Chapter 1

Analytic extensions

1.1 Preliminaries

In this thesis $G$ is a connected real semisimple Lie group with finite centre (actually from p. 3 on it will be assumed that the centre is trivial) and $K$ is a maximal compact subgroup of $G$. We denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. The orthocomplement of $\mathfrak{k}$ with respect to the Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ is denoted by $\mathfrak{s}$, and the Cartan involution corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ is denoted by $\theta$.

Let $a$ be a maximal abelian subspace of $\mathfrak{s}$, $\Delta$ the set of roots of the pair $(\mathfrak{g}, a)$, $W$ the corresponding Weyl group. Fix a choice $\Delta^+$ of positive roots, and let $a^+$ be the corresponding positive Weyl chamber in $a$.

Roots $\alpha \in \Delta$ with $\langle \alpha, \mu \rangle \neq 0$ are called indivisible; let $\Delta^{++}$ denote the set of indivisible positive roots. If $\mu \in a^*$ ($a^*$ denotes the dual of the real linear space $a$) we write $\mathfrak{g}_\mu$ for the space $\bigcap_{H \in a} \ker (\text{ad} H - \mu(H)I)$ (ad denotes the adjoint representation of $\mathfrak{g}$). If $\alpha \in \Delta^{++}$ we write $n_\alpha$ for $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $\bar{n}_\alpha$ for $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, and finally we write:

$$n = \sum_{\alpha \in \Delta^+} n_\alpha, \quad \bar{n} = \sum_{\alpha \in \Delta^+} \bar{n}_\alpha.$$

The sets $A = \exp a$, $N = \exp n$, $\bar{N} = \exp \bar{n}$ are closed subgroups of $G$ with Lie algebras $a$, $n$, $\bar{n}$ respectively. Let $M, m$ be the centralizers of $a$ in $K, \mathfrak{k}$ respectively; $m$ is the Lie algebra of $M$.

$G$ admits the Iwasawa decomposition $G = KAN$. Here the map $K \times A \times N \to G$, $(k, a, n) \to \text{kan}$ is a real analytic diffeomorphism. The
maps $\kappa, h, \nu$ of $G$ into $K, A, N$ respectively are defined by:

$$x = \kappa(x)h(x)\nu(x) \quad (x \in G).$$

Thus $\kappa, h, \nu$ are real analytic maps $G \to K, G \to A, G \to N$. The exponential map $\exp: a \to A$ is a real analytic diffeomorphism, its inverse is denoted by $\log$. We define the map $H: G \to a$ by:

$$H(x) = \log h(x) \quad (x \in G).$$

Let $\mathbb{D}(G)^K$ be the algebra of left $G$-right $K$-invariant differential operators on $G$. A $C^\infty$ function $\phi: G \to \mathbb{C}$ which is bi-$K$-invariant, a simultaneous eigenfunction for $\mathbb{D}(G)^K$, and satisfies $\phi(e) = 1$, is called an elementary spherical function of the pair $(G, K)$. As Harish-Chandra proved in his paper [1], the elementary spherical functions of $(G, K)$ are the functions $\phi_\lambda (\lambda \in a_*^*, \text{ the complexified dual of } a)$ defined by:

$$\phi_\lambda(x) = \int_K \mathrm{e}^{(i\lambda - \rho)H(xk)} \, dk \quad (x \in G). \quad (1)$$

Here $\rho = \frac{1}{2} \sum a(a)a$ (summation over $A^+$), $m(a) = \dim a$, and $dk$ is the Haar measure of $K$ normalized by $\int_K dk = 1$.

One has $\phi_\lambda = \phi_{\bar{\mu}}$ iff $\lambda, \mu$ are conjugate under $W$ (cf. Harish-Chandra [1]).

We write $\text{Aut}(\mathfrak{a})$ for the group of automorphisms of a Lie algebra $\mathfrak{a}$. If $L$ is a Lie group, its component of the identity is denoted by $L^0$. Let $\text{Ad}$ be the adjoint representation of $G$ in $\mathfrak{g}$. Since $G$ is a connected semisimple Lie group, $\text{Ad}$ is a Lie group homomorphism of $G$ onto $(\text{Aut } \mathfrak{g})^0$; its kernel is $Z(G)$, the centre of $G$. Now $Z(G) \subseteq K$, so by the bi-$K$-invariance of the $\phi_\lambda$, we may pass to $\text{Ad } G = (\text{Aut } \mathfrak{g})^0$ and study the elementary spherical functions of the pair $(\text{Ad } G, \text{Ad } K)$. The Lie algebra $\text{ad } \mathfrak{g}$ of $\text{Ad } G$ is isomorphic to $\mathfrak{g}$ under $\text{ad}$ and so there is no loss of generality if we assume...
that \( Z(G) = \{e\} \).

