eisenstein integrals make their appearance in harmonic analysis through a residue calculus in a way we shall explain in the sequel. It is only at the end of the analysis that they are identified as matrix coefficients of the generalized principal series.

Let us return to the Eisenstein integral $E^\circ(\lambda : x)$ associated with the minimal $\sigma$-parabolic subgroup $P_0$. Put $\underline{\mu} = \underline{\mu}_{P_0}$. In view of Proposition 11.7, the action of the invariant differential operators on the Eisenstein integral is described by

$$DE^\circ(\lambda : \cdot\,) = E^\circ(\lambda : \cdot\,) \circ \underline{\mu}(D : \lambda), \qquad (D \in \mathbb{D}(\mathrm{X})), \tag{12.1}$$

as a meromorphic identity in the variable $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. From this combined with the fact that it is a $(1 \otimes \tau)$-spherical function on X it follows that the Eisenstein integral $E^\circ(\lambda : x)$ has a particular asymptotic behavior as $x$ tends to infinity in X. The structure of directions to infinity in X is best understood in terms of the decomposition (3.10) in Corollary 3.7. Let $A_{\mathrm{q}}^+$ and $\mathcal{W}$ be as in the mentioned corollary. Then the differential equations (12.1) give rise to so-called radial differential equations on $v^{-1}A_{\mathrm{q}}^+ v$, for every $v \in \mathcal{W}$. These radial equations form a cofinite system. Moreover, let $\Lambda$ be a basis of $\mathfrak{a}_{\mathrm{q}}^*$ containing the set $\Delta$ of simple roots in $\Sigma^+$. Then the functions $a \mapsto a^{-v^{-1}\alpha}$ for $\alpha \in \Lambda$, form a system of coordinates at infinity on $v^{-1}A_{\mathrm{q}}^+ v$, with respect to which the system of radial differential equations becomes of the regular singular type at zero. As a consequence it follows that the Eisenstein integrals have an asymptotic behavior that can be described in terms of power series in these coordinates. More precisely, let $D$ be the unit disk in $\mathbb{C}$. Then we have the following result from [14], Thm. 11.1. For the case of the group the result is due to Harish-Chandra [52], see also [37] and [4] for the relation of the system (12.1) with the theory of regular singularities. We agree to write $\rho := \rho_{P_0}$, $\tau := \tau_\delta$, $\tau_M := \tau_{P_0} = \tau|_{K\cap M}$ and

$$^\circ\mathcal{C}(\tau) := \mathcal{A}_{2,P_0}(\tau) = \oplus_{v\in\mathcal{W}}\, C^\infty(M/M \cap vHv^{-1} : \tau_M),$$

see (11.7). Note that $M/M \cap vHv^{-1}$ is compact, for $v \in \mathcal{W}$, by minimality of $P_0$ in $\mathcal{P}_\sigma$.

**Theorem 12.1** *Let $v \in \mathcal{W}$. There exist meromorphic* $\mathrm{End}\,({}^\circ\mathcal{C}(\tau))$*-valued functions $\lambda \mapsto C^\circ(s:\lambda)$ on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, for $s \in W$, and a function $\Psi_v : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \times D^\Delta \to$* $\mathrm{End}\,(V_\tau^{M\cap K\cap vHv^{-1}})$*, meromorphic in the first and holomorphic in the second variable, with $\Psi_v(\lambda, 0) = I$, such that*

$$E^\circ(\lambda : av)\psi = \sum_{s\in W} a^{s\lambda-\rho}\ \Psi_v(s\lambda, (a^{-\alpha}))\ [C^\circ(s:\lambda)\psi]_v(e),$$

*for every $\psi \in {}^\circ\mathcal{C}(\tau)$, all $a \in A_{\mathrm{q}}^+$ and generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.*

The function $\Psi_v(\lambda : \cdot\,)$ has a power series expansion on $D^\Delta$, with coefficients

$$\Gamma_\mu(v,\lambda) \in \mathrm{End}\,(V_\tau^{M\cap K\cap vHv^{-1}}),$$

for $\mu \in \mathbb{N}\Delta$, which depend meromorphically on $\lambda$; moreover, the constant term is given by $\Gamma_0(v,\lambda) = I$. Accordingly, the Eisenstein integral $E^\circ(\lambda : av)$ has the following series expansion which describes its asymptotic behavior as $a$ tends to infinity in the chamber $A_{\mathrm{q}}^+$,

$$E^\circ(\lambda : av)\psi = \sum_{s\in W}\sum_{\mu\in\mathbb{N}\Delta} a^{s\lambda-\rho-\mu}\ \Gamma_\mu(v,\lambda)\ [C^\circ(s:\lambda)\psi]_v(e).$$

Observe that the $C$-functions defined here were denoted by $C^\circ_{P_0|P_0}(s:\lambda)$ earlier, see Theorem 11.18. In particular,

$$C^\circ(1:\lambda) = I, \tag{12.2}$$

and we have the Maass–Selberg relations, for $s \in W$,

$$C^\circ(s:-\bar\lambda)^* C^\circ(s:\lambda) = I. \tag{12.3}$$

In the present setting of a minimal $\sigma$-parabolic subgroup, the Maass–Selberg relations are due to [6]. The proof given in [6] depends on a careful study, [5], [8], of the action of the standard intertwining operators, introduced in (10.2), on the $H$-fixed generalized vectors of the minimal principal series, introduced in Section 8. See the remarks following Theorem 11.22 for further comments on the history.

**Proof of Theorem 11.27** The proof of Theorem 11.27, which amounts to the most continuous part of the Plancherel decomposition, is given in [16]. We sketch some of the main ideas occurring in that proof.

First, we agree to use the notation

$$\Phi_v(\lambda, a) = a^{\lambda-\rho}\Psi_v(\lambda, (a^{-\alpha})).$$

Let $f \in C_c^\infty(\mathrm{X}:\tau)$. Writing $\mathcal{F} := \mathcal{F}_{P_0}$ and $\mathcal{J} := \mathcal{J}_{P_0}$ and ignoring singularities in the variable $\lambda$ as well as convergence of integrals for the moment, we see that

$$\mathcal{J}\mathcal{F}f(av) \;=\; \int_{i\mathfrak{a}_{\mathrm{q}}^*}\sum_{s\in W}\Phi_v(s\lambda, a)[C^\circ(s:\lambda)\mathcal{F}f(\lambda)]_v(e)\, d\lambda \tag{12.4}$$

$$
\begin{aligned}
&= \sum_{s\in W}\int_{i\mathfrak{a}_{\mathrm{q}}^*} \Phi_v(s\lambda,a)[C^\circ(s:\lambda)\mathcal{F}f(\lambda)]_v(e)\,d\lambda && (12.5)\\
&= \sum_{s\in W}\int_{i\mathfrak{a}_{\mathrm{q}}^*} \Phi_v(s\lambda,a)[\mathcal{F}f(s\lambda)]_v(e)\,d\lambda && (12.6)\\
&= |W|\int_{i\mathfrak{a}_{\mathrm{q}}^*} \Phi_v(\lambda,a)[\mathcal{F}f(\lambda)]_v(e)\,d\lambda && (12.7)\\
&= |W|\int_{i\mathfrak{a}_{\mathrm{q}}^*+\eta} \Phi_v(\lambda,a)[\mathcal{F}f(\lambda)]_v(e)\,d\lambda \quad + \text{residual integrals}
\end{aligned}
$$

for a generic $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ that is antidominant, i.e., $\langle \eta\,,\,\alpha\rangle < 0$ for all $\alpha \in \Delta$. There are two major problems with this procedure. The first is that the integrands may have singularities as a function of $\lambda$. The second is that the integrands need to be estimated in order to justify the passage from (12.4) to (12.5) and to apply Cauchy's theorem to the integral in (12.7). Both of these problems are dealt with in [6]. Both factors of the integrand in (12.7) are shown to have singularities along a locally finite union of hyperplanes of the form $\langle \lambda\,,\,\alpha\rangle = c$, $(c \in \mathbb{R})$, for $\alpha \in \Sigma$, of order independent of $f$ and $a$. Moreover, none of these hyperplanes meets $\eta + i\mathfrak{a}_{\mathrm{q}}^*$, if $\eta$ is sufficiently far out in the antidominant direction. The required estimates can be described as follows. Let $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$ be a bounded subset. Then there exists a polynomial function $q = q_\omega : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$, which is a product of factors of the form $\langle\,\cdot\,,\,\alpha\rangle - c$, with $\alpha \in \Sigma$ and $c \in \mathbb{R}$, such that for some $N \in \mathbb{N}$,

$$\|q(\lambda)\Phi_v(\lambda,a)\| = \mathcal{O}((1+|\lambda|)^N),$$

locally uniformly in $a \in A_{\mathrm{q}}^+$, for $\lambda$ in the strip $\omega + i\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, and such that, for every $n \in \mathbb{N}$,

$$\|q(\lambda)\mathcal{F}f(\lambda)\| = \mathcal{O}((1+|\lambda|)^{-n}),$$

for $\lambda$ in the same strip.

If $D \in \mathbb{D}(\mathrm{X})$, then $\mathcal{F}(Df)(\lambda) = \underline{\mu}(D:\lambda)\mathcal{F}f(\lambda)$, by Lemma 11.17. In [16] it is shown that there exists an operator $D_\tau \in \mathbb{D}(\mathrm{X})$ depending on $\tau$, such that the zeros of $\underline{\mu}(D_\tau:\,\cdot\,)$ annihilate all singularities of the integrands in (12.4)–(12.7), and such that

$$\det \underline{\mu}(D_\tau:\,\cdot\,) \neq 0. \tag{12.8}$$

It follows that all equalities in the array are valid with $D_\tau f$ instead of $f$, for any $f \in C_c^\infty(\mathrm{X}:\tau)$ and all $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ antidominant; moreover, the residual integrals vanish. This leads to the equalities

$$\mathcal{J}\mathcal{F}D_\tau f(av) = \int_{\eta+i\mathfrak{a}_{\mathrm{q}}^*} \Phi_v(\lambda,a)[\underline{\mu}(D_\tau:\lambda)\mathcal{F}f(\lambda)]_v(e)\,d\lambda, \tag{12.9}$$

for each $v \in \mathcal{W}$ and all $a \in A_{\mathrm{q}}^+$. In the Riemannian case, where $H = K$, $\mathcal{W} = \{1\}$ and $\tau = 1$, one can show that $D_\tau = 1$ fulfills the requirements. Moreover, the procedure just described corresponds to the shift procedure applied by Helgason [60].

**Proposition 12.2** *The operator $\mathcal{JF}D_\tau$ is a support preserving continuous linear endomorphism of $C_c^\infty(\mathrm{X} : \tau)$.*

*Sketch of proof.* From estimates for the Eisenstein integral it can be shown that there exists a polynomial function $q : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$, which is a product of factors of the form $\langle \cdot \,, \alpha\rangle - c$, with $\alpha \in \Sigma$ and $c \in \mathbb{R}$, such that

$$\|q(\lambda)\mathcal{F}f(\lambda)\| = \mathcal{O}((1 + \|\lambda\|)^{-n} e^{\langle \mathrm{Re}\,\lambda\,,\,\mu\rangle})$$

in the region of points $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with $\langle \mathrm{Re}\,\lambda\,,\,\alpha\rangle \leq 0$ for all $\alpha \in \Sigma^+$. Here $\mu$ is any dominant element of $\mathfrak{a}_{\mathrm{q}}^*$ with the property that $\mu \leq 1$ on $\mathcal{A}$, for $\mathcal{A} \subset \mathfrak{a}_{\mathrm{q}}^+$ any subset with $\mathcal{A}v \supset A_{\mathrm{q}}^+ v \cap \mathrm{supp}\, f$. The functions $\Phi_v$ have series expansions with estimates similar to those obtained by Gangolli [47] for the Riemannian case, see [14] for details. One can now apply a Paley–Wiener shift argument with $\eta = -t\mu$, $t \to \infty$, to conclude that the smooth function $\mathcal{JF}D_\tau f$ has a support $S$ satisfying $\mu \leq 1$ on $\log[(S \cap A_{\mathrm{q}}^+ v)v^{-1}]$. Collecting these observations for $v \in \mathcal{W}$, we conclude: if $X_0 \in \mathfrak{a}_{\mathrm{q}}^+$ and $f \in C_c^\infty(\mathrm{X} : \tau)$ is a function with support contained in $K\exp(\mathrm{conv}\, WX_0)H$, then $\mathcal{JF}D_\tau f$ is a smooth function with support contained in the same set. Here $\mathrm{conv}\,(WX_0)$ denotes the convex hull of the Weyl group orbit $WX_0$ in $\mathfrak{a}_{\mathrm{q}}$.

The operator $D_\tau$ can be chosen formally symmetric, so that $\mathcal{JF}D_\tau$ is symmetric with respect to the $L^2$-type inner product on $C_c^\infty(\mathrm{X} : \tau)$, see also the text following Proposition 11.12. In combination with this symmetry, the support properties just mentioned imply that $\mathcal{JF}D_\tau$ is support preserving. $\square$

We can now finish our sketch of the proof of Theorem 11.27. It follows from the above proposition, combined with the commutativity of the algebra $\mathbb{D}(\mathrm{X})$ and Lemma 11.17, that $\mathcal{T} := \mathcal{JF}D_\tau$ is a support preserving endomorphism commuting with the algebra $\mathbb{D}(\mathrm{X})$. The property of support preservation implies that the operator $\mathcal{T}$ is related to a differential operator. More precisely, we define the operator

$$T^\uparrow :\ C_c^\infty(A_{\mathrm{q}}^+, V_\tau^{M\cap K\cap H}) \to C_c^\infty(\mathrm{X} : \tau)$$

by $T^\uparrow f(ka) = \tau(k)f(a)$, for $k \in K$ and $a \in A_{\mathrm{q}}^+$, and by $T^\uparrow f = 0$ outside $KA_{\mathrm{q}}^+$. In the converse direction, we define the operator

$$T^\downarrow :\ C^\infty(\mathrm{X} : \tau) \to C^\infty(A_{\mathrm{q}}^+, V_\tau^{M\cap K\cap H})$$

by restriction. Then $\mathcal{T}^{\mathrm{rad}} := T^\downarrow \circ \mathcal{T} \circ T^\uparrow$ is a support preserving continuous linear endomorphism of $C_c^\infty(A_{\mathrm{q}}^+, V_\tau^{M\cap K\cap H})$, hence a linear partial differential operator with coefficients in $C^\infty(A_{\mathrm{q}}^+) \otimes \mathrm{End}\,(V_\tau^{M\cap K\cap H})$. We observe that for $D \in \mathbb{D}(\mathrm{X})$,

$$T^\downarrow \circ D \circ T^\uparrow = D^{\mathrm{rad}},$$

the radial part of $D$ along $A_{\mathrm{q}}^+$. Thus, the fact that $\mathcal{T}$ commutes with $\mathbb{D}(\mathrm{X})$ implies that

$$[\mathcal{T}^{\mathrm{rad}}, D^{\mathrm{rad}}] = 0, \qquad (D \in \mathbb{D}(\mathrm{X})). \tag{12.10}$$

These commutation relations form a cofinite system of differential equations for the coefficients of the differential operator $\mathcal{T}^{\mathrm{rad}}$, with regular singularities at infinity. Moreover, the associated system of indicial equations has only the trivial exponent as solution, so that $\mathcal{T}^{\mathrm{rad}}$ is completely determined by its top order asymptotic behavior at infinity. Let $\langle\,\cdot\,,\,\cdot\,\rangle_J$ denote the $L^2$-inner product on $C_c^\infty(A_\mathrm{q}^+, V_\tau^{M\cap K\cap H})$ associated with the measure $Jda$ on $A_\mathrm{q}^+$. Then it follows from Theorem 3.9 that

$$\langle f\,,\,g\rangle_J = \langle T^\uparrow f\,,\,T^\uparrow g\rangle_{L^2(\mathrm{X}:\tau)}. \tag{12.11}$$

Using the asymptotic behavior of the Eisenstein integral, as described in Theorem 12.1, combined with the Maass–Selberg relations (12.3), it can be shown that

$$\langle \mathcal{T}^{\mathrm{rad}} f_n\,,\,g_n\rangle_J \sim \langle D_\tau^{\mathrm{rad}} f_n\,,\,g_n\rangle_J, \qquad (n\to\infty),$$

for $f_n, g_n$ sequences of functions in $C_c^\infty(A_\mathrm{q}^+, V_\tau^{M\cap K\cap H})$ with the property that $T^\uparrow f_n$ and $T^\uparrow g_n$ have $L^2$-norm 1 and that the compact sets $\operatorname{supp} f_n$ and $\operatorname{supp} g_n$ tend to infinity in $A_\mathrm{q}^+$, for $n\to\infty$. From this it can be deduced that $\mathcal{T}^{\mathrm{rad}}$ has the same top order behavior as $D_\tau^{\mathrm{rad}}$. The latter operator satisfies the same commutation relations (12.10) as the operator $\mathcal{T}^{\mathrm{rad}}$. Therefore,

$$\mathcal{T}^{\mathrm{rad}} = D_\tau^{\mathrm{rad}}.$$

Applying a similar argument involving chambers of the form $v^{-1}A_\mathrm{q}^+ v$, for $v\in\mathcal{W}$, it follows that $\mathcal{T} = D_\tau$ on functions from $C_c^\infty(\mathrm{X}:\tau)$ supported by a compact subset of $\cup_{v\in\mathcal{W}} KA_\mathrm{q}^+ vH$. Since the latter union is open and dense in X it follows that $\mathcal{T} = D_\tau$. This completes the proof of part (b) of Theorem 11.27.