**Remark.** In the remainder of this thesis it will be assumed that the centre of \( G \) is trivial.

This case has the advantage that \( G \) is isomorphic to \( (\text{Aut } \mathfrak{g})^0 \) under \( \text{Ad} \). Denote the complexification of the Lie algebra \( \mathfrak{g} \) by \( \mathfrak{g}_c \).

\( (\text{Aut } \mathfrak{g})^0 \) embeds in \( (\text{Aut } \mathfrak{g}_c)^0 \) under the map \( L \rightarrow L_c \), \( L(\mathfrak{g}, \mathfrak{g}) \rightarrow L(\mathfrak{g}_c, \mathfrak{g}_c) \) (if \( L \in L(\mathfrak{g}, \mathfrak{g}) \), \( L_c \) denotes its complexification \( \mathfrak{g}_c \rightarrow \mathfrak{g}_c \)). \( (\text{Aut } \mathfrak{g}_c)^0 \) is a connected complex semisimple Lie group with trivial centre and with Lie algebra \( (\text{ad } \mathfrak{g})_c \), and \( (\text{Aut } \mathfrak{g})^0 \) is the connected analytic subgroup of \( (\text{Aut } \mathfrak{g}_c)^0 \) with Lie algebra \( \text{ad } \mathfrak{g} \). This shows that without loss of generality we may assume that we work already in a connected complex semisimple Lie group \( G_c \) with trivial centre and with Lie algebra \( \mathfrak{g}_c \), and that \( G \) is the analytic subgroup of \( G_c \) generated by \( \exp(\mathfrak{g}) \).

### 1.2 Analytic extensions of \( G, K, A, M, N, \overline{N} \)

If \( E \) is a real linear subspace of \( \mathfrak{g} \), we denote its complexification in \( \mathfrak{g}_c \) by \( E_c \). With this notation let \( K_c, A_c, N_c, \overline{N}_c \) be the connected subgroups of \( G_c \) with Lie algebras \( k_c, a_c, n_c, \overline{n}_c \) respectively. In this section we shall study the images of these groups under the isomorphism \( \text{Ad}: G_c \rightarrow (\text{Aut } \mathfrak{g}_c)^0 \) (the adjoint representation of \( G_c \) in \( \mathfrak{g}_c \)).

We define the inner product \( ( \ , \ ) \) on \( \mathfrak{g} \) by

\[
(X, Y) = -<X, \mathfrak{g}Y> \quad (X, Y \in \mathfrak{g}).
\]

We denote its extension to a complex bilinear form \( \mathfrak{g}_c \times \mathfrak{g}_c \rightarrow \mathbb{C} \) by \( ( \ , \ ) \) as well. Provide \( \mathfrak{g}^* \) with a lexicographic ordering with respect
to some choice of linear coordinates, and let \( \alpha_1, \ldots, \alpha_d \) be the induced ordering of \( \Delta^+ \). Select a (, )-orthonormal basis \( (e_k)_{k=1, \ldots, n} \) \((n = \dim g)\) for \( g \), subordinate to the direct sum decomposition

\[
g = g_{d_1} + \ldots + g_{\alpha_1} + m + a + g_{-\alpha_1} + \ldots + g_{-d_1}
\]

and such that the ordering \( e_1, \ldots, e_n \) is compatible with the ordering of the spaces at the right hand side of \((3)\). We identify any map \( L \in L(g_c, g_c) \) with its matrix with respect to the basis \( (e_k) \), and we write \( L_{ij} \) for the entry of that matrix in the \( i \)-th row and \( j \)-th column. Let \( G = (GL(g))^0 \), and let \( K \) be the subgroup of (, )-orthogonal maps in \( G \). Denote the group of diagonal matrices with positive diagonal entries by \( A \), and the group of upper (lower) triangular matrices with diagonal entries equal to 1 by \( N \) (\( \overline{N} \)).

We write \( \mathfrak{g}, \mathfrak{l}, \mathfrak{a}, \mathfrak{n}, \mathfrak{\overline{n}} \) for the Lie algebras of \( G, K, A, N, \overline{N} \) respectively.

**Proposition 1.1** If \( Q \) is any of the groups \( K, A, N, \overline{N} \) and if \( g \) denotes its Lie algebra then:

\[
ad(g) = \mathfrak{g} \cap \text{ad}(g), \quad \text{Ad}(Q) = \mathfrak{g} \cap \text{Ad}(G).
\]

For a proof and a more detailed discussion of the above construction we refer the reader to Wallach [1, Ch. 5].

Observe that \( \mathfrak{l} \) consists of the real anti-symmetric matrices in \( L(g, g) \), \( \mathfrak{a} \) consists of the real diagonal matrices, and \( \mathfrak{n} \) (\( \mathfrak{\overline{n}} \)) consists of the real upper (lower) triangular matrices with diagonal entries equal to 0.

We denote the complexifications of \( \mathfrak{l}, \mathfrak{a}, \mathfrak{n}, \mathfrak{\overline{n}} \) in \( L(g_c, g_c) \) by \( \mathfrak{l}_c, \mathfrak{a}_c, \mathfrak{n}_c, \mathfrak{\overline{n}}_c \) respectively. Thus \( \mathfrak{l}_c \) consists of all complex anti-symmetric matrices, \( \mathfrak{a}_c \) consists of the complex diagonal matrices and
\( \mathfrak{m}_c \) (\( \mathfrak{n}_c \)) consists of the complex upper (lower) triangular matrices with zeros on the diagonal. Let \( K_c, A_c, N_c, N_c^{\ast} \) be the connected subgroups of \( \text{GL}(\mathfrak{g}_c) \) with Lie algebras \( \mathfrak{l}_c, \mathfrak{a}_c, \mathfrak{n}_c, \mathfrak{n}_c^{\ast} \) respectively. Then \( K_c \) consists of the complex matrices \( M \) with \( M'M = MM' = I \) and \( \det(M) = 1 \) (where \( M' \) denotes the matrix defined by \( (M')_{ij} = M_{ji} \)), \( A_c \) consists of the complex diagonal matrices with non-zero diagonal entries, and \( N_c (N_c^{\ast}) \) consists of the complex upper (lower) triangular matrices with diagonal entries equal to 1.

Since \( K_c \cap \text{Ad}(G_c) \) has \( \mathfrak{l}_c \cap \text{ad}(\mathfrak{g}_c) = \text{ad}(\mathfrak{f}_c) \) as its Lie algebra, it follows that \( \text{Ad}(K_c) = (K_c \cap \text{Ad}(G_c))^0 \).

Furthermore \( \exp: \mathfrak{l}(\mathfrak{g}_c, \mathfrak{a}_c) \to \text{GL}(\mathfrak{g}_c) \) maps \( \mathfrak{a}_c \) onto \( A_c \), and it maps \( \mathfrak{m}_c \) and \( \mathfrak{n}_c \) diffeomorphically onto \( N_c \) and \( N_c^{\ast} \) respectively. Thus we obtain the following proposition.

**Proposition 1.2.** We have

\[
\text{Ad}(K_c) = (K_c \cap \text{Ad}(G_c))^0,
\]

and if \( Q \) is any of the groups \( A, N, N \) we have:

\[
\text{Ad}(Q_c) = Q_c \cap \text{Ad}(G_c).
\]

Moreover the maps \( \exp: \mathfrak{m}_c \to N_c \) and \( \exp: \mathfrak{n}_c \to N_c^{\ast} \) are diffeomorphisms, and \( \exp \) maps \( \mathfrak{a}_c \) onto \( A_c \).

### 1.3 Analytic extensions of \( \kappa, h, \nu \)

The map \( K \times A \times N \to G, (k, a, n) \to \text{kan} \) is a real analytic diffeomorphism. Let \( \kappa: G \to K, \ h: G \to A, \ \nu: G \to N \) be the maps defined by:

\[
x = \kappa(x)h(x)\nu(x) \quad (x \in G).
\]

Then \( \kappa, h, \nu \) are real analytic maps and in view of Proposition 1.1
the following diagrams commute.

\[ \begin{array}{ccc}
G & \xrightarrow{Ad} & G \\
\downarrow{K} & & \downarrow{K} \\
\downarrow{Ad} & & \downarrow{Ad} \\
K & \xrightarrow{\cdot} & K \\
\end{array} \]

\[ \begin{array}{ccc}
G & \xrightarrow{Ad} & G \\
\downarrow{h} & & \downarrow{h} \\
A & \xrightarrow{Ad} & A \\
\downarrow{N} & & \downarrow{N} \\
N & \xrightarrow{\cdot} & N \\
\end{array} \]

\[ \begin{array}{ccc}
G & \xrightarrow{Ad} & G \\
\downarrow{v} & & \downarrow{v} \\
A & \xrightarrow{Ad} & A \\
\downarrow{N} & & \downarrow{N} \\
N & \xrightarrow{\cdot} & N \\
\end{array} \] (4)

In this section we will study analytic extensions of \( k, h, v \).