The proof of part (a) is now based on the following result from [13]. Let $\mathfrak{b}$ be a $\theta$-invariant Cartan subspace of $\mathfrak{q}$, containing $\mathfrak{a}_\mathrm{q}$. Let $\Sigma(\mathfrak{b})$ be the root system of $\mathfrak{b}$ in $\mathfrak{g}_\mathbb{C}$, and let $\Sigma^+(\mathfrak{b})$ be a positive system for $\Sigma(\mathfrak{b})$ that is compatible with $\mathfrak{a}_\mathrm{q}$, i.e., the nonzero restrictions $\alpha|_{\mathfrak{a}_\mathrm{q}}$, for $\Sigma^+(\mathfrak{b})$ form a positive system for $\Sigma$. Let $\mathfrak{n}_\mathfrak{m}$ be the sum of the root spaces in $\mathfrak{g}_\mathbb{C}$ for the roots of $\Sigma^+(\mathfrak{b})$ that vanish on $\mathfrak{a}_\mathrm{q}$. We define the linear functional $\rho_\mathfrak{m}\in\mathfrak{b}_\mathbb{C}^*$ by

$$\rho_\mathfrak{m}(X) = \frac{1}{2}\mathrm{tr}\,(\mathrm{ad}\,(X)|_{\mathfrak{n}_\mathfrak{m}}).$$

**Theorem 12.3** *Let $D\in\mathbb{D}(\mathrm{X})$ and let $D^*$ denote its formal adjoint. Assume that the polynomial function $\lambda\mapsto\gamma(D^*:\lambda+\rho_\mathfrak{m})$ is nontrivial on $\mathfrak{a}_\mathrm{q}^*$. Then $D$ is injective on $C_c^\infty(\mathrm{X})$.*

The proof of this result, due to [13], is based on an application of Holmgren's uniqueness theorem from the theory of linear partial differential equations with analytic coefficients, see [64], Thm. 5.3.1.

Part (a) of Theorem 11.27 follows because the operator $D_\tau$ can be constructed in such a way that it satisfies the condition of Theorem 12.3.

The following is an immediate consequence of Theorem 11.27.

**Corollary 12.4** *The Fourier transform $\mathcal{F} = \mathcal{F}_{P_0}$ is injective on $C_c^\infty(\mathrm{X}:\tau)$.*

**Two problems** In view of Corollary 12.4, it is natural to consider the following two problems.

**(a)** Find an inversion formula expressing $f \in C_c^\infty(\mathrm{X})$ in terms of its most-continuous Fourier transform $\mathcal{F}f$. This is the problem of *Fourier inversion.*

**(b)** Give a characterization of the image of $C_c^\infty(\mathrm{X} : \tau)$ under the most-continuous Fourier transform $\mathcal{F}$. The solution of this problem would amount to a *Paley–Wiener theorem.*

The solution to the first of these problems will be presented in the next section. It is an important step towards both the Plancherel and the Paley–Wiener theorem. The application to the Paley–Wiener theorem will be discussed elsewhere in this volume, by H. Schlichtkrull.

# 13 Fourier inversion

**Partial Eisenstein integrals** We retain the notation of the previous section. For the formulation of the Fourier inversion theorem it will be convenient to introduce the concept of the so-called partial Eisenstein integral. First, we recall from Corollary 3.7, that the open dense subset $\mathrm{X}_+$ of X can be written as the disjoint union

$$\mathrm{X}_+ = \cup_{v \in \mathcal{W}} \, KA_{\mathrm{q}}^+ vH.$$

Since $\mathrm{X}_+$ is left invariant under $K$, it makes sense to define the space $C^\infty(\mathrm{X}_+ : \tau)$ of smooth $\tau$-spherical functions $f : \mathrm{X}_+ \to V_\tau$, by imposing the rule (11.1) for $x \in \mathrm{X}_+$ and $k \in K$. Accordingly, via restriction to $\mathrm{X}_+$, the space $C^\infty(\mathrm{X} : \tau)$ may be viewed as the subspace of functions in $C^\infty(\mathrm{X}_+ : \tau)$ that have a smooth extension to all of X.

In the following definition, we transfer the tensor product representation $1 \otimes \tau$ of $K$ in ${}^\circ\mathcal{C}(\tau)^* \otimes V_\tau$ to a representation of $K$ in $\mathrm{Hom}\,({}^\circ\mathcal{C}(\tau), V_\tau)$, via the obvious natural isomorphism.

**Definition 13.1** Let $s \in W$. The *partial Eisenstein integral* $E_{+,s}(\lambda : \cdot)$, for generic $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$, is defined to be the $1 \otimes \tau$-spherical function in $C^\infty(\mathrm{X}_+) \otimes \mathrm{Hom}\,({}^\circ\mathcal{C}(\tau), V_\tau)$, given by

$$E_{+,s}(\lambda : kav)\psi = \tau(k)\Phi_v(s\lambda, a)[C^\circ(s : \lambda)\psi]_v(e),$$

for $\psi \in {}^\circ\mathcal{C}(\tau)$, $k \in K$, $a \in A_{\mathrm{q}}^+$ and $v \in \mathcal{W}$.

The partial Eisenstein integral is viewed as a function depending meromorphically on the parameter $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. In addition, we agree to write

$$E_+(\lambda : \cdot\,) := E_{+,1}(\lambda : \cdot\,).$$

In view of (12.2), the partial Eisenstein integral $E_{+,s}$ is related to the one above by

$$E_{+,s}(\lambda : \cdot\,) = E_+(s\lambda : \cdot\,)C^\circ(s : \lambda), \tag{13.1}$$

for each $s \in W$.

With this notation, the equalities (12.9), for $v \in \mathcal{W}$, can be rephrased as the single equality

$$\mathcal{J}\mathcal{F}D_\tau f(x) = |W| \int_{\eta + i\mathfrak{a}_\mathrm{q}^*} E_+(\lambda : x)\underline{\mu}(D_\tau, \lambda)\mathcal{F}f(\lambda)\, d\lambda, \tag{13.2}$$

valid for $f \in C_c^\infty(\mathrm{X} : \tau)$, $x \in \mathrm{X}_+$ and $\eta \in \mathfrak{a}_\mathrm{q}^*$ sufficiently antidominant. Given $f \in C_c^\infty(\mathrm{X} : \tau)$, we now agree to write

$$\mathcal{T}_\eta \mathcal{F}f(x) := |W| \int_{\eta + i\mathfrak{a}_\mathrm{q}^*} E_+(\lambda : x)\mathcal{F}f(\lambda)\, d\lambda, \qquad (x \in \mathrm{X}_+), \tag{13.3}$$

for every $\eta \in \mathfrak{a}_\mathrm{q}^*$ for which the integrand is regular on $\eta + i\mathfrak{a}_\mathrm{q}^*$. Then $\mathcal{T}_\eta\mathcal{F}f$ defines an element of $C^\infty(\mathrm{X}_+ : \tau)$ which is independent of $\eta$, as long as $\eta + i\mathfrak{a}_\mathrm{q}^*$ varies in a connected open subset of $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ on which the integrand is regular, in view of Cauchy's theorem.

The Fourier inversion theorem asserts that the formula (13.2) is actually valid with $D_\tau$ replaced by 1. For $R \in \mathbb{R}$ we define

$$\mathfrak{a}_\mathrm{q}^*(P_0, R) = \{\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \mid \langle \mathrm{Re}\,\lambda\,,\, \alpha\rangle < R \;\; (\forall \alpha \in \Sigma^+)\}.$$

**Theorem 13.2** (Fourier inversion) *There exists a constant $R \in \mathbb{R}$ such that both functions $\lambda \mapsto E_+(\lambda : \cdot\,)$ and $\lambda \mapsto E^*(\lambda : \cdot\,)$ are holomorphic in the open set $\mathfrak{a}_\mathrm{q}^*(P_0, R)$. If $f \in C_c^\infty(\mathrm{X} : \tau)$, then, for every $\eta \in \mathfrak{a}_\mathrm{q}^*(P_0, R)$,*

$$f = \mathcal{T}_\eta \mathcal{F}f \quad \text{on} \quad \mathrm{X}_+. \tag{13.4}$$

Since $\mathfrak{a}_\mathrm{q}^*(P_0, R)$ is convex, hence connected, $\mathcal{T}_\eta\mathcal{F}f$ is independent of the particular choice of $\eta \in \mathfrak{a}_\mathrm{q}^*(P_0, R)$. We will first establish the theorem under the assumption that $\mathcal{T}_\eta\mathcal{F}f$ extends smoothly to all of X. Assume this to be true. Then by the Paley–Wiener shift technique discussed in the previous section, it can be shown that $\mathcal{T}_\eta\mathcal{F}f$ is compactly supported, hence belongs to $C_c^\infty(\mathrm{X} : \tau)$. The partial Eisenstein integral is readily seen to satisfy the differential equations (12.1) on $\mathrm{X}_+$. It follows from this that

$$D_\tau \mathcal{T}_\eta \mathcal{F}f = \mathcal{T}_\eta(\underline{\mu}(D_\tau : \cdot\,)\mathcal{F}f) = \mathcal{T}_\eta(\mathcal{F}D_\tau f),$$

on $\mathrm{X}_+$, hence on X. In view of (13.2) and Theorem 11.27 (b) we now see that $f - \mathcal{T}_\eta\mathcal{F}f$ is a function in $C_c^\infty(\mathrm{X} : \tau)$, annihilated by $D_\tau$. Hence, (13.4) follows by application of Theorem 11.27 (a).

Thus, to complete the proof of Theorem 13.2, we must show that $\mathcal{T}_\eta\mathcal{F}f$ extends smoothly from $\mathrm{X}_+$ to all of X. This will be achieved by a shift of integration, where $\eta \to 0$. The shift will give rise to residual integrals that are to be shifted again according to a certain rule.

We shall describe some ideas of this residue calculus, which is developed in the paper [18]. The notion of residue will be defined in terms of the notion of Laurent functional, which generalizes the idea of taking linear combinations of coefficients in Laurent series in one variable theory.

**Laurent functionals** Let $V$ be a finite-dimensional real linear space and let $X \subset V^* \setminus \{0\}$ be a finite subset. For any point $a \in V_\mathbb{C}$, we define a polynomial function $\pi_a : V_\mathbb{C} \to \mathbb{C}$ by

$$\pi_a := \prod_{\xi \in X} (\xi - \xi(a)).$$

The ring of germs of meromorphic functions at $a$ is denoted by $\mathcal{M}(V_\mathbb{C}, a)$, and the subring of germs of holomorphic functions by $\mathcal{O}_a$. In terms of this subring we define the subring

$$\mathcal{M}(V_\mathbb{C}, a, X) := \cup_{N \in \mathbb{N}} \ \pi_a^{-N} \mathcal{O}_a.$$

We use the notation $\mathrm{ev}_a$ for the linear functional on $\mathcal{O}_a$ that assigns to any $f \in \mathcal{O}_a$ its value $f(a)$ at $a$. We agree to write $S(V)$ for the symmetric algebra of $V_\mathbb{C}$, and identify it with the algebra of constant coefficient complex differential operators on $V_\mathbb{C}$.

**Definition 13.3** An $X$-*Laurent functional* at $a \in V_\mathbb{C}$ is a linear functional $\mathcal{L} \in \mathcal{M}(V_\mathbb{C}, a, X)^*$ such that for any $N \in \mathbb{N}$ there exists a $u_N \in S(V)$ such that

$$\mathcal{L} = \mathrm{ev}_a \circ u_N \circ \pi_a^N \qquad \text{on} \quad \pi_a^{-N} \mathcal{O}_a. \tag{13.5}$$

The space of all Laurent functionals on $V_\mathbb{C}$, relative to $X$, is defined as the algebraic direct sum of linear spaces

$$\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}} := \bigoplus_{a \in V_\mathbb{C}} \mathcal{M}(V_\mathbb{C}, X, a)^*_{\mathrm{laur}}. \tag{13.6}$$

For $\mathcal{L}$ in the space (13.6), the finite set of $a \in V_\mathbb{C}$ for which the component $\mathcal{L}_a$ is nonzero is called the support of $\mathcal{L}$ and is denoted by $\mathrm{supp}\,\mathcal{L}$.

According to the above definition, any $\mathcal{L} \in \mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}}$, may be decomposed as

$$\mathcal{L} = \sum_{a \in \mathrm{supp}\,\mathcal{L}} \mathcal{L}_a.$$

Let $\mathcal{M}(V_\mathbb{C}, X)$ denote the space of meromorphic functions $\varphi$ on $V_\mathbb{C}$ with the property that the germ $\varphi_a$ at any point $a \in V_\mathbb{C}$ belongs to $\mathcal{M}(V_\mathbb{C}, a, X)$. Then we have the natural bilinear map $\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}} \times \mathcal{M}(V_\mathbb{C}, X) \to \mathbb{C}$, given by

$$(\mathcal{L}, \varphi) \mapsto \mathcal{L}\varphi := \sum_{a \in \mathrm{supp}\,\mathcal{L}} \mathcal{L}_a \varphi_a.$$

This bilinear map naturally induces an embedding of the space $\mathcal{M}(V_\mathbb{C}, X)^*_{\mathrm{laur}}$ onto a linear subspace of the dual space $\mathcal{M}(V_\mathbb{C}, X)^*$. For more details concerning these definitions, we refer the reader to [20], Sect. 12.

**A residue calculus for root systems** We consider the nonrestricted root system $\Sigma$ of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$. In this subsection we shall describe a residue calculus entirely in terms of the given root system, without reference to the harmonic analysis on X. More details can be found in [18]. At the end of Section 13 we shall discuss the application of the residue calculus to harmonic analysis on the symmetric space.

We equip $\mathfrak{a}_{\mathrm{q}}$ with a $W$-invariant positive definite inner product $\langle \cdot\, , \,\cdot \rangle$. It induces a real linear isomorphism $\mathfrak{a}_{\mathrm{q}} \simeq \mathfrak{a}_{\mathrm{q}}^*$ via which we shall identify these spaces. Accordingly it makes sense to speak of the spaces $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)$ and $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \Sigma)_{\mathrm{laur}}^*$. Let $\mathcal{H}$ be a locally finite collection of real $\Sigma$-hyperplanes, i.e., hyperplanes $H \subset \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ given by an equation of the form $\langle \alpha_H\, , \,\cdot \rangle = c_H$, with $\alpha_H \in \Sigma$ and $c_H \in \mathbb{R}$. We define $\mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H})$ to be the space of meromorphic functions $\varphi$ on $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ with singular locus contained in $\cup\mathcal{H}$. Moreover, we define $\mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H})$ to be the subspace of functions $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H})$ with fast decrease along strips in the imaginary directions. More precisely, this requirement of fast decrease means that for every compact subset $\omega \subset \mathfrak{a}_{\mathrm{q}}^*$, there exists a polynomial function $q_\omega : \mathfrak{a}_{\mathrm{q}\mathbb{C}}^* \to \mathbb{C}$ that is a product of powers of linear factors of the form $\langle \alpha_H\, , \,\cdot \rangle - c_H$, for $H \in \mathcal{H}$, such that the function $\varphi$ satisfies the estimate

$$\sup_{\lambda \in \omega + i\mathfrak{a}_{\mathrm{q}}^*} (1 + |\lambda|)^n |q_\omega(\lambda)\varphi(\lambda)| < \infty,$$

for every $n \in \mathbb{N}$.