Restriction of the obtained extensions to \( \text{Ad}(G_c) \) will lead to extensions of \( k, h, v \).

If \( k \in K, a \in A, n \in N, x = \text{kan} \), we have that
\[ x'x = n'a'k'k' = n'a'^2n \in \overline{N_A N}. \]

We shall first study the set \( \overline{N_A N} \) in \( L(\theta_c, \theta_c) \). A matrix \( y \in L(\theta_c, \theta_c) \) belongs to \( \overline{N_A N} \) iff there exists a matrix \( u \in N_c \) such that \( yu \) is lower triangular with non-zero diagonal elements. Translating the latter statement into equations for the matrix coefficients \( u_{jk} \) and applying Cramer's rule to the obtained equations we obtain that \( y \) belongs to \( \overline{N_A N} \) iff all minors

\[ D_k(y) = \det \left( (y_{ij})_{1 \leq i, j \leq k} \right) \quad (1 \leq k \leq n) \]

are different from zero, and then there exists a unique
\[ u = u(y) \in N_c \] such that \( yu(y) \in \overline{N_A N} \). Its entries are given by

\[ u_{jk}(y) = \frac{D_{k-j}(y)}{D_{k-1}(y)} \quad (1 \leq j < k \leq n), \]

where \( D_{k-j}(y) \) is the minor of \( D_k(y) \) obtained by omitting the \( k \)-th row and \( j \)-th column. Moreover, if \( 1 \leq k \leq n \), then the minor consisting of the first \( k \) rows and columns of \( yu(y) \) is equal to \( D_k(y) \). Writing \( v_i(y) \) for the \( i \)-th diagonal entry of \( yu(y) \) we thus have:
\begin{align*}
v_i(y) &= D_i(y) = y_{11}, \quad (5) \\
v_k(y) &= \frac{D_k(y)}{D_{k-1}(y)} \quad (1 < k \leq n). \quad (6)
\end{align*}

We have proved:

**Lemma 1.3.** Let $D: L(\mathfrak{g}_C, \mathfrak{g}_C) \to \mathfrak{g}_C$ be the polynomial function given by:

$$D(y) = D_1(y) \ldots \ldots D_n(y).$$

Then the polynomial map $\mathbb{N}_C \times \mathbb{A} \times \mathbb{N}_C \to \mathbb{S}_C$, $(\tilde{n}, a, n) \mapsto \tilde{n}a_n$ is a diffeomorphism onto $L(\mathfrak{g}_C, \mathfrak{g}_C) \setminus D^{-1}(0)$. Its inverse is the rational map

$$y \to (yu(y)v(y)^{-1}, v(y), u(y)^{-1}),$$

where $v(y)$ denotes the diagonal matrix with entries $v_i(y)$ \[(i = 1, \ldots, n),\] and where $u(y), v_i(y)$ are given by the formulas (5), (6) and (7).

**Remark.** The above computations can be found in more detail in Gelfand-Neumark [1, Ch. I, §3]

Now consider the map $g: L(\mathfrak{g}_C, \mathfrak{g}_C) \to L(\mathfrak{g}_C, \mathfrak{g}_C)$, defined by $g(x) = x'x$. Observe that $g(x)$ is equal to the Gram matrix

$$g(x) = ( (x(e_i), x(e_j)) )_{i < j < n}.$$ 

Obviously, $g$ maps $\mathbb{G}$ onto the set of positive definite symmetric matrices in $L(\mathfrak{g}_C, \mathfrak{g}_C)$. Hence, if $x \in \mathbb{G}$, then all minors $D_k(g(x))$ \[(1 < k < n)\] are strictly positive, and by the above lemma there exists a unique $(\tilde{n}, b, n) \in \mathbb{N} \times \mathbb{A} \times \mathbb{N}$ such that $y = \tilde{n}b_n$. By symmetry of
$g(x)$ we must have $\bar{n} = n'$. Let $a$ be the unique element of $A$ such that $b = a^2$, and write $k = x a^{-1} a^{-1}$. Then $k' = a^{-1}(n')^{-1}x' = a^{-1}(n')^{-1}g(x)x^{-1} = anx^{-1} = k^{-1}$, and $\det k > 0$, showing that $k \in K$. This proves that the map $K \times A \times N \to G$, $(k,a,n) \to kan$ is a diffeomorphism (as was said before), and the corresponding maps $\kappa$, $h$, $\nu$ are determined by:

$$
\nu(x) = u(g(x))^{-1}, \\
h(x)^2 = v(g(x)), \\
\kappa(x)h(x) = u(g(x)).
$$

(8)

Let the polynomial function $F$ on $L(\theta_C, \theta_C)$ be defined by

$$
F(x) = D(g(x)) \quad (x \in L(\theta_C, \theta_C)),
$$

then by the above discussion we have:

**Theorem 1.4.** The maps $F, \kappa, h, F, h^2$ and $F, \nu$ are polynomials in the entries $(x(e_i), x(e_j))$ of the Gram matrix $g(x)$. Consequently, $\kappa, h, h^2, \nu$ extend holomorphically to $G_C \backslash F^{-1}(0)$. Moreover, writing $b_i$ for the $i$-th diagonal entry of $h^2$, we have:

$$
b_1(x) = D_1(g(x)),
$$

(9)

$$
b_k(x) = \frac{D_k(g(x))}{D_{k-1}(g(x))} \quad (1 < k \leq n).
$$

(10)

**Remark.** By the same type of computations, formulas similar to (9), (10) are obtained in Gelfand-Neumark [1, Ch. II, §8, (8.15)]. The above computation of $h$ has also been used by T.S.Bhanu Murti to determine the Harish-Chandra $c$-function explicitly for the groups $SL(n, \mathbb{R})$ and $Sp(n, \mathbb{R})$ (cf. Bhanu Murti [1], [2]).
The following theorem is about multi-valued analytic extensions of the maps $\kappa: G \to K$, $h: G \to A$ to $G_c \backslash F^{-1}(0)$. The discrete group $\hat{M}$ defined by

$$\hat{M} = K \cap \exp i\mathfrak{a}$$

plays a main role in it. For the terminology of multi-valued analytic maps we refer the reader to the appendix to this chapter.

**Theorem 1.5.** The maps $\kappa: G \to K$, $h: G \to A$ have extensions to multi-valued analytic maps $\kappa_c: G_c \backslash F^{-1}(0) \to K_c$, $h_c: G_c \backslash F^{-1}(0) \to A_c$ (with respect to the base point 1). Moreover, if $\kappa_1$ and $\kappa_2$ are two branches of $\kappa_c$ at a point $x_0 \in G_c \backslash F^{-1}(0)$, and if $h_1$, $h_2$ are the corresponding branches of $h_c$ at $x_0$, then $\kappa_1 = \kappa_2 d$ and $h_1 = h_2 d = d h_2$ for some $d \in \hat{M}$.

**Remark.** In particular the monodromy groups associated with the multi-valued maps $h_c$, $\kappa_c$ are isomorphic to subgroups of $\hat{M}$. Observe that the group $\hat{M}$ consists of the diagonal matrices with diagonal entries equal to $\pm 1$, with an even number of $-$ signs.

**Proof of Theorem 1.5.** Let $\sqrt{}$ be the extension of the map $x \to x^{\frac{1}{2}}$, $\mathbb{R}^+ \to \mathbb{R}^+$ to a multi-valued map $\mathcal{C}\{0\} \to \mathcal{C}$ with base point 1. Writing $b_{k,c}$ for the holomorphic extensions of the functions $b_k$ ($1 \leq k \leq n$) given in Theorem 1.4 we define the multi-valued analytic map $h_c: G_c \backslash F^{-1}(0) \to A_c$ by:

$$h_c(x) = \begin{pmatrix} \sqrt{b_{1,c}(x)} & \vdots \\ \vdots & \sqrt{b_{n,c}(x)} \end{pmatrix},$$

(11)
Obviously $\kappa_c$ is the multi-valued analytic extension of $\kappa$. Writing $\nu_c$ for the holomorphic extension of $\nu$ to $G_c \setminus F^{-1}(0)$, we define the multi-valued analytic map $\kappa_c : G_c \setminus F^{-1}(0) \to L(\mathfrak{g}_c, \mathfrak{g}_c)$ by

$$\kappa_c(x) = x \nu_c(x)^{-1} h_c(x)^{-1}. \quad (12)$$

It is the multi-valued analytic extension of $\kappa$ to $G_c \setminus F^{-1}(0)$. $\kappa_c$ is a Zariski closed subset of $L(\mathfrak{g}_c, \mathfrak{g}_c)$. Therefore, if $I$ is the ideal of polynomial functions vanishing on $\kappa_c$, we have $\kappa_c = I^{-1}(0)$.