By a root space in $\mathfrak{a}_{\mathrm{q}}^*$ we mean any finite intersection of root hyperplanes of the form $\alpha^\perp$, for $\alpha \in \Sigma$. The collection of root spaces is denoted by $\mathcal{R} = \mathcal{R}_\Sigma$. It is understood that $\mathfrak{a}_{\mathrm{q}}^* \in \mathcal{R}$. The map $P \mapsto \mathfrak{a}_{P\mathrm{q}}^*$ is surjective from $\mathcal{P}_\sigma$ onto $\mathcal{R}$. For this map, the fiber $\mathcal{P}_\sigma(\mathfrak{b})$ of an element $\mathfrak{b}$ consists of all $P \in \mathcal{P}_\sigma$ with $\mathfrak{a}_{P\mathrm{q}}^* = \mathfrak{b}$. We agree to equip $\mathfrak{b}$ with the Euclidean Lebesgue measure associated with the dual inner product $\langle \cdot\, , \,\cdot \rangle$ of $\mathfrak{a}_{\mathrm{q}}^*$. For each $\eta \in \mathfrak{a}_{\mathrm{q}}^*$, the image of this measure under the map $\nu \mapsto \eta + i\nu$, $\mathfrak{b} \to \eta + i\mathfrak{b}$, is denoted by $d\mu_{\mathfrak{b}}$.

If $\mathfrak{b} \in \mathcal{R}$, then $\mathfrak{b}^\perp$ is called a Levi subspace. This terminology has the following explanation. If $P \in \mathcal{P}_\sigma(\mathfrak{b})$, then $\mathfrak{b}^\perp$ equals ${}^*\mathfrak{a}_{P\mathrm{q}}^*$, which is the analogue of $\mathfrak{a}_{\mathrm{q}}^*$ for the Levi component $M_P$ of $P$. We note that

$$\Sigma_{\mathfrak{b}^\perp} := \Sigma \cap \mathfrak{b}^\perp \tag{13.7}$$

is a root system in $\mathfrak{b}^\perp$. The map $P \mapsto \Sigma(P)$ maps the collection $\mathcal{P}_\sigma^{\min}$ of minimal elements in $\mathcal{P}_\sigma$ bijectively onto the collection of positive systems for $\Sigma$. If $P \in \mathcal{P}_\sigma^{\min}$ and $\mathfrak{b} \in \mathcal{R}$, then $\Sigma(P) \cap \mathfrak{b}^\perp$ is a positive system for (13.7).

If $\mathfrak{b} \in \mathcal{R}$, then $\mathfrak{b}^{\mathrm{reg}}$ is defined to be the intersection of all sets $\mathfrak{b} \setminus \alpha^\perp$, for $\alpha \in \Sigma \setminus \Sigma_{\mathfrak{b}^\perp}$. We observe that $\mathfrak{b}^{\mathrm{reg}}$ is the disjoint union of the chambers $\mathfrak{a}_{P\mathrm{q}}^{*+}$, for $P \in \mathcal{P}_\sigma(\mathfrak{b})$.

For $\mathfrak{b} \in \mathcal{R}$ and $\lambda \in \mathfrak{a}_{\mathrm{q}}^*$ we denote by $\mathcal{H}_{\lambda+\mathfrak{b}}$ the collection of $\Sigma$-hyperplanes in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$ containing $\lambda + \mathfrak{b}_{\mathbb{C}}$. Clearly, $\mathcal{H}_{\lambda+\mathfrak{b}} = \lambda + \mathcal{H}_{\mathfrak{b}}$. Moreover, if $P \in \mathcal{P}_\sigma(\mathfrak{b})$, then $\mathcal{H}_{\mathfrak{b}} = \{\alpha_{\mathbb{C}}^\perp \mid \alpha \in \Sigma_P\}$.

By a residue weight on $\Sigma$ we mean a function $t : \mathcal{P}_\sigma \to [0, 1]$ such that for all $\mathfrak{b} \in \mathcal{R}$,

$$\sum_{P \in \mathcal{P}_\sigma(\mathfrak{b})} t(P) = 1. \tag{13.8}$$

Observe that by Corollary 6.8 and (6.3), the map $P \mapsto \mathfrak{a}_{P\mathrm{q}}^{+}$ is a bijection from $\mathcal{P}_\sigma$ onto the Coxeter complex $\mathcal{P}(\Sigma)$, so that the residue weight is a notion completely defined in terms of the root system. Accordingly, we shall sometimes view $t$ as a map $\mathcal{P}(\Sigma) \to [0,1]$ and write $t(\mathfrak{a}_{P\mathrm{q}}^{+})$ instead of $t(P)$.

We now have the following result, which characterizes residual Laurent operators in terms of a collection of integral shifts governed by a particular choice of residue weight. If $\xi$ is a point and $\mathfrak{b}$ a root space in $\mathfrak{a}_\mathrm{q}^*$, then by $\xi_{\mathfrak{b}^\perp}$ we denote the orthogonal projection of $\xi$ onto $\mathfrak{b}^\perp$. The collection of complexified root hyperplanes $\alpha_\mathbb{C}^\perp$, for $\alpha \in \Sigma$, is denoted by $\mathcal{H}_\Sigma(0)$.

**Proposition 13.4** *Let $t$ be a residue weight on $\Sigma$, let $P \in \mathcal{P}_\sigma^{\min}$, and let $\xi \in \mathfrak{a}_\mathrm{q}^*$. Then there exist unique Laurent functionals*

$$\mathrm{Res}_{\xi+\mathfrak{b}}^{\mathrm{P,t}} \in \mathcal{M}(\mathfrak{b}_\mathbb{C}^\perp, \xi_{\mathfrak{b}^\perp}, \Sigma_{\mathfrak{b}^\perp})_{\mathrm{laur}}^*,$$

*for $\mathfrak{b} \in \mathcal{R}$, such that the following is valid for $\eta \in \mathfrak{a}_\mathrm{q}^*$ sufficiently $\Sigma(\bar{P})$-dominant. For every $\varphi \in \mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \xi + \mathcal{H}_\Sigma(0))$,*

$$\int_{\eta+i\mathfrak{a}_\mathrm{q}^*} \varphi(\lambda)\, d\lambda = \sum_{\mathfrak{b}\in\mathcal{R}} \sum_{P\in\mathcal{P}_\sigma(\mathfrak{b})} t(P) \int_{\mathrm{pt}\,(\mathfrak{a}_{P\mathrm{q}}^{*+})+i\mathfrak{b}} \mathrm{Res}_{\xi+\mathfrak{b}}^{\mathrm{P,t}}[\varphi(\,\cdot\,+\nu)]\, \mathrm{d}\mu_\mathfrak{b}(\nu), \qquad (13.9)$$

*where* $\mathrm{pt}\,(\mathfrak{a}_{P\mathrm{q}}^{*+})$ *denotes an arbitrary choice of point in $\mathfrak{a}_{P\mathrm{q}}^{*+}$, for each $P \in \mathcal{P}_\sigma$.*

For the proof of this proposition we refer the reader to [18], Thm. 1.13 and Sect. 3. The idea is that the integral on the left-hand side of (13.9) is shifted to a similar integral with $\eta$ close to zero; the latter integral is distributed over the open Weyl chambers according to the residue weights. The shift is along a path that intersects the singular hyperplanes for the integrand one at a time. By applying the classical residue calculus with respect to a one-dimensional variable transversal to an encountered singular hyperplane $\eta + \mathfrak{b}_\mathbb{C}$ one obtains a residual integral along a codimension 1 hyperplane of the form $\eta + i\mathfrak{b}$, with $\eta \in \xi + \mathfrak{b}$. Such an integral is shifted in the manner described above, in order to move $\eta$ inside $\xi + \mathfrak{b}$ to a position close to $\xi_{\mathfrak{b}^\perp}$. The latter point may be characterized as the central point of $\xi + \mathfrak{b}$, i.e., the point closest to the origin. The shifted integral is distributed over the chambers $\xi_{\mathfrak{b}^\perp} + \mathfrak{a}_{P\mathrm{q}}^{+}$, for $P \in \mathcal{P}_\sigma(\mathfrak{b})$, with weights determined by the residue weight $t$; this explains condition (13.8). In the process of shifting, residual integrals split off. These are treated in a similar fashion, leading to residual integrals over affine spaces of lower and lower dimension. As a result one ends up with the sum on the right of (13.9). This idea of shifting is present in the work of R.P. Langlands [68] on automorphic forms, see also [71], and in that of J. Arthur [1] on the Paley–Wiener theorem for real reductive groups.

Another key idea needed in the proof of Proposition 13.4 is that the residue operators are uniquely determined by the requirement that the formula be valid on the large indicated space of test functions. This idea goes back to G.J. Heckman and E.M. Opdam, [59]. It is this idea that allows one to develop the residue calculus in terms of the root system only, without reference to the harmonic analysis on X.

If $S$ is a finite subset of a linear space $V$, then by $\Gamma^-(S)$ we denote the closed convex cone spanned by 0 and the points of $-S$. In particular, $\Gamma^-(\emptyset) = \{0\}$.

**Proposition 13.5** *Let $t$ be a residue weight on $\Sigma$, $P \in \mathcal{P}_\sigma^{\min}$, $\xi \in \mathfrak{a}_{q\mathbb{C}}^*$ and $\mathfrak{b} \in \mathcal{R}$, and assume that*

$$\mathrm{Res}_{\xi+\mathfrak{b}}^{P,t} \neq 0. \tag{13.10}$$

*Then $\xi_{\mathfrak{b}^\perp}$ is contained in the closed convex cone $\Gamma^-(\Sigma(P) \cap \mathfrak{b}^\perp)$.*

*Proof.* We explain the idea of the proof, referring to [18] for details.

In the proof of Proposition 13.4 it is seen that the nontrivial contributions to the residual operator in (13.10) come from successively taking residues in variables transversal to hyperplanes of the form $\xi + \alpha_{\mathbb{C}}^\perp$ that contain $\xi + \mathfrak{b}$. Each step involves a residual integral along $\eta + i\mathfrak{c}$, with $\mathfrak{c} \in \mathcal{R}$ and with $\eta \in \xi + \mathfrak{c}$. Moreover, in such a step $\eta$ crosses a hyperplane of the form $\xi + (\mathfrak{c} \cap \alpha^\perp)$, for $\alpha \in \Sigma(P) \cap \Sigma_{\mathfrak{b}^\perp} \setminus \Sigma_{\mathfrak{c}^\perp}$. At the moment of crossing, $\eta$ comes from the region $\langle \eta - \xi\,,\, \alpha\rangle < 0$ in $\xi + \mathfrak{c}$. Moreover, the crossing only needs to take place if $\xi_{\mathfrak{c}^\perp}$ is not in the same region in $\xi + \mathfrak{c}$, i.e.,

$$\langle \xi_{\mathfrak{c}^\perp} - \xi\,,\, \alpha\rangle \geq 0. \tag{13.11}$$

The crossing causes a residual integral along $\eta_0 + i\mathfrak{c}_0$ to split off, with $\mathfrak{c}_0 = \mathfrak{c} \cap \alpha^\perp$ and $\eta_0 \in \mathfrak{c}_0$.

From (13.10) it follows that a crossing as above occurs with $\mathfrak{c}_0 = \mathfrak{b}$ and with $\mathrm{Res}_{\xi+\mathfrak{c}}^{P,t} \neq 0$. Applying induction with respect to codim $\mathfrak{b}$ we may assume that

$$\xi_{\mathfrak{c}^\perp} \in \Gamma^-(\Sigma(P) \cap \mathfrak{c}^\perp). \tag{13.12}$$

On the other hand, it is clear that $\xi - \xi_{\mathfrak{c}^\perp}$ is perpendicular to the roots from $\Sigma(P) \cap \mathfrak{c}^\perp$. The positive system $\Sigma(P) \cap \mathfrak{b}^\perp$ for $\Sigma_{\mathfrak{b}^\perp}$ has precisely one simple root not perpendicular to $\mathfrak{c}$. If we combine this with the inequality (13.11) for a certain root $\alpha$ from $\Sigma(P) \cap \Sigma_{\mathfrak{b}^\perp} \setminus \Sigma_{\mathfrak{c}^\perp}$, we see that $\langle \xi_{\mathfrak{c}^\perp} - \xi\,,\, \beta\rangle \geq 0$ for all $\beta \in \Sigma(P) \cap \mathfrak{b}^\perp$. This implies that $\xi - \xi_{\mathfrak{c}^\perp}$ lies in the negative chamber in $\mathfrak{b}^\perp$ associated with the positive system $\Sigma(P) \cap \mathfrak{b}^\perp$. The chamber in turn is contained in $\Gamma^-(\Sigma(P) \cap \mathfrak{b}^\perp)$, so that

$$\xi_{\mathfrak{b}^\perp} - \xi_{\mathfrak{c}^\perp} \in \Gamma^-(\Sigma(P) \cap \mathfrak{b}^\perp).$$

Combining this with (13.12) we deduce that

$$\xi_{\mathfrak{b}^\perp} \in \Gamma^-(\Sigma(P) \cap \mathfrak{b}^\perp) + \Gamma^-(\Sigma(P) \cap \mathfrak{c}^\perp) \subset \Gamma^-(\Sigma(P) \cap \mathfrak{b}^\perp).$$

$\square$

**Transitivity of residues** We shall now describe a result which ensures that the residue operators behave well with respect to parabolic induction. Via the natural isomorphism $\mathfrak{a}_q \simeq \mathfrak{a}_q^*$ we shall view the Coxeter complex $\mathcal{P} = \mathcal{P}(\Sigma)$ as the set of facets in $\mathfrak{a}_q^*$. Accordingly, $P \mapsto \mathfrak{a}_{Pq}^{*+}$ defines a bijection from $\mathcal{P}_\sigma$ onto $\mathcal{P}$.

Let $\mathfrak{b} \in \mathcal{R}$; then the map $\mathfrak{s} \mapsto \mathfrak{s} + \mathfrak{b}$ is a bijection from the collection $\mathcal{R}(\Sigma_{\mathfrak{b}^\perp})$ of root spaces in $\mathfrak{b}^\perp$ for the root system $\Sigma_{\mathfrak{b}^\perp}$ onto the collection

$$\mathcal{R}_{\supset \mathfrak{b}} := \{\mathfrak{c} \in \mathcal{R} \mid \mathfrak{c} \supset \mathfrak{b}\}.$$

If $\mathfrak{c} \in \mathcal{R}_{\supset \mathfrak{b}}$, then the associated root space in $\mathfrak{b}^\perp$ is given by ${}^*\mathfrak{c} = \mathfrak{c} \cap \mathfrak{b}^\perp$. In particular, it follows from the above considerations that

$$\mathfrak{c}^{\mathrm{reg}} \subset ({}^*\mathfrak{c})^{\mathrm{reg}} + \mathfrak{b} \subset \mathfrak{c}.$$

From this in turn we see that for each open chamber $C$ in $\mathfrak{c}$, there exists a unique open chamber in ${}^\star\mathfrak{c}$, relative to the root system $\Sigma_{\mathfrak{b}^\perp}$, such that $C \subset {}^*C + \mathfrak{b}$. Let $\mathcal{P}_{\supset \mathfrak{b}}$ denote the collection of facets in $\mathcal{P}$ whose linear span is a root space containing $\mathfrak{b}$. Then $C \mapsto {}^*C$ defines a surjective map from $\mathcal{P}_{\supset \mathfrak{b}}$ onto the Coxeter complex $\mathcal{P}_{\mathfrak{b}^\perp}$ of the root system $(\mathfrak{b}^\perp, \Sigma_{\mathfrak{b}^\perp})$.

Keeping the above in mind we see that any residue weight $t$ on $\Sigma$ induces a residue weight ${}^*t$ on $\Sigma_{\mathfrak{b}^\perp}$ given by

$${}^*t(D) = \sum_{C \in \mathcal{P}_{\supset \mathfrak{b}}\,, {}^*C = D} t(C).$$

If $\mathfrak{b} \in \mathcal{R}$, then the group $M_Q$ is independent of the particular choice of $Q \in \mathcal{P}_\sigma(\mathfrak{b})$; we denote it by $M_{\mathfrak{b}}$. It is invariant under both the involutions $\theta$ and $\sigma$; moreover, the space $\mathfrak{b}^\perp$ is the analogue of $\mathfrak{a}_{\mathrm{q}}^*$ for $M_{\mathfrak{b}}$. We denote by $\mathcal{P}_\sigma(M_{\mathfrak{b}})$ the analogue of the set $\mathcal{P}_\sigma$ for $M_{\mathfrak{b}}$. The map $Q \mapsto \mathfrak{a}_{Q\mathrm{q}}$ defines a bijection from $\mathcal{P}_\sigma(M_{\mathfrak{b}})$ onto $\mathcal{P}_{\mathfrak{b}^\perp}$. Let $\mathcal{P}_{\sigma, \supset \mathfrak{b}}$ be the collection of $P \in \mathcal{P}_\sigma$ with $\mathfrak{a}_{P\mathrm{q}}^* \supset \mathfrak{b}$. Then the map $P \mapsto \mathfrak{a}_{P\mathrm{q}}^*$ is a bijection from $\mathcal{P}_{\sigma, \supset \mathfrak{b}}$ onto $\mathcal{P}_{\supset \mathfrak{b}}$. Via the mentioned bijections, we transfer the map $C \mapsto {}^*C$ described above to a surjection $P \mapsto {}^*P$ from $\mathcal{P}_{\sigma, \supset \mathfrak{b}}$ onto $\mathcal{P}_\sigma(M_{\mathfrak{b}})$. We note that for all $P \in \mathcal{P}_{\sigma, \supset \mathfrak{b}}$ we have ${}^*P = P \cap M_{\mathfrak{b}}$. If $P$ is a minimal element of $\mathcal{P}_\sigma$, then $P \in \mathcal{P}_{\sigma, \supset \mathfrak{b}}$ and ${}^*P$ is a minimal element in $\mathcal{P}_\sigma(M_{\mathfrak{b}})$. We note that $\Sigma({}^*P) = \Sigma(P) \cap \mathfrak{b}^\perp$.

**Lemma 13.6** *Let $P \in \mathcal{P}_\sigma^{\min}$, $t$ a residue weight on $\Sigma$, $\xi \in \mathfrak{a}_{\mathrm{q}}^*$ and $\mathfrak{b} \in \mathcal{R}$. Then for every root space $\mathfrak{c}$ containing $\mathfrak{b}$ we have*

$$\mathrm{Res}^{\mathrm{P,t}}_{\xi + \mathfrak{c}} = \mathrm{Res}^{{}^*\mathrm{P}, {}^*\mathrm{t}}_{\xi_{\mathfrak{b}^\perp} + {}^*\mathfrak{c}}.$$

This result is a rather straightforward consequence of the characterization of the residual operators in Proposition 13.4. We refer to [18] for details. Taking $\mathfrak{c} = \mathfrak{b}$ we see that each residual operator may be viewed as a point residual operator in a suitable Levi subspace.

**Action by the Weyl group** Another crucial aspect of the residual calculus is that it behaves well under the action of the Weyl group. The Weyl group $W$ acts naturally on the set of root spaces $\mathcal{R}$, on the Coxeter complex $\mathcal{P}$ and on the collection of residue weights $\mathrm{WT}(\Sigma)$. Moreover, if $w \in W$ and $\mathfrak{b} \in \mathcal{R}$, then the map $w : \mathfrak{b}^\perp \to w(\mathfrak{b})^\perp$ naturally induces a map

$$w_* : \mathcal{M}(\mathfrak{b}_{\mathbb{C}}^\perp, \Sigma_{\mathfrak{b}^\perp})^*_{\mathrm{laur}} \to \mathcal{M}(w\mathfrak{b}_{\mathbb{C}}^\perp, \Sigma_{w\mathfrak{b}^\perp})^*_{\mathrm{laur}}.$$

It readily follows from the characterization in Proposition 13.4 that the residual operators transform naturally for the mentioned actions of the Weyl group. Thus, let $P, t, \xi, \mathfrak{b}$ be as in Lemma 13.6; then

$$w_* \mathrm{Res}^{\mathrm{P,t}}_{\xi+\mathfrak{b}} = \mathrm{Res}^{\mathrm{wP,wt}}_{\mathrm{w}\xi+\mathrm{w}\mathfrak{b}} \qquad (\mathrm{w} \in \mathrm{W}). \tag{13.13}$$

Here we have written $wP$ for the $w$-conjugate of $P$; it is given by $wP = \bar{w}P\bar{w}^{-1}$, with $\bar{w} \in N_K(\mathfrak{a}_{\mathrm{q}})$ a representative for $w$. In addition to this transformation formula we have the following result.

**Lemma 13.7** *Let $P \in \mathcal{P}_\sigma^{\min}$, $t$ a residue weight on $\Sigma$, $\xi \in \mathfrak{a}_{\mathrm{q}}^*$ and $\mathfrak{b} \in \mathcal{R}$. Moreover, let $w \in W$ be such that $w(\Sigma(P) \cap \mathfrak{b}^\perp) \subset \Sigma(P)$. Then the operators in (13.13) equal $\mathrm{Res}^{\mathrm{P,wt}}_{\mathrm{w}\xi+\mathrm{w}\mathfrak{b}}$.*

*Proof.* The hypothesis implies that $w(\Sigma(P)) \cap (w\mathfrak{b})^\perp \subset \Sigma(P) \cap (w\mathfrak{b})^\perp$. Since both intersections are positive systems for $\Sigma_{w\mathfrak{b}^\perp}$ and are nested, they are equal. Hence, ${}^*P = {}^*(wP)$. Now apply Lemma 13.6 with $w\mathfrak{b}$ in place of $\mathfrak{b}$. □

We now come to a result which makes the residue calculus introduced above available in many situations. We assume that $\mathcal{H}$ is a locally finite collection of real $\Sigma$-hyperplanes in $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$. We denote by $\mathcal{L} = \mathcal{L}(\mathcal{H})$ the intersection lattice of $\mathcal{H}$, i.e., the collection of all finite intersections of hyperplanes from $\mathcal{H}$, ordered by inclusion. The intersection of the empty collection is understood to be $\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

The configuration $\mathcal{H}$ is said to be $P$-bounded from below if there exists a constant $R > 0$ such that for any hyperplane of the form $\langle \alpha\,,\, \lambda \rangle = s$ contained in $\mathcal{H}$ we have $s \geq -R$. If $\mathcal{H}$ is $P$-bounded from below, then there exists a constant $M > 0$ such that for every $\eta \in \mathfrak{a}_{\mathrm{q}}^*$,

$$\forall \alpha \in \Sigma(P)\ \langle \eta\,,\, \alpha \rangle < -M \quad \Rightarrow \eta \notin \cup\mathcal{H}.$$

**Theorem 13.8** *Let $P \in \mathcal{P}_\sigma^{\min}$ and let $\mathcal{H}$ be a locally finite collection of real $\Sigma$-hyperplanes that is $P$-bounded from below. Then the collection $\Pi$ of pairs $(\xi, \mathfrak{b}) \in \mathfrak{a}_{\mathrm{q}}^* \times \mathcal{R}$ with $\xi \in \mathfrak{b}^\perp$, $\xi + \mathfrak{b} \in \mathcal{L}(\mathcal{H})$ and $\mathrm{Res}^{\mathrm{P,t}}_{\xi+\mathfrak{b}} \neq 0$ is contained in a finite set only depending on $P, \mathcal{H}$. Moreover, for every $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ sufficiently $\bar{P}$-dominant and each collection of $\varepsilon_Q \in \mathfrak{a}_{Q\mathrm{q}}^{*+}$ sufficiently close to $0$ for all $Q \in \mathcal{P}_\sigma$, the following holds. For every $\varphi \in \mathcal{P}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*, \mathcal{H})$,*

$$\int_{\eta + i\mathfrak{a}_{\mathrm{q}}^*} \varphi(\lambda)\, d\lambda = \sum_{(\xi,\mathfrak{b})\in\Pi}\ \sum_{Q \in \mathcal{P}_\sigma(\mathfrak{b})} t(Q) \int_{\varepsilon_Q + i\mathfrak{b}} \mathrm{Res}^{\mathrm{P,t}}_{\xi+\mathfrak{b}} \varphi(\,\cdot\, + \nu)\, \mathrm{d}\mu_{\mathfrak{b}}(\nu).$$

For the proof of this result we refer the reader to [18]. The $\varepsilon_Q$ are sufficiently small perturbations of 0 inside $\mathfrak{a}_{Q\mathrm{q}}^{*+}$; they make sure that each of the integrations is performed over an affine space that is disjoint from $\cup\mathcal{H}$, hence also from the singular locus of the integrand.

We now fix $P_0 \in \mathcal{P}_\sigma^{\min}$ and put $\Sigma^+ := \Sigma(P_0)$. The associated set of simple roots is denoted by $\Delta$. For each subset $F \subset \Delta$ we denote by $\mathfrak{a}_{F\mathrm{q}}^*$ the intersection of the root hyperplanes $\alpha^\perp$, for $\alpha \in F$ and by $\mathfrak{a}_{F\mathrm{q}}^{*+}$ the positive chamber determined by the remaining simple roots $\Delta \setminus F$. Then there exists a unique $P_F \in \mathcal{P}_\sigma$

whose associated positive chamber equals $\mathfrak{a}_{F\mathrm{q}}^{*+}$. The group $P_F$ is called the standard parabolic subgroup determined by $F$. In the rest of this chapter we shall adopt the convention to replace an index $P_F$ by $F$. In particular, the Langlands decomposition of $P_F$ is denoted by $P_F = M_F A_F N_F$, and the centralizer of $\mathfrak{a}_{F\mathrm{q}}$ in $W$ is denoted by $W_F$.

Let $W^F$ denote the set of elements $w \in W$ for which $w(F) \subset \Sigma^+$. Then it is well known that $W^F$ consists of the cosets representatives of $W/W_F$ which are of minimal length. Moreover, the multiplication map $W^F \times W_F \to W$ is a bijection.

It follows from the standard theory of root systems that for each $P \in \mathcal{P}_\sigma \simeq \mathcal{P}$ there exists a unique $F \subset \Delta$ such that $P$ is $W$-conjugate to $P_F$. Moreover, there exists a unique $v \in W^F$ such that $P = vP_Fv^{-1}$.

**Lemma 13.9** *Let $\mathcal{H}$ be a locally finite collection of real $\Sigma$-hyperplanes that is $P_0$-bounded from below. Then for each $F \subset \Delta$ there exists a finite subset $\Lambda_F = \Lambda_F(\mathcal{H})$ of the closed convex cone $\Gamma^-(F)$ spanned by $0$ and $-F$, such that the following holds. For every $W$-invariant residue weight $t$ on $\Sigma$ the collection of elements $\xi \in (\mathfrak{a}_{F\mathrm{q}}^*)^\perp$ with*

$$\mathrm{Res}^{\mathrm{P,t}}_{\xi+\mathfrak{a}^*_{\mathrm{Fq}}} \neq 0 \quad \text{and} \ \exists \mathrm{w} \in \mathrm{W}^{\mathrm{F}} \ : \ \mathrm{w}(\xi + \mathfrak{a}^*_{\mathrm{Fq}}) \in \mathcal{L}(\mathcal{H})$$

*is contained in $\Lambda_F$.*

*Proof.* For $w \in W^F$, let $X_{F,w}$ denote the set of $\xi \in \Lambda_F$ with the property that $w(\xi + \mathfrak{a}^*_{F\mathrm{q}}) \in \mathcal{L}(\mathcal{H})$. For such $\xi$ it follows from Lemma 13.7 that $\mathrm{Res}^{\mathrm{P,t}}_{\mathrm{w}\xi+\mathrm{w}\mathfrak{b}} \neq 0$. Hence $(w\xi, w\mathfrak{b})$ is contained in the set $\Pi$ of Theorem 13.8 with $P = P_0$. It follows that $X_{F,w}$ is contained in a finite subset $\Lambda_{F,w}$ depending only on $P_0, \mathcal{H}$. Moreover, it follows from Proposition 13.5 that for $\xi \in X_{F,w}$ we have $w\xi \in \Gamma^-(\Sigma(P_0) \cap w\mathfrak{a}^{*\perp}_{F\mathrm{q}}) = w\Gamma^-(F)$, hence $\xi \in \Gamma^-(F)$. Thus, the above holds with $\Lambda_{F,w}$ a finite subset of $\Gamma^-(F)$ only depending on $P_0$ and $\mathcal{H}$. Now $\Lambda_F$ may be taken to be the union of the sets $\Lambda_{F,w}$, for $w \in W^F$, and the lemma follows. □

In the formulation of the following result, we use the abbreviation $d\mu_F$ for the normalized Lebesgue measure $d\mu_{\mathfrak{a}^*_{F\mathrm{q}}}$ on $i\mathfrak{a}^*_{F\mathrm{q}}$ and its translates, for each $F \subset \Delta$.

**Theorem 13.10** *Let $t$, $\mathcal{H}$ and the set $\Lambda_F$ be as in Lemma 13.9. Then for every $\eta \in \mathfrak{a}^*_\mathrm{q}$ sufficiently $\bar{P}$-dominant and each collection of $\varepsilon_F \in \mathfrak{a}^{*+}_{F\mathrm{q}}$ sufficiently close to $0$ for all $F \subset \Delta$, the following holds. For every $\varphi \in \mathcal{P}(\mathfrak{a}^*_{\mathrm{q}\mathbb{C}}, \mathcal{H})$,*

$$\int\limits_{\eta+i\mathfrak{a}^*_\mathrm{q}} \varphi(\lambda)\, d\lambda = \sum_{F\subset\Delta} \sum_{\xi\in\Lambda_F} t(P_F) \int\limits_{\varepsilon_F+i\mathfrak{a}^*_{F\mathrm{q}}} \mathrm{Res}^{\mathrm{P,t}}_{\xi+\mathfrak{a}^*_{\mathrm{Fq}}} \sum_{\mathrm{w}\in\mathrm{W}^{\mathrm{F}}} \varphi(\mathrm{w}(\,\cdot\, + \nu))\, \mathrm{d}\mu_{\mathrm{F}}(\nu).$$

*Proof.* The formula can be derived from the formula displayed in Theorem 13.8 as follows. Every $Q$ in the formula is of the form $w(P_F)$ for a unique $F \subset \Delta$ and a unique $w \in W^F$. Moreover, $(\xi, Q) \in \Pi \Rightarrow w^{-1}\xi \in \Lambda_F$. The desired formula now follows by application of Lemma 13.7 and the $W$-invariance of the residue weight. □

We come to the completion of the proof of the Fourier inversion theorem, which depends on the above result in a crucial way.

**Residues and Fourier inversion** As said, to complete the proof of Theorem 13.2, it suffices to show that, for $f \in C_c^\infty(\mathrm{X} : \tau)$, the function $\mathcal{T}_\eta \mathcal{F} f$, defined by (13.3), extends smoothly from $\mathrm{X}_+$ to all of $\mathrm{X}$. In [17] this is proved by using the residue calculus described above. For the application of the residue calculus we need the following result, for which we refer the reader to [17].

**Proposition 13.11** *The union $\mathcal{H}$ of the collections of singular hyperplanes of the functions $\lambda \mapsto E^*(\lambda : \cdot\,)$ and $\lambda \mapsto E_{1,+}(\lambda : \cdot\,)$ is a locally finite collection of real $\Sigma$-hyperplanes that is $P_0$-bounded from below.*

From now on we fix $\mathcal{H}$ as in Proposition 13.11 above and we fix a finite set $\Lambda_F \subset \Gamma^-(F)$ meeting the requirements of Lemma 13.9. Moreover, we fix any residue weight $t$ on $\Sigma$ that is $W$-invariant and in addition *even,* i.e., $t(\bar{P}) = t(P)$ for all $P \in \mathcal{P}_\sigma$.

*Completion of the proof of Theorem 13.2.* In view of Definition 11.10 with $P = P_0$, the singular locus of the meromorphic function $\mathcal{F}f$ is contained in $\cup\mathcal{H}$ as well. We may therefore apply Theorem 13.10 with $\varphi(\lambda) = E_+(\lambda : x)\mathcal{F}f(\lambda)$, where $x \in \mathrm{X}_+$, and obtain that

$$\begin{aligned} &\mathcal{T}_\eta \mathcal{F} f(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (13.14)\\ &\quad = |W| \sum_{F \subset \Delta} t(P_F) \int_{\varepsilon_F + i\mathfrak{a}_{F\mathrm{q}}^*} \mathcal{R}_F^t \, [\sum_{w \in W^F} E_+(w(\,\cdot\, + \nu) : x)\, \mathcal{F}f(w(\,\cdot\, + \nu))\,]\, d\mu_F(\nu), \end{aligned}$$

where, for each $F \subset \Delta$, we have used the notation $\mathcal{R}_F^t$ for the Laurent functional in $\mathcal{M}(\mathfrak{a}_{F\mathrm{q}}^{*\perp}, \Sigma_F)^*_{\mathrm{laur}}$ given by the formula

$$\mathcal{R}_F^t := \sum_{\xi \in \Lambda_F} \mathrm{Res}^{\mathrm{P,t}}_{\xi + \mathfrak{a}_{\mathrm{Fq}}^*}. \tag{13.15}$$

For later applications it is important to note that the support of this Laurent functional is contained in the finite set $\Lambda_F$, which in turn is contained in the closed convex cone $\Gamma^-(F)$ spanned by 0 and $-F$.

Using (13.14), (13.1) and (11.14) with $P = Q = P_0$, we obtain that

$$\begin{aligned} &E_+(w(\,\cdot\, + \nu) : x)\, \mathcal{F}f(w(\,\cdot\, + \nu)) \\ &\qquad = \quad E_{w,+}(\,\cdot\, + \nu : x)\mathcal{F}f(\,\cdot\, + \nu) \\ &\qquad = \quad \int_{\mathrm{X}} E_{w,+}(\,\cdot\, + \nu : x) E^*(\,\cdot\, + \nu : y) f(y)\, dy. \end{aligned}$$

We now define the kernel function

$$K_F^t(\nu : y : x) = \mathcal{R}_F^t[\sum_{w \in W^F} E_{w,+}(\,\cdot\, + \nu : x)E^*(\,\cdot\, + \nu : y)\,], \tag{13.16}$$

for $y \in \mathrm{X}$, $x \in \mathrm{X}_+$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. This kernel depends meromorphically on the variable $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. After this (13.14) becomes

$$\mathcal{T}_\eta \mathcal{F} f(x) = |W| \sum_{F \subset \Delta} t(P_F) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_{\mathrm{X}} K_F(\nu : x : y) f(y)\, dy\, d\mu_F(\nu).$$

For fixed generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, the kernel function $K^t_F(\nu : \cdot : \cdot)$ is a smooth function on $\mathrm{X}_+ \times \mathrm{X}$, with values in $\mathrm{End}\,(V_\tau) \simeq V_\tau \otimes V_\tau^*$. Moreover, it is spherical for the $K \times K$- representation $\tau \otimes \tau^*$. Finally, it is $\mathbb{D}(\mathrm{X})$-finite in both variables. It follows that $K^t_F(\nu : \cdot : \cdot)$ belongs to a tensor product space of the form ${}^1E_\nu \otimes {}^2E_\nu$, with ${}^1E_\nu$ and ${}^2E_\nu$ finite-dimensional linear subspaces of $C^\infty(\mathrm{X}_+ : \tau)$ and $C^\infty(\mathrm{X}_+ : \tau^*)$, respectively. For each $j = 1, 2$, let ${}^jE'_\nu$ denote the subspace of functions in ${}^jE_\nu$ that extend smoothly to all of X. Then by the symmetry property formulated in the result below, the kernel function $K^t_F(\nu : \cdot : \cdot)$ belongs to the intersection of ${}^1E'_\nu \otimes {}^2E_\nu$ and ${}^1E_\nu \otimes {}^2E'_\nu$, which equals ${}^1E'_\nu \otimes {}^2E'_\nu$. From this we see that $K^t_F(\nu : \cdot : \cdot)$ extends smoothly to all of $\mathrm{X} \times \mathrm{X}$. □

**Proposition 13.12** *Let $x, y \in \mathrm{X}_+$. Then*

$$K^t_F(\nu : x : y) = K^t_F(-\bar{\nu} : y : x)^*, \tag{13.17}$$

*as a meromorphic* $\mathrm{End}\,(V_\tau)$*-valued identity in the variable $\nu$.*

*Sketch of proof.* The result is proved by induction on $\dim \mathfrak{a}_\mathrm{q}$, the split rank of X. For $\dim \mathfrak{a}_\mathrm{q} = 0$, the symmetric space X is compact and the result is obvious. Thus, assume that the result has already been established for spaces of lower split rank.

We first assume that $F \subsetneq \Delta$. Then the equality follows from the symmetry of the kernels for the spaces $\mathrm{X}_{F,v}$, for $v \in {}^F\mathcal{W}$, which are of lower split rank. The proof of this part of the induction step is based on the principle of induction of relations, which we shall explain in the next subsection. For now we assume that the symmetry holds for $F \subsetneq \Delta$ and we will show how to derive it for $F = \Delta$.

We first observe that for $a, b$ in $A_{\Delta\mathrm{q}}$, the vectorial part of the center of $G$ modulo $H$, we have $K^t_\Delta(\nu : ax : by) = a^\nu b^{-\nu} K^t_\Delta(\nu : x : y)$. By factoring out this part we reduce to the case that $\mathfrak{a}^*_{\Delta\mathrm{q}} = \{0\}$. Suppressing the variable from this space, we put $K^t_\Delta(x : y) = K^t_\Delta(0 : x : y)$.

The argument in the proof of Theorem 13.2 can now be modified in such a way that the symmetry (13.17) is only needed for proper subsets $F$ of $\Delta$. This goes as follows. The kernel $K^t_\Delta(\cdot : y)$ is annihilated by a cofinite ideal $I$ of $\mathbb{D}(\mathrm{X})$, independent of $y$.

The function $g := f - \mathcal{T}_\eta f$ is smooth on $\mathrm{X}_+$ and has support that is bounded in X. Given $F \subset \Delta$ we write

$$T^t_F f := |W|\, t(P_F) \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_{\mathrm{X}} K_F(\nu : x : y) f(y)\, dy\, d\mu_F(\nu). \tag{13.18}$$

Then by the residue shift in the proof of Theorem 13.2, $g = f - \sum_F T^t_F f$. Let $D \in I$. Then $Dg = Df - \sum_{F \subsetneq \Delta} T^t_F f$; by application of Proposition 13.12 for

$F \subsetneq \Delta$, as in the final part of the proof of Theorem 13.2, the function $Dg$ is seen to extend to a smooth function on all of X. Its support is compact, as said above. Let now $D_\tau$ be the differential operator of Theorem 11.27 with $P = P_0$. Then $D_\tau Dg = DD_\tau g = 0$. It follows that $Dg = 0$ on $\mathrm{X}_+$, for each $D \in I$. This implies that the function $g$ is real analytic on $\mathrm{X}_+$, with a support that is bounded in X. By analytic continuation it follows that $g = 0$, whence Theorem 13.2. At the same time it follows that $T^t_\Delta f$ is the smooth function on X given by

$$T^t_\Delta f = f - \sum_{F \subsetneq \Delta} T^t_F f. \tag{13.19}$$

Let $s_0$ be the longest element of $W$. Then for each set $F \subsetneq \Delta$, the set $F' := -s_0(F)$ is properly contained in $\Delta$. Moreover, from the symmetry of the kernels, (13.17), combined with their definition, (13.16), the $W$-equivariance of the residue calculus and the fact that the weight $t$ is $W$-invariant and even, it follows that $T^t_F$ and $T^t_{F'}$ are adjoint to each other as operators $C^\infty_c(\mathrm{X} : \tau) \to C^\infty(\mathrm{X} : \tau)$. This implies that $T^t_\Delta$ is symmetric, from which in turn it follows that (13.17) is valid for $F = \Delta$. □

**Induction of relations** In this subsection we shall describe the principle of induction of relations, as developed in [20]. The principle says that relations of a certain type between partial Eisenstein associated with a symmetric space $\mathrm{X}_{F,v}$, for $F \subset \Delta$ and $v \in {}^F\mathcal{W}$, induce relations between corresponding partial Eisenstein integrals for the space X.

Keeping to the convention of replacing index $P_F$ by $F$, we have

$$\mathrm{X}_{F,v} = M_F / M_F \cap vHv^{-1}.$$

The space ${}^*\mathfrak{a}_{F\mathrm{q}} := \mathfrak{a}_{F\mathrm{q}}^\perp$ is the analogue of $\mathfrak{a}_\mathrm{q}$ for the reductive pair $(M_F, M_F \cap vHv^{-1})$. Moreover, $\Sigma_F$ and $W_F$ are the associated root system and Weyl group, respectively, and ${}^*P_0 := M_F \cap P_0$ is the minimal $\sigma$-parabolic subgroup of $M_F$ determined by the positive system $\Sigma_F^+ := \Sigma_F \cap \Sigma^+$. Let $K_F = K \cap M_F$ and $\tau_F := \tau|_{K_F}$. If $t \in W_F$, we denote by

$$E_{+,t}(\mathrm{X}_{F,v} : \mu : m) \in \mathrm{Hom}\,({}^\circ\mathcal{C}(\mathrm{X}_{F,v}, \tau), V_\tau), \qquad (\mu \in {}^*\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, m \in \mathrm{X}_{F,v}),$$

the analogue for the symmetric pair $(M_F, M_F \cap vH_Fv^{-1})$ (and the parabolic subgroup ${}^*P_0$) of the partial Eisenstein integral $E_{+,t}(\mathrm{X} : \lambda : x)$. Here ${}^\circ\mathcal{C}(\mathrm{X}_{F,v} : \tau)$ denotes the analogue of the space ${}^\circ\mathcal{C}(\tau)$ for $\mathrm{X}_{F,v}$. Via the bijection (3.9), the natural action of $W$ on $W/W_{K\cap H}$ is transferred to an action on $\mathcal{W}$. Accordingly, the space ${}^\circ\mathcal{C}(\mathrm{X}_{F,v} : \tau)$ is the direct sum of the spaces $C^\infty(M/M \cap wHw^{-1} : \tau_M)$, for $w \in W_Fv$, hence may be naturally embedded into ${}^\circ\mathcal{C}(\tau)$, which is a similar direct sum for $w \in \mathcal{W}$. The natural inclusion map is denoted by $\mathrm{i}_{F,v}$; its transpose, the natural projection map, by $\mathrm{pr}_{F,v}$.

**Theorem 13.13** *Let $\mathcal{L}_t \in \mathcal{M}({}^*\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}, \Sigma_F)^*_{\mathrm{laur}} \otimes {}^\circ\mathcal{C}(\tau)$ be given Laurent functionals, for each $t \in W_F$, and assume that, for every $v \in {}^F\mathcal{W}$,*

$$\sum_{t \in W_F} \mathcal{L}_t[E_{+,t}(\mathrm{X}_{F,v} : \cdot : m) \circ \mathrm{pr}_{F,v}] = 0, \qquad (m \in \mathrm{X}_{F,v,+}). \tag{13.20}$$

*Then, for each $s \in W^F$, the following meromorphic identity in the variable $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ is valid:*

$$\sum_{t \in W_F} \mathcal{L}_t[E_{+,st}(\mathrm{X} : \, \cdot \, + \nu : x)] = 0, \qquad (x \in \mathrm{X}_+). \tag{13.21}$$

*Conversely, if for a fixed $s \in W^F$ the equality (13.21) holds for all $\nu$ in a nonempty open subset of $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, then (13.20) holds for each $v \in {}^F\mathcal{W}$.*

This result is proved in [20], Thm. 16.1. The proof in turn is based on a more general vanishing theorem, see [20], Thm. 12.10. The vanishing theorem asserts that a suitably restricted meromorphic family $\mathfrak{a}^*_{F\mathrm{q}\mathbb{C}} \ni \nu \mapsto f_\nu \in C^\infty(\mathrm{X}_+ : \tau)$ of eigenfunctions for $\mathbb{D}(\mathrm{X})$ is completely determined by the coefficient of $a^{\nu - \rho_F}$ in its asymptotic expansion towards infinity along $A^+_{F\mathrm{q}}v$, for each $v \in {}^F\mathcal{W}$. This coefficient is a spherical function on $\mathrm{X}_{F,v,+}$, depending meromorphically on $\nu$. In particular, if the coefficients, one for each $v \in {}^F\mathcal{W}$, are zero, then $f_\nu = 0$ for all $\nu$. This explains the name vanishing theorem. Part of the mentioned restriction on families is the so-called asymptotic globality condition. It requires that certain asymptotic coefficients in the expansions of $f_\nu$ along certain codimension one walls have smooth behavior as functions in the variables transversal to these walls. The precise condition is given in [20], Def. 9.5.

We shall now indicate how the vanishing theorem is applied to prove Theorem 13.13. Let $f^s_\nu$, for $s \in W^F$, denote the expression on the left-hand side of (13.21). The sum $f_\nu = \sum_{s \in W^F} f^s_\nu$ defines a family for which the vanishing theorem holds; the summation over $W^F$ is needed for the family to satisfy the asymptotic globality condition mentioned above. The coefficient of $a^{\nu - \rho_F}$ in the expansion of $f_\nu$ along $A^+_{F\mathrm{q}}v$, for $v \in {}^F\mathcal{W}$, is given by the expression on the left-hand side of (13.20) with the same $v \in W^F$. By the vanishing theorem, the vanishing expressed in (13.20), for all $v \in W^F$, implies that $f = 0$. For each $v \in {}^F\mathcal{W}$, the sets of exponents of $f^s_\nu$ in the expansion along $A^+_{F\mathrm{q}}v$ are mutually disjoint for distinct $s \in W^F$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. Thus, the vanishing of $f_\nu$ implies the vanishing of each individual function $f^s_\nu$ and (13.21) follows.

The converse statement of Theorem 13.13 is proved as follows. First, the asymptotic globality condition connects the vanishing of distinct sets of exponents, from which it follows that the vanishing of an individual term $f^s_\nu$ implies that of $f_\nu$. The validity of (13.20) then follows by taking the coefficient of $a^{\nu - \rho_F}$ in the asymptotic expansion of $f_\nu$ along $A^+_\mathrm{q}v$.

*Completion of the proof of Proposition 13.12.* The Laurent functional $\mathcal{R}^t_F$ has real support; moreover, it is scalar, and can be shown to be real in the sense that at each point of its support it can be represented by a string $\{u_N\} \subset S({}^*\mathfrak{a}^*_{F\mathrm{q}})$ as in Definition 13.3, with $u_N$ real for all $N \in \mathbb{N}$. Using these facts it can be shown that the adjoint kernel in (13.17) can be expressed as

$$K^t_F(-\bar{\nu} : y : x)^* = \mathcal{R}^t_F \left( \sum_{s \in W^F} E^\circ(\nu - \, \cdot \, : x) E^*_{+,s}(\nu - \, \cdot \, : y) \right), \tag{13.22}$$

where the dual partial Eisenstein integral is defined by

$$E^*_{+,s}(\lambda : y) := E_{+,s}(-\bar{\lambda} : y)^*.$$

The residue weight $t$ on $\Sigma$ induces a residue weight ${}^*t$ on $\Sigma_F$. The set $F$ is a simple system for $\Sigma_F$. If $v \in {}^F\mathcal{W}$, we denote the kernel for the space $\mathrm{X}_{F,v}$ associated with the data ${}^*t, F$ by $K^{{}^*t}_F(\mathrm{X}_{F,v} : m : m')$. In this notation the spectral parameter $\nu$ has been suppressed, as it is zero-dimensional. The inductive hypothesis, which asserts the symmetry of the kernels for spaces of lower split rank, implies that, for each $v \in {}^F\mathcal{W}$,

$$K^{{}^*t}_F(\mathrm{X}_{F,v} : m : m') = K^{{}^*t}_F(\mathrm{X}_{F,v} : m' : m)^*, \qquad (m, m' \in \mathrm{X}_{F,v}). \tag{13.23}$$

In view of the first part of the proof, applied to the present dual kernel, using transitivity of residues, see Lemma 13.6, and taking into account that $(W_F)^F = \{1\}$, the equation (13.23) is seen to be equivalent to

$$\begin{aligned} &\mathcal{R}^t_F[\, E_+(\mathrm{X}_{F,v} : \,\cdot\, : m) E^*(\mathrm{X}_{F,v} : \,\cdot\, : m')\,] \\ &\quad = \ \mathcal{R}^t_F[\, E^\circ(\mathrm{X}_{F,v} : -\cdot\, : m) E^*_+(\mathrm{X}_{F,v} : -\cdot\, : m')\,]. \end{aligned}$$

In view of (13.16) and (13.22) the equality (13.17) can now be deduced by applying induction of relations, Theorem 13.13, first with respect to the variable $x$, and then with respect to the variable $y$. More details can be found in [17], Sect. 8.

## 14 The proof of the Plancherel theorem

**The generalized Eisenstein integral** We shall now give a sketch of the proof of the spherical Plancherel theorem, Theorem 11.26, as given in [21], indicating some of the main ideas. The starting point of the proof is the following formula, obtained in the proof of the inversion theorem in the previous section, for $f \in C^\infty_c(\mathrm{X} : \tau)$,

$$f = \sum_{F \subset \Delta} t(P_F)\,|W| \int_{\varepsilon_F + i\mathfrak{a}^*_{F\mathrm{q}}} \int_{\mathrm{X}} K^t_F(\nu : x : y)\, f(y)\, dy\, d\mu_F(\nu). \tag{14.1}$$

Here we recall that $t$ is any choice of $W$-invariant and even residue weight on $\Sigma$. The leading idea in the proof of the Plancherel theorem is to show that this formula, initially only valid for $\varepsilon_F \in \mathfrak{a}^{*+}_{F\mathrm{q}}$ sufficiently close to zero for all $F \subset \Delta$, is actually true with $\varepsilon_F = 0$ for all $F$. This in turn is achieved by showing that the kernel functions $K^t_F(\nu : \cdot : \cdot)$ are regular for imaginary values of $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$.

The regularity of the kernels is established in the course of a long inductive argument in [21]. The nature of this inductive argument will be explained in the next subsection. To prepare for it, we first indicate how the symmetry of the kernels leads to the introduction of the so-called *generalized Eisenstein integrals*. For details we refer to [17].

Let $F \subset \Delta$ and $v \in {}^F\mathcal{W}$. Using the kernel $K^t_F(\mathrm{X}_{F,v} : \cdot : \cdot)$ defined in (13.16), we define the following subspace of $C^\infty_c(\mathrm{X}_{F,v} : \tau_F)$:

$$\mathcal{A}^{{}^*t}_{F,v} = \mathcal{A}^{{}^*t}(\mathrm{X}_{F,v} : \tau_F) := \mathrm{span}\{K^{{}^*t}_F(\mathrm{X}_{F,v} : \,\cdot\, : m')u \mid m' \in \mathrm{X}_{F,v,+}, u \in V_\tau\}.$$

This space is annihilated by a cofinite ideal of $\mathbb{D}(\mathrm{X}_{F,v})$, hence finite-dimensional. For $\psi \in \mathcal{A}^{*t}_{F,v}$, we define the generalized Eisenstein integral $E^{\circ}_{F,v}(\nu : \cdot)\psi$ as a meromorphic $C^{\infty}(\mathrm{X} : \tau)$-valued function of $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$, as follows. If

$$\psi = \sum_i K^{*t}_F(\mathrm{X}_{F,v} : \cdot\ m'_i)u_i,$$

with $m'_i \in \mathrm{X}_{F,v,+}$ and $u_i \in V_\tau$, then

$$E^{\circ}_{F,v}(\nu : x)\psi := \sum_i \mathcal{R}^t_F[\sum_{s \in W^F} E_{+,s}(\nu + \cdot)E^*(\mathrm{X}_{F,v} : \cdot : m'_i)u_i], \tag{14.2}$$

for generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$ and all $x \in \mathrm{X}_+$. It follows by application of Theorem 13.13 that the expression on the right-hand side of (14.2) is independent of the particular representation of $\psi$, so that the definition is unambiguous.

Finally, we define the space

$$\mathcal{A}^{*t}_F := \oplus_{v \in {}^F\mathcal{W}}\ \mathcal{A}^{*t}_{F,v}, \tag{14.3}$$

and for $\psi = (\psi_v) \in \mathcal{A}^{*t}_F$ we define the generalized Eisenstein integral

$$E^{\circ}_F(\nu : x)\psi := \sum_{v \in {}^F\mathcal{W}} E^{\circ}_{F,v}(\nu : x)\psi_v.$$

In the course of [17] it is shown that for any choice of inner product on $\mathcal{A}^{*t}_F$, for which the decomposition (14.3) is orthogonal, the symmetry of the kernel $K^t_F$, see Proposition 13.12, implies the existence of a unique selfadjoint endomorphism $\alpha_F$ of $\mathcal{A}^{*t}_F$ such that

$$K^t_F(\nu : x : y) = E^{\circ}_F(\nu : x) \circ \alpha_F \circ E^{\circ}_F(-\bar{\nu} : y)^*.$$

**The inductive argument** So far, in the proof of the Fourier inversion argument, the theory of the discrete series has played no role. However, in the inductive argument leading up to the regularity of the kernels, this changes fundamentally, as we shall now explain.

A reductive symmetric pair $(G, H)$ of the Harish-Chandra class is said to be of *residue type* if $G$ has compact center modulo $H$ and if in addition the following holds. For any choice of $W$-invariant and even residue weight $t$, the operator $T^t_\Delta$ defined as in (13.18) with $F = \Delta$, is required to be equal to the restriction to $C^{\infty}_c(\mathrm{X} : \tau)$ of the orthogonal projection $P_{\mathrm{ds}} : L^2(\mathrm{X} : \tau) \to L^2_{\mathrm{d}}(\mathrm{X} : \tau)$. The second space denotes the discrete part, see (2.20). In particular, it follows from the requirement that $T^t_\Delta$ is independent of the choice of residue weight. Equivalently, the latter condition means that $K^t_\Delta(x : y)$ is the kernel of the orthogonal projection $P_{\mathrm{ds}}$. Moreover, the assumption straightforwardly implies that

$$\mathcal{A}^t_\Delta(\mathrm{X} : \tau) = L^2_{\mathrm{d}}(\mathrm{X} : \tau).$$

In particular it follows that the space on the left-hand side is independent of the choice of residue weight, and that the space on the right-hand side is a finite-dimensional space which can be realized by means of point residues of

Eisenstein integrals from the minimal principal series for X. Its elements are $\mathbb{D}(\mathrm{X})$-finite functions. In view of this it also follows that $L^2_{\mathrm{d}}(\mathrm{X} : \tau)$ equals $\mathcal{A}_2(\mathrm{X} : \tau)$, the space of $\mathbb{D}(\mathrm{X})$-finite spherical Schwartz functions on X; see [4], Thm. 6.4, for details.

The inductive argument proceeds by induction on $\dim A_{\mathrm{q}}$, the $\sigma$-split rank of $G$. Its purpose is to establish that every pair $(G, H)$ is of residue type as soon as $G$ has compact center modulo $H$. A parabolic subgroup $Q \in \mathcal{P}_\sigma$ is said to be of residue type if all pairs $(M_Q, M_Q \cap vHv^{-1})$ are of residue type, for $v \in {}^Q\mathcal{W}$. A subset $F \subset \Delta$ is said to be of residue type if the associated standard parabolic subgroup $P_F$ is of this type.

In the course of the induction step, the induction hypothesis guarantees that each $Q \in \mathcal{P}_\sigma$ different from $G$ is of residue type. Moreover, if the center of $G$ is not compact modulo $H$, then $M_G = M_\Delta$ is of strictly smaller $\sigma$-split rank than $G$, so that $G$, viewed as a parabolic subgroup, is of residue type as well.

We will now proceed to describe the induction step. In what follows we assume that the occurring subset $F \subset \Delta$ is of residue type. Let $v \in {}^F\mathcal{W}$. Then the assumption implies that

$$\mathcal{A}^{*t}_{F,v} = L^2_{\mathrm{d}}(\mathrm{X}_{F,v} : \tau_F) = \mathcal{A}_2(\mathrm{X} : \tau).$$

Accordingly,

$$\mathcal{A}^{*t}_F = \mathcal{A}_{2,F} := \oplus_{v \in {}^F\mathcal{W}} \ \mathcal{A}_2(\mathrm{X}_{F,v} : \tau_F)$$

is independent of $t$ and may be equipped with the direct sum of the $L^2$-inner products. For this choice of inner product it can be shown that $\alpha_F$ equals $|W_F|^{-1}$ times the identity operator. Thus, we obtain

$$K^{*t}_F(\nu : x : y) = |W_F|^{-1}\, E^\circ_F(\nu : x) E^*_F(\nu : y),$$

where the dual generalized Eisenstein integral is defined by

$$E^*_F(\nu : y) = E^\circ_F(-\bar{\nu} : y)^* \in \mathrm{Hom}\,(V_\tau, \mathcal{A}_{2,F}),$$

for $y \in \mathrm{X}$ and generic $\nu \in \mathfrak{a}^*_{F\mathrm{q}\mathbb{C}}$. From the induction hypothesis that $F$ is of residue type, it follows that the kernels $K^{*t}_F(\mathrm{X}_{F,v} : \cdot : \cdot)$ do not depend on $t$. In view of their construction in (14.2) it follows that the generalized Eisenstein integrals do not depend on the choice of $t$ either.

Each parabolic subgroup from $\mathcal{P}_\sigma$ is a standard one for a particular choice of positive roots. It follows that the notion of generalized Eisenstein integral can be defined for every $Q \in \mathcal{P}_\sigma$ of residue type. More precisely, we define $\mathcal{A}_{2,Q}$ as in (11.7). For each $\psi \in \mathcal{A}_{2,Q}$ we have an associated Eisenstein integral

$$E^\circ(Q : \nu : \cdot)\psi \in C^\infty(\mathrm{X} : \tau),$$

which depends meromorphically on the parameter $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$. This in turn allows us to define kernels by the formula

$$K_Q(\nu : x : y) := |W_Q|^{-1} E^\circ(Q : \nu : x) E^*(Q : \nu : y). \tag{14.4}$$

The definitions are such that for $F \subsetneq \Delta$ all objects with parameter $P_F$ coincide with their analogues with index $F$.

**Tempered estimates** We emphasize the observation that the Eisenstein integrals just introduced do not enter harmonic analysis as matrix coefficients of generalized principal series representations for X; in fact, the connection with representation theory is only made in the final stage of the development of the theory. Instead, the general Eisenstein integrals enter the analysis as residues of Eisenstein integrals connected with the minimal principal series for X. This has a two-fold advantage.

First, certain moderate estimates that are uniform with respect to the parameter $\nu \in \mathfrak{a}^*_{Q\mathrm{q}\mathbb{C}}$ are inherited from the similar estimates for minimal Eisenstein integrals, which were established in [6], by using the functional equation for $j(Q : \xi : \lambda)$ mentioned in Remark 8.6. The minimal Eisenstein integrals are easier to handle as they require no knowledge of the discrete series of noncompact symmetric spaces of lower $\sigma$-split rank.

Second, the location of the asymptotic exponents of the general Eisenstein integrals is determined by the supports of the residual operators. By application of the text following (13.15) it can be shown that the Eisenstein integrals are tempered functions for imaginary $\nu$.

Combining these two facts with the structure of the differential equations satisfied by the Eisenstein integrals, see Proposition 11.7, the initial moderate estimates for the Eisenstein integrals $E^\circ(Q : \nu : x)$ can be sharpened to tempered estimates that are of a uniform nature in the parameter $\nu \in i\mathfrak{a}^*_{Q\mathrm{q}}$. For details we refer the reader to [21], Sect. 15. The mentioned technique of sharpening estimates goes back to Wallach, [87]. In the formulation of the following result we use notation introduced in the text preceding Theorem 11.16.

**Theorem 14.1** *There exist constants $\varepsilon > 0$ and $s > 0$ and a polynomial function $q : \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}} \to \mathbb{C}$ that is a product of linear factors of the form $\langle \alpha\,,\,\cdot\,\rangle - c$, with $\alpha \in \Sigma(P)$ and $c \in \mathbb{R}$, such that the function $f_\nu = q(\nu)E^\circ(P : \nu : \,\cdot\,)$ depends holomorphically on $\nu$ in the region $\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}(\varepsilon) = \{\lambda \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}} \mid |\lambda| < \varepsilon\}$ and satisfies the following estimates. For every $u \in U(\mathfrak{g})$ there exist constants $n \in \mathbb{N}$ and $C > 0$ such that*

$$|L_u f_\nu(x)| \leq C\,(1 + |\nu|)^n (1 + l_{\mathrm{X}}(x))^n\, \Theta(x)\, e^{s|\mathrm{Re}\,\nu| l_{\mathrm{X}}(x)}.$$

For minimal $\sigma$-parabolic subgroups this result is due to [6]; for general $\sigma$-parabolic subgroups it was first established by [38]. Both papers rely on the same idea, described above. First, a functional equation for the $j(Q : \xi : \lambda)$ is obtained. These yield uniform moderate estimates, which can be sharpened to uniform tempered estimates. As said, the case of general parabolics is harder, since it involves the discrete series of spaces of lower $\sigma$-split rank. The reduction of the general case to the minimal one by means of the residue calculus is due to [21].

The above result is absolutely crucial for the further development of the theory, as it admits application of the theory of the constant term, developed by Harish-Chandra [56] for the case of the group and by J. Carmona [33] for reductive symmetric spaces. Theorems 5.4 and 5.6 on the discrete series are also indispensable ingredients of this theory. By the mentioned theory of the constant term, one deduces that the leading part of the asymptotic expansion of the Eisenstein integral has the form given in Theorem 11.18. The $C$-functions entering this description satisfy Proposition 11.19; this follows readily from the definition of the generalized Eisenstein integral. In the following subsection we shall indicate how the general Maass–Selberg relations formulated in Theorem 11.22 follow from those for the $C$-functions associated with minimal $\sigma$-parabolic subgroups.

**The Maass–Selberg relations** It is an important observation that the Maass–Selberg relations of Theorem 11.22 can be reformulated as an invariance property of the kernel functions.

**Theorem 14.2** *Let $P, Q \in \mathcal{P}_\sigma$ be associated parabolic subgroups and let $s \in W(\mathfrak{a}_{Q\mathrm{q}} \mid \mathfrak{a}_{P\mathrm{q}})$. Then the following two assertions are equivalent.*

(a) *$K_Q(s\nu : x : y) = K_P(\nu : x : y)$, for all $x, y \in \mathrm{X}$ and generic $\nu \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.*

(b) *$C^\circ_{Q|P}(s : \nu) C^\circ_{Q|P}(s : -\bar{\nu})^* = I_{\mathcal{A}_{2,Q}}$, as an identity of meromorphic functions in the variable $\nu \in \mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$.*

*Sketch of proof.* Assume (a) and express the kernels in terms of Eisenstein integrals according to (14.4). Next, substitute $x = mav$ and $y = m'bv$ and let $a, b \to \infty$ in $A^+_{Q\mathrm{q}}$. Comparing the coefficients of $a^{\nu-\rho_Q} b^{-\nu-\rho_Q}$ on both sides of the equation, for every $v \in {}^Q\mathcal{W}$, we infer that the expression on the left-hand side of the equality in (b) equals $C^\circ_{Q|Q}(1 : s\nu) C^\circ_{Q|Q}(1 : -s\bar{\nu})^*$, which in turn equals $I_{\mathcal{A}_{2,Q}}$, by Proposition 11.19. Thus, (b) follows. The converse reasoning is also valid, in view of the vanishing theorem of [20], described in the text following Theorem 13.13. □

In [17] it is shown that the Weyl group invariance property of the kernel $K_F$ follows from the similar invariance of the kernel $K_\emptyset$ because the residue operators behave well with respect to the action of the Weyl group, see Lemma 13.7. In view of Theorem 14.2 it follows that the Maass–Selberg relations for the $C$-functions associated with minimal $\sigma$-parabolic subgroups imply those for the $C$-functions associated with general $\sigma$-parabolic subgroups. For historical comments on the proofs of these relations, see the remarks following Theorem 11.22 as well as those following (12.3).

As said in the text preceding Theorem 11.22, the Maass–Selberg relations constitute the essential step towards the regularity theorem for the Eisenstein integrals, Theorem 11.8. This theorem in turn implies that the meromorphic $C^\infty(\mathrm{X} \times \mathrm{X}, \mathrm{End}\,(V_\tau))$-valued kernel functions $\nu \mapsto K_P(\nu : \cdot : \cdot)$, are regular on $i\mathfrak{a}^*_{P\mathrm{q}\mathbb{C}}$, for all $P \in \mathcal{P}_\sigma$ of residue type.

**Conclusion of the induction** We now come to the end of the induction argument. In (13.18) one may take $\varepsilon_F = 0$ for all $F \subset \Delta$ that are of residue type, by regularity of the kernels. Moreover, in view of (14.4) with $Q = P_F$ it follows that, for every $f \in C^\infty_c(\mathrm{X} : \tau)$, and each subset $F \subset \Delta$ of residue type,

$$T^t_F f(x) =$$

$$
\begin{aligned}
&= t(P_F)\,[W:W_F] \int_{i\mathfrak{a}^*_{F\mathrm{q}}} \int_{\mathrm{X}} E^\circ_F(\nu : x) E^*_F(\nu : y) f(y)\, dy d\mu_F(\nu) \\
&= t(P_F)\,[W:W_F]\ \mathcal{J}_F \mathcal{F}_F f(x).
\end{aligned}
$$

In view of Theorem 11.16 it follows that for $F$ of residue type, the operator $T^t_F$ maps $C^\infty_c(\mathrm{X} : \tau)$ into the Schwartz space $\mathcal{C}(\mathrm{X} : \tau)$. The induction hypothesis implies that each subset $F \subsetneq \Delta$ is of residue type, so that by application of (13.19) we see that $T^t_\Delta$ maps into the Schwartz space as well.

The induction step is now finished as follows. If the center of $G$ is not compact modulo $H$, nothing remains to be done. Therefore, let us assume that $G$ has compact center modulo $H$. Then it follows that $T^t_\Delta$ is defined by means of point residues, hence maps into a subspace of $\mathbb{D}(\mathrm{X})$-finite functions. In the above we established that it maps into the Schwartz space, hence it maps into $\mathcal{A}_2(\mathrm{X} : \tau)$. By using the action of $\mathbb{D}(\mathrm{X})$ it is easily seen that the image of $\mathcal{J}_F$ is perpendicular to $\mathcal{A}_2(\mathrm{X} : \tau)$, for each $F \subsetneq \Delta$. This implies that $T^t_\Delta$ is the restriction of the orthogonal projection $L^2(\mathrm{X} : \tau) \to L^2_{\mathrm{d}}(\mathrm{X} : \tau)$. Hence, $(G, H)$ is of residue type and the induction is finished. □

Now that the inductive argument has been completed, it follows that all parabolic subgroups are of residue type, so that the results obtained under this assumption are valid in full generality.

**Completion of the proof the Plancherel theorem** It follows from the functional equation for the Eisenstein integral, combined with the Maass–Selberg relations for the $c$-function, that $\mathcal{J}_P \circ \mathcal{F}_P$ depends on $P \in \mathcal{P}_\sigma$ through the conjugacy class of $\mathfrak{a}_{P\mathrm{q}}$ for the Weyl group $W$. Let $2^\Delta$ denote the collection of subsets of $\Delta$ and let $\sim$ be the equivalence relation on $2^\Delta$ defined by $F \sim F'$ if and only if $\mathfrak{a}_{F\mathrm{q}}$ and $\mathfrak{a}_{F'\mathrm{q}}$ are conjugate under $W$. Let $F \subset \Delta$ and let $[F]$ denote the associated class in $2^\Delta/\sim$. Then we have the following lemma.

**Lemma 14.3**
$$
\sum_{F' \in [F]} t(P_{F'}) = |W(\mathfrak{a}_{F\mathrm{q}})|^{-1}.
$$

*Proof.* The proof is basically a counting argument. Let $\mathcal{P}(\mathfrak{a}_{F\mathrm{q}})$ be the set of $P \in \mathcal{P}_\sigma$ with $\mathfrak{a}_{P\mathrm{q}} = \mathfrak{a}_{F\mathrm{q}}$. For each parabolic subgroup $P \in \mathcal{P}(\mathfrak{a}_{F\mathrm{q}})$ there exists a unique subset $F_P \subset \Delta$ such that $P$ is $W$-conjugate to $P_{F_P}$. Clearly, $F_P \sim F$ and the map $p : P \mapsto F_P$ is surjective from $\mathcal{P}(\mathfrak{a}_{F\mathrm{q}})$ onto $[F]$. For each $F' \in [F]$, the natural map $W(\mathfrak{a}_{F'\mathrm{q}} \mid \mathfrak{a}_{F\mathrm{q}}) \to \mathcal{P}_\sigma$ given by $w \mapsto w P_F w^{-1}$ is surjective onto the fiber $p^{-1}(P_{F'})$. Since the action of $W(\mathfrak{a}_{F\mathrm{q}})$ on $W(\mathfrak{a}_{F'\mathrm{q}} \mid \mathfrak{a}_{F\mathrm{q}})$ by right composition is simply transitive, it follows that each fiber $p^{-1}(P_{F'})$ consists of $|W(\mathfrak{a}_{F\mathrm{q}})|$ elements. The disjoint union of these fibers is $\mathcal{P}(\mathfrak{a}_{F\mathrm{q}})$. By $W$-invariance of the residue weight it follows that

$$
1 = \sum_{P \in \mathcal{P}(\mathfrak{a}_{F\mathrm{q}})} t(P) = \sum_{F' \in [F]} \sum_{P \in p^{-1}(F')} t(P) = \sum_{F' \in [F]} |W(\mathfrak{a}_{F\mathrm{q}})|\, t(P_{F'}),
$$

whence the result. □

We now observe that $|W||W_F|^{-1}|W(\mathfrak{a}_{F\mathrm{q}})|^{-1} = [W : W_F^*]$. By application of the above lemma we may thus rewrite (14.1) as

$$f = \sum_{[F] \in 2^\Delta/\sim} [W : W_F^*]\, \mathcal{J}_F \mathcal{F}_F f. \tag{14.5}$$

In particular, this expression is independent of the choice of the residue weight $t$. We can now clarify the role of the residue weight in the argumentation leading up to (14.5). The residue weight $t$ determines the weight by which each $F'$ from $[F]$ contributes to the term corresponding to $[F]$ in the summation in (14.5).

To get the full statement of Theorem 11.26 it remains to study the operators $\mathcal{F}_Q \circ \mathcal{J}_P$ from $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}) \otimes \mathcal{A}_{2,P}$ to $\mathcal{S}(i\mathfrak{a}_{Q\mathrm{q}}) \otimes \mathcal{A}_{2,Q}$. The key observation is that this operator is continuous linear and intertwines the natural $\mathbb{D}(\mathrm{X})$-module structures of these spaces determined by $\mu_P$ and $\mu_Q$, respectively, by Lemma 11.17. Using Theorems 5.4 and 5.6 on the discrete series it can be deduced that the composition $\mathcal{F}_Q \circ \mathcal{J}_P$ is zero unless $P$ and $Q$ are associated. Moreover, if $P = Q$, then the composition equals

$$\mathcal{F}_P \circ \mathcal{J}_P = [W : W_P^*]^{-1} \mathrm{P}_{W(\mathfrak{a}_{P\mathrm{q}})},$$

where $\mathrm{P}_{W(\mathfrak{a}_{P\mathrm{q}})}$ is the orthogonal projection from $\mathcal{S}(i\mathfrak{a}_{P\mathrm{q}}^*) \otimes \mathcal{A}_{2,P}$ onto the subspace of $W(\mathfrak{a}_{P\mathrm{q}})$-invariants, see (11.16). If we combine this with the inversion formula (14.5), the remaining assertion of Theorem 11.26 follows.

**The relation with representation theory** In the theory exposed above, the generalized Eisenstein integrals are obtained as residues from Eisenstein integrals associated with the minimal $\sigma$-principal series. To establish the Plancherel theorem in the sense of representation theory the generalized Eisenstein integrals must still be identified with matrix coefficients of the generalized principal series representations defined in Section 7; we recall that the definition of these matrix coefficients for the generalized principal series is due to [34]. For the minimal principal series it is due to [6]. The identification of Eisenstein integrals as matrix coefficients, described in Section 11, is established in the paper [22]. We shall briefly outline the argument. For $\delta \in \widehat{K}$ we define the representation $(\tau_\delta, V_{\tau_\delta})$ of $K$ as in Section 11. Thus, $V_{\tau_\delta} = V_\delta^* \otimes V_\delta$. Let $\delta_e : V_{\tau_\delta} \to \mathbb{C}$ denote the natural contraction map $v^* \otimes v \mapsto v^*(v)$. Let $Q \in \mathcal{P}_\sigma$, $\xi \in \mathrm{X}^\wedge_{Q,*,\mathrm{ds}}$. Then for generic $\nu$ we define a linear map $J_{Q,\xi,\nu,\delta} : \overline{V(Q,\xi)} \otimes L^2(K : \xi)_\delta \to C^\infty(\mathrm{X})_\delta)$ by the formula

$$J_{Q,\xi,\nu,\delta}(T)(x) = \delta_e[E_\delta^\circ(Q : \nu : x)\psi_T], \tag{14.6}$$

for $T \in \overline{V(Q,\xi)} \otimes L^2(K : \xi)_\delta$ and $x \in \mathrm{X}$. Here the index $\delta$ on the Eisenstein integral indicates that we have taken the Eisenstein integral for $\tau = \tau_\delta$. Moreover, the map $T \mapsto \psi_T$ from $\overline{V(Q,\xi)} \otimes L^2(K : \xi)_\delta$ to $\mathcal{A}_{2,Q}$ is defined as in the text preceding Definition 11.2. If we compare this with the definition just mentioned, we see that the function $J_{Q,\xi,\nu,\delta}(T)$ is our candidate for the matrix coefficient (11.3).

It is readily seen that the map $J_{Q,\xi,\nu,\delta}$ is $K$-equivariant. We define the map

$$J_{Q,\xi,\nu} : \overline{V(Q,\xi)} \otimes L^2(K : \xi)_K \to C^\infty(\mathrm{X})_K$$

by taking the direct sum of the $J_{Q,\xi,\nu,\delta}$, for $\delta \in \widehat{K}$.

**Proposition 14.4** *The map $J_{Q,\xi,\nu}$ is $(\mathfrak{g}, K)$-equivariant for the infinitesimal representations associated with $1 \otimes \pi_{Q,\xi,-\nu}$ and $L$.*

This proposition is proved in [22] by studying left derivatives of Eisenstein integrals. These can be identified with Eisenstein integrals for different $K$-types by an asympotic analysis involving the use of the vanishing theorem from [20].

Keeping (11.4) in mind, we see that Proposition 14.4 allows us to define a $(\mathfrak{g}, K)$-equivariant Fourier transform by transposition as follows. For $f \in C_c^\infty(\mathrm{X})_K$ we define $\hat{f}(Q : \xi : \nu) \in \overline{V(Q,\xi)} \otimes L^2(K : \xi)_K$ by

$$\langle \hat{f}(Q : \xi : \nu)\, , T\rangle = \int_{\mathrm{X}} f(x) \overline{J_{Q,\xi,\bar{\nu}}(T)(x)}\, dx,$$

for all $T \in \overline{V(Q,\xi)} \otimes L^2(K : \xi)_K$. By using Theorem 11.26, the Plancherel theorem for spherical functions, combined with the relation (11.10), it is then shown that the Fourier transform $f \mapsto \hat{f}(Q, \xi)$ extends to a $G$-equivariant partial isometry from $L^2(\mathrm{X})$ to the space $\mathfrak{L}^2(Q, \xi)$ defined in (10.11). Moreover, Theorem 10.21 can be derived from Theorem 11.26 along the lines indicated in Section 11.

Finally, at the end of [22], the Eisenstein integrals are identified as matrix coefficients of the principal series. First, by application of the automatic continuity theorem due to Casselman and Wallach, see [89], Thm. 11.6.7, it is shown that the map $J_{Q,\xi,\nu}$ extends to a continuous linear map from $\overline{V(Q,\xi)} \otimes C^\infty(K : \xi)$ to $C^\infty(\mathrm{X})$, intertwining $1 \otimes \pi_{Q,\xi,-\nu}$ with $L$. Therefore,

$$\mathrm{ev}_e \circ J_{Q,\xi,\nu} \in \overline{V(Q,\xi)}^* \otimes C^{-\infty}(Q : \xi : \bar{\nu})^H.$$

By asymptotic analysis based on the known asymptotic behavior of the Eisenstein integral it is then shown that

$$\mathrm{ev}_e \circ J_{Q,\xi,\nu}(\eta \otimes \varphi) = \langle \varphi\, , \, j^\circ(Q : \xi : \bar{\nu})\eta \rangle,$$

for $\eta \otimes \varphi \in \overline{V(Q,\xi)} \otimes C^\infty(K : \xi)$. This implies that $J_{Q,\xi,\nu} = \mathrm{M}_{Q,\xi,-\nu}$. Combining this with (14.6) we obtain the equality of Definition 11.2, expressing the Eisenstein integral as a spherical generalized matrix coefficient.

# 15 Appendix: Harish-Chandra's class of groups

A Lie group $G$ is said to be real reductive if its Lie algebra $\mathfrak{g}$ is a real reductive Lie algebra. This in turn means that $\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}]$ is a semisimple real Lie algebra and that

$$\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1,$$

with $\mathfrak{c}$ the center of $\mathfrak{g}$.

**Definition 15.1** A Lie group $G$ is said to belong to *Harish-Chandra's class* if it is real reductive and satisfies the following conditions.

(a) $G$ has finitely many connected components.

(b) The image of $G$ under the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}_{\mathbb{C}})$ is contained in the identity component of $\mathrm{Aut}\,(\mathfrak{g}_{\mathbb{C}})$.

(c) The analytic subgroup $G_1$ with Lie algebra $\mathfrak{g}_1$ has finite center.

We shall use the abbreviation $\mathcal{H}$ for this class of groups. Clearly, a connected semisimple Lie group belongs to the class $\mathcal{H}$ if and only if it has finite center. The class $\mathcal{H}$ was introduced by Harish-Chandra [58] for a two-fold reason. First, all main facts from the structure theory of connected semisimple groups extend to groups of the Harish-Chandra class, as we shall indicate below. Second, Harish-Chandra's class behaves well with respect to a certain type of induction, see the text following Proposition 6.10.

We shall describe those properties of groups of the Harish-Chandra class that explain how to extend the familiar Cartan decomposition $G = K \exp \mathfrak{p}$ for connected semisimple groups with finite center to all groups from $\mathcal{H}$.

We start with some general observations, the proofs of which are not difficult. It is readily seen that any Lie group with finitely many connected components and abelian Lie algebra is in $\mathcal{H}$. Moreover, any connected compact Lie group belongs to $\mathcal{H}$. The product of two groups from $\mathcal{H}$ belongs to $\mathcal{H}$ again. If $p : \tilde{G} \to G$ is a surjective homomorphism of Lie groups with finite kernel, one readily sees that $G$ belongs to $\mathcal{H}$ if and only if $\tilde{G}$ does.

The following facts are somewhat more difficult to establish. We shall not give proofs here, referring to [84], pp. 192–201, instead. We assume that $G$ belongs to $\mathcal{H}$.

The first important fact is that $G_1$ is a closed subgroup of $G$. We note that $G_1$ is connected semisimple with finite center, hence belongs to $\mathcal{H}$. Let $C = \ker \mathrm{Ad}$. Then $C_e$ is a closed central subgroup of $G_e$ with Lie algebra $\mathfrak{c}$. Let $\mathfrak{t}$ be the linear span of the kernel of $\exp : \mathfrak{c} \to C$ and let $\mathfrak{v} \subset \mathfrak{c}$ be a complementary linear subspace. Then $T = \exp \mathfrak{t}$ is a maximal compact subgroup and $V = \exp \mathfrak{v}$ a maximal closed vector subgroup of $C_e$, and $C_e \simeq T \times V$ via the natural multiplication map. One readily sees that $T$ is the unique maximal compact subgroup of $C_e$. A maximal closed vector subgroup of $C_e$ is called a *split component* for $G$. It is readily verified that every split component of $G$ arises as above for some choice of $\mathfrak{v}$. From now on we assume a split component $V$ of $G$ to be fixed.

We define $X(G)$ to be the group of continuous multiplicative characters $G \to \mathbb{R}^*$ and put

$$^\circ G := \cap_{\chi \in X(G)} \ \ker |\chi|.$$

The idea behind this definition is that $^\circ G$ contains any compact subgroup of $G$, as well as any closed connected semisimple subgroup. Moreover, it has trivial intersection with $V$. Taking this into account, the following result is not surprising.

**Lemma 15.2** *The group $^\circ G$ belongs to $\mathcal{H}$ and $G \simeq {}^\circ G \times V$ via the natural multiplication map.*

**Corollary 15.3** *Every compact subgroup of $G$ is contained in ${}^\circ G$. Moreover, every maximal compact subgroup of ${}^\circ G$ is maximal compact in $G$.*

*Proof.* These statements follow from the above lemma since $G/{}^\circ G \simeq V$ has no compact subgroups but $\{1\}$. $\square$

The maximal compact subgroups of ${}^\circ G$, hence those of $G$, can be found from those of $G_1$.

**Proposition 15.4** *Let $K$ be a maximal compact subgroup of ${}^\circ G$. Then*

$$ {}^\circ G = KG_1. \tag{15.1}$$

*Moreover, $K_1 := K \cap G_1$ is a maximal compact subgroup of $G_1$. Conversely, let $K_1$ be a maximal compact subgroup of $G_1$ with Lie algebra $\mathfrak{k}_1$. Then $K := {}^\circ G \cap \mathrm{Ad}^{-1}(\mathrm{Ad}(K_1))$ is a maximal compact subgroup of ${}^\circ G$, hence of $G$, with Lie algebra $\mathfrak{t} + \mathfrak{k}_1$.*

*Finally, the map $K \mapsto K \cap G_1$ sets up a bijective correspondence between the maximal compact subgroups of $G$ and those of $G_1$.*

From the theory of semisimple groups we now recall the fact that $G_1$ has a maximal compact subgroup and that all maximal compact subgroups of $G_1$ are conjugate. Combining this with the above proposition we see that all maximal compact subgroups of $G$ are conjugate by an element of $G_1$. In fact, this statement can be refined by using the notion of a Cartan involution.

**Definition 15.5** A *Cartan involution* of $G$ is an involution $\theta$ of $G$ for which the associated group of fixed points $G^\theta$ is maximal compact in $G$.

If $\theta$ is a Cartan involution of $G$, with fixed point group $K$, then clearly $\theta$ leaves $G_1$ invariant. Moreover, the group of fixed points of the restricted involution $\theta_1 = \theta|_{G_1}$ equals $K \cap G_1$, which is maximal compact in $G_1$, so that $\theta_1$ is a Cartan involution of $G_1$.

Conversely, we will show that every Cartan involution $\theta_1$ of $G_1$, with fixed point group $K_1$, extends to a Cartan involution $\theta$ of $G$. In view of Proposition 15.4, its group of fixed points must then be the unique maximal compact subgroup of $G$ containing $K_1$. To find $\theta$ we proceed as follows. Let the infinitesimal involution associated with $\theta_1$ be denoted by the same symbol. Let

$$\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$$

be the associated infinitesimal Cartan decomposition whose summands are the $+1$ and $-1$ eigenspaces of $\theta_1$, respectively. Then $G_1 = K_1 \exp(\mathfrak{p}_1)$, the map $(k, X) \mapsto k \exp X$ being a diffeomorphism $K_1 \times \mathfrak{p}_1 \to \mathfrak{g}_1$.

Let $K \subset {}^\circ G$ be the unique maximal compact subgroup of ${}^\circ G$ with $K \cap G_1 = K_1$. Then it follows from the Cartan decomposition for $G_1$ combined with (15.1) that the map $K \times \mathfrak{p}_1 \to {}^\circ G$, $(k, X) \mapsto k \exp X$ is a diffeomorphism. Moreover, $\mathrm{Ad}(K)$ normalizes $\mathfrak{p}_1$. It follows that we may extend $\theta_1$ to an involution $\theta$ of ${}^\circ G$ by requiring it to be the identity on $K$. It is now readily seen that $\theta$ is the unique extension of $\theta_1$ to a Cartan involution of ${}^\circ G$.

Finally, using Lemma 15.2, we may extend $\theta$ to a Cartan involution of $G$ by requiring that $\theta(a) = a^{-1}$ for $a \in V$. This extension is not unique, since it depends on the choice of $V$. There is a resulting infinitesimal Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k} = \mathfrak{t} + \mathfrak{k}_1$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{v}$. Moreover, on the level of the group we find that

$$G = K \exp \mathfrak{p},$$

the map $(k, X) \mapsto k \exp X$, $K \times \mathfrak{p} \to G$ being a diffeomorphism. Finally, given $K$ as above, the map $X \mapsto \exp X K \exp(-X)$ defines a bijection from $\mathfrak{p}_1$ onto the set of maximal compact subgroups of $G$.

## References


[1] *J. Arthur,* A Paley–Wiener theorem for real reductive groups. Acta Math. **150** (1983), 1–89.

[2] *J. Arthur,* A local trace formula. Publ. Math. I.H.E.S. No. **73** (1991), 5–96.

[3] *E.P. van den Ban,* Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula. Ark. Mat. **25** (1987), 175–187.

[4] *E.P. van den Ban,* Asymptotic behaviour of matrix coefficients related to reductive symmetric spaces. Nederl. Akad. Wetensch. Indag. Math. **49** (1987), 225–249.

[5] *E.P. van den Ban,* The principal series for a reductive symmetric space, I. $H$-fixed distribution vectors. Ann. Scient. Éc. Norm. Sup. **21** (1988), 359–412.

[6] *E.P. van den Ban,* The principal series for a reductive symmetric space II. Eisenstein integrals. J. Funct. Anal. **109** (1992), 331–441.

[7] *E.P. van den Ban,* Induced Representations and the Langlands classification. Proc. Edinburgh '96. Symposia in Pure Math. AMS.

[8] *E.P. van den Ban,* The action of intertwining operators on spherical vectors in the minimal principal series of a reductive symmetric space. Indag. Math. **145** (1997), 317–347.

[9] *E.P. van den Ban,* Eisenstein integrals and induction of relations, pp. 487 – 509 in: Noncommutative Harmonic Analysis, In Honor of Jacques Carmona, P. Delorme & M. Vergne, eds., Progress in Math. 220, Birkhäuser, Boston, 2004.

[10] *E.P. van den Ban, J. Carmona, P. Delorme,* Paquets d' ondes dans l'espace de Schwartz d'un espace symétrique réductif. J. Funct. Anal. **139** (1996), 225–243.

[11] *E.P. van den Ban, M. Flensted-Jensen, H. Schlichtkrull,* Harmonic analysis on semisimple symmetric spaces: a survey of some general results. Representation theory and automorphic forms (Edinburgh, 1996), 191–217, Proc. Sympos. Pure Math., **61**, Amer. Math. Soc., Providence, RI, 1997.

[12] *E.P. van den Ban, H. Schlichtkrull,* Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces. J. Reine Angew. Math. **380** (1987), 108–165.

[13] *E.P. van den Ban, H. Schlichtkrull,* Convexity for invariant differential operators on a semisimple symmetric space. Compos. Math. **89** (1993), 301–313.

[14] *E.P. van den Ban, H. Schlichtkrull,* Expansions for Eisenstein integrals on semisimple symmetric spaces. Ark. Mat. **35** (1997), 59–86.

[15] *E.P. van den Ban, H. Schlichtkrull,* Fourier transforms on a semisimple symmetric space. Invent. Math. **130** (1997), 517–574.

[16] *E.P. van den Ban, H. Schlichtkrull,* The most continuous part of the Plancherel decomposition for a reductive symmetric space. Annals Math. **145** (1997), 267–364.

[17] *E.P. van den Ban, H. Schlichtkrull,* Fourier inversion on a reductive symmetric space. Acta Math. **182** (1999), 25–85.

[18] *E.P. van den Ban, H. Schlichtkrull,* A residue calculus for root systems. Compositio Math. **123** (2000), 27–72.

[19] *E.P. van den Ban, H. Schlichtkrull,* Harmonic Analysis on reductive symmetric spaces. Proc. third European Congress of Mathematics, 2000.

[20] *E.P. van den Ban, H. Schlichtkrull,* Analytic families of eigenfunctions on a reductive symmetric space. Represent. Theory **5** (2001), 615–712.

[21] *E.P. van den Ban, H. Schlichtkrull,* The Plancherel decomposition for a reductive symmetric space, I. Spherical functions. arXiv.math.RT/0107063.

[22] *E.P. van den Ban, H. Schlichtkrull,* The Plancherel decomposition for a reductive symmetric space, II. Representation theory. arXiv.math.RT/0111304.

[23] *E.P. van den Ban, H. Schlichtkrull,* A Paley–Wiener theorem for reductive symmetric spaces. arXiv.math.RT/0302232.

[24] *M. Berger,* Les espaces symétriques non compacts. Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.

[25] *J.N. Bernstein,* On the support of Plancherel measure. J. Geom. Phys. **5** (1988), 663–710.

[26] *F.V. Bien,* $\mathcal{D}$-modules and spherical representations. Mathematical Notes, 39. Princeton University Press, Princeton, NJ, 1990.

[27] *N. Bopp, P. Harinck,* Formule de Plancherel pour $\mathrm{GL}(n, C)/\mathrm{U}(p, q)$. J. Reine Angew. Math. **428** (1992), 45–95.

[28] *A. Borel, N. Wallach,* Continuous cohomology, discrete subgroups, and representations of reductive groups. Second edition. Mathematical Surveys and Monographs, **67**. American Mathematical Society, Providence, RI, 2000.

[29] *N. Bourbaki,* Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.

[30] *F. Bruhat,* Sur les représentations induites des groupes de Lie. Bull. Soc. Math. France **84** (1956), 97–205.

[31] *J.-L. Brylinski, P. Delorme,* Vecteurs distributions $H$-invariants pour les séries principales généralisées d'espaces symétriques réductifs et prolongement méromorphe d'intégrales d'Eisenstein. Invent. Math. **109** (1992), 619–664.

[32] *O.A. Campoli,* Paley–Wiener type theorems for rank-1 semisimple Lie groups, Rev. Union Mat. Argent. **29** (1980), 197–221.

[33] *J. Carmona,* Terme constant des fonctions tempérées sur un espace symétrique réductif. J. Reine Angew. Math. **491** (1997), 17–63.

[34] *J. Carmona, P. Delorme,* Base méromorphe de vecteurs distributions H-invariants pour les séries principales généralisées d'espaces symétriques réductifs: Equation fonctionelle. J. Funct. Anal. **122** (1994), 152–221.

[35] *J. Carmona, P. Delorme,* Transformation de Fourier sur l'espace de Schwartz d'un espace symétrique réductif. Invent. Math. **134** (1998), 59–99.

[36] *W. Casselman,* Canonical extensions of Harish-Chandra modules to representations of $G$. Canad. J. Math. **41** (1989), 385–438.

[37] *W. Casselman, D. Miličić,* Asymptotic behavior of matrix coefficients of admissible representations. Duke Math. J. **49** (1982), 869–930.

[38] *P. Delorme,* Intégrales d'Eisenstein pour les espaces symétriques réductifs: tempérance, majorations. Petite matrice B. J. Funct. Anal. **136** (1994), 422–509.

[39] *P. Delorme,* Troncature pour les espaces symétriques réductifs. Acta Math. **179** (1997), 41–77.

[40] *P. Delorme,* Formule de Plancherel pour les espaces symétriques réductifs. Annals Math. **147** (1998), 417–452.

[41] *P. Delorme,* Harmonic analysis on real reductive symmetric spaces. Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 545–554. Higher Ed. Press, Beijing, 2002.

[42] *J. Dixmier,* Les $C^*$-algèbres et leurs représentations. Reprint of the second (1969) edition. Les Grands Classiques Gauthier-Villars. Éditions Jacques Gabay, Paris, 1996.

[43] *J. Faraut,* Distributions sphériques sur les espaces hyperboliques. J. Math. Pures Appl. (9) **58** (1979), 369–444.

[44] *M. Flensted-Jensen,* Spherical functions of a real semisimple Lie group. A method of reduction to the complex case. J. Funct. Anal. **30** (1978), no. 1, 106–146.

[45] *M. Flensted-Jensen,* Discrete series for semisimple symmetric spaces. Annals Math. **111** (1980), 253–311.

[46] *M. Flensted-Jensen,* Analysis on non-Riemannian symmetric spaces. CBMS Regional Conference Series in Mathematics, **61**. AMS, Providence, RI, 1986.

[47] *R. Gangolli,* On the Plancherel formula and the Paley–Wiener theorem for spherical functions on semisimple Lie groups. Ann. of Math. **93** (1971), 150-165.

[48] *S.G. Gindikin, F.I. Karpelevič,* Plancherel measure for symmetric Riemannian spaces of non-positive curvature. (Russian) Dokl. Akad. Nauk SSSR **145** (1962), 252–255.

[49] *P. Harinck,* Fonctions orbitales sur $G_{\mathbb{C}}/G_{\mathbb{R}}$. Formule d'inversion des intégrales orbitales et formule de Plancherel. J. Funct. Anal. **153** (1998), 52–107.

[50] *Harish-Chandra,* Spherical functions on a semisimple Lie group. I. Amer. J. Math. **80** (1958), 241–310.

[51] *Harish-Chandra,* Spherical functions on a semisimple Lie group. II. Amer. J. Math. **80** (1958), 553–613.

[52] *Harish-Chandra,* Differential equations and semisimple Lie groups. Unpublished manuscript (1960), in: Collected Papers Vol. 3, Springer-Verlag, New York 1984.

[53] *Harish-Chandra,* Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions. Acta Math. **113** (1965), 241–318.

[54] *Harish-Chandra,* Discrete series for semisimple Lie groups. II. Explicit determination of the characters. Acta Math. **116** (1966), 1–111.

[55] *Harish-Chandra,* On the theory of the Eisenstein integral. Lecture Notes in Math. Vol. **266**, 123–149, Springer-Verlag, New York, 1972.

[56] *Harish-Chandra,* Harmonic analysis on real reductive groups I. The theory of the constant term. J. Funct. Anal. **19** (1975), 104–204.

[57] *Harish-Chandra,* Harmonic analysis on real reductive groups II. Wave packets in the Schwartz space. Invent. Math. **36** (1976), 1–55.

[58] *Harish-Chandra,* Harmonic analysis on real reductive groups III. The Maass–Selberg relations and the Plancherel formula. Annals of Math. **104** (1976), 117–201.

[59] *G.J. Heckman, E.M. Opdam,* Yang's system of particles and Hecke algebras. Ann. of Math. **145** (1997), 139–173. Erratum: Ann. of Math. **146** (1997), 749–750.

[60] *S. Helgason,* An analogue of the Paley–Wiener theorem for the Fourier transform on certain symmetric spaces. Math. Ann. **165** (1966), 297–308.

[61] *S. Helgason,* A duality for symmetric spaces with applications to group representations. Advances in Math. **5** (1970), 1–154.

[62] *S. Helgason,* A duality for symmetric spaces with applications to group representations. II. Differential equations and eigenspace representations. Advances in Math. **22** (1976), no. 2, 187–219.

[63] *S. Helgason,* Groups and Geometric Analysis. Academic Press, Orlando, FL, 1984.

[64] *L. Hörmander,* Linear Partial Differential Operators. Springer Verlag, Berlin, 1963.

[65] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, T. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space. Ann. of Math. **107** (1978), 1–39.

[66] *A.W. Knapp, E.M. Stein,* Intertwining operators for semisimple groups. II. Invent. Math. **60** (1980), 9–84.

[67] *J.A.C. Kolk, V.S. Varadarajan,* On the transverse symbol of vectorial distributions and some applications to harmonic analysis. Indag. Math. (N.S.) **7** (1996), 67–96.

[68] *R.P. Langlands,* On the functional equations satisfied by Eisenstein series. Springer Lecture Notes, Vol. **544**, Springer-Verlag, Berlin, 1976.

[69] *T. Matsuki,* The orbits of affine symmetric spaces under the action of minimal parabolic subgroups. J. Math. Soc. Japan **31** (1979), 331–357.

[70] *T. Matsuki,* A description of discrete series for semisimple symmetric spaces. II. Representations of Lie groups, Kyoto, Hiroshima, 1986, 531–540, Adv. Stud. Pure Math., 14, Academic Press, Boston, MA, 1988.

[71] *C. Moeglin, J.-L. Waldspurger,* Spectral decomposition and Eisenstein series. Cambridge Tracts in Mathematics, 113. Cambridge University Press, Cambridge, 1995.

[72] *V. F. Molčanov,* An analog of Plancherel's formula for hyperboloids. (Russian) Dokl. Akad. Nauk SSSR **183** (1968), 288–291.

[73] *G.D. Mostow,* Some new decomposition theorems for semi-simple groups. Mem. Amer. Math. Soc. no. **14**, (1955), 31–54.

[74] *G. Ólafsson,* Fourier and Poisson transformation associated to a semisimple symmetric space. Invent. Math. **90** (1987), 605–629.

[75] *T. Oshima,* A realization of semisimple symmetric spaces and construction of boundary value maps. Adv. Studies in Pure Math. Vol. **14** (1988), 603–650.

[76] *T. Oshima,* A calculation of $c$-functions for semisimple symmetric spaces. Lie groups and symmetric spaces, 307–330, Amer. Math. Soc. Transl. Ser. 2, 210, Amer. Math. Soc., Providence, RI, 2003.

[77] *T. Oshima, T. Matsuki,* A description of discrete series for semisimple symmetric spaces. Adv. Stud. Pure Math. Vol. **4** (1984), 331–390.

[78] *T. Oshima, J. Sekiguchi,* Eigenspaces of invariant differential operators on an affine symmetric space. Invent. Math. **57** (1980), 1–81.

[79] *M. Poel, G. van Dijk,* The Plancherel formula for the pseudo-Riemannian space $\mathrm{SL}(n,R)/\mathrm{GL}(n-1,R)$. Compositio Math. **58** (1986), 371–397.

[80] *W. Rossmann,* The structure of semisimple symmetric spaces. Canad. J. Math. **31** (1979), 157–180.

[81] *W. Rossmann,* Analysis on real hyperbolic spaces. J. Funct. Anal. **30** (1978), 448–477.

[82] *H. Schlichtkrull,* Harmonic Analysis on semisimple symmetric spaces. Part II in: G. Heckman, H. Schlichtkrull, Harmonic Analysis and Special Functions on Symmetric Spaces. Persp. in Math., Vol. 16, Acad. Press, San Diego, 1994.

[83] *H. Schlichtkrull,* Hyperfunctions and harmonic analysis on symmetric spaces. Progress in Mathematics, Vol. 49. Birkhäuser Boston, Inc., Boston, MA, 1984.

[84] *V.S. Varadarajan,* Harmonic analysis on real reductive groups. Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977.

[85] *D.A. Vogan,* Irreducibility of discrete series representations for semisimple symmetric spaces. Representations of Lie groups, Kyoto, Hiroshima, 1986, 191–221, Adv. Stud. Pure Math., Vol. 14, Academic Press, Boston, MA, 1988.

[86] *D.A. Vogan, N.R. Wallach,* Intertwining operators for real reductive groups. Adv. Math. **82** (1990), 203–243.

[87] *N.R. Wallach,* Asymptotic expansions of generalized matrix entries of representations of real reductive groups. Lie group representations, I (College Park, Md., 1982/1983), 287–369, Lecture Notes in Math., Vol. 1024, Springer, Berlin, 1983.

[88] *N.R. Wallach,* Real Reductive Groups I. Academic Press, Inc., San Diego, 1988.

[89] *N.R. Wallach,* Real Reductive Groups II. Academic Press, Inc., San Diego, 1992.

E.P. van den Ban
Mathematisch Instituut
Universiteit Utrecht
PO Box 80 010
3508 TA Utrecht
Netherlands
E-mail: ban@math.uu.nl