If $f \in I$, then $(f \circ \kappa)(x) = 0$ for all $x \in G$. Now $G_c$ is connected, hence the complement $G_c \setminus F^{-1}(0)$ of the analytic null set $F^{-1}(0)$ is (this local property is an easy consequence of the Weierstrass preparation theorem, cf. Griffiths-Harris [1, p. 8]). By analytic continuation it follows that $f \circ \kappa_c = 0$ on $G_c \setminus F^{-1}(0)$. Consequently $\kappa_c$ maps $G_c \setminus F^{-1}(0)$ into $K_c$.

Now let $x_0 \in G_c \setminus F^{-1}(0)$, and let $c_1, c_2$ be two continuous curves $[0,1] \to G_c \setminus F^{-1}(0)$ with $c_i(0) = 1$, $c_i(1) = x_0$ ($i = 1, 2$). Write $a_{k, c}$ for the multi-valued analytic function $\sqrt{\mathfrak{d}_{k, c}}$ ($1 \leq k \leq n$), and let $a_{k, i}$, $\kappa_i$, $h_i$ denote the branches of $a_{k, c}$, $\kappa_c$, $h_c$ obtained by continuation of $a_{k, i}$, $\kappa$, $h$ along $c_i$ ($1 \leq k \leq n$, $i = 1, 2$). We have that $a_{k, 1} = d_k a_{k, 2}$ with $d_k = \pm 1$ ($1 \leq k \leq n$), hence, writing $d$ for the diagonal matrix with $k$-th diagonal entry $d_k$ we obtain that $h_1 = d h_2 = h_2 d$. Obviously $d \in \exp(i \mathfrak{a})$. On the other hand, since $\kappa_c h_c$ is single valued, we have $\kappa_1 = \kappa_2 d^{-1} = \kappa_2 d$. In particular it follows that $\det(d) = 1$. Consequently $d \in \mathfrak{M}$.

We now turn our attention to the maps $\kappa$, $h$, $\nu$. Let $M = A d^{-1}(\mathfrak{M})$. Then obviously

$$M = K \cap \exp i \mathfrak{a} = K \cap \mathfrak{a}_c.$$
The following properties of $\mathcal{M}$ will be of use later. First we have $\mathcal{M} = \mathcal{M}^0$ (cf. Warner [1, pp. 28, 29]). Next we have the following characterization of the set $\{ H \in a_c; \exp H \in \mathcal{M} \}$ (cf. for instance Warner [1, p. 213]). If $\alpha \in \Lambda^{++}$, let $H_{\alpha,0}$ be the element of $a$, $< , >$-orthogonal to $\ker \alpha$, with $\alpha(H_{\alpha,0}) = 1$. Then:

the lattice $\{ H \in a_c; \exp H \in \mathcal{M} \}$ is generated over $\mathbb{Z}$ by the vectors $2\pi i H_{\alpha,0}$ ($\alpha \in \Lambda^{++}$).

**Theorem 1.6.** Let $S$ be the null set of $F = F^* \text{Ad}$ in $G_c$. Then:

(i) The maps $\kappa: G \rightarrow K$, $h: G \rightarrow A$, $\nu: G \rightarrow N$ have extensions to multi-valued analytic maps $\kappa_c: G_c \backslash S \rightarrow K_c$, $h_c: G_c \backslash S \rightarrow A_c$ and $\nu_c: G_c \backslash S \rightarrow N_c$ respectively (all with respect to the base point $e$).

(ii) The maps $\kappa_c h_c$, $h_c^2$ and $\nu_c$ are single valued. Moreover, if $\kappa_1$ and $\kappa_2$ are two branches of $\kappa_c$ at a point $x_0 \in G_c \backslash S$, and if $h_1, h_2$ are the corresponding branches of $h_c$ at $x_0$, then $\kappa_1 = \kappa_2^d$, $h_1 = dh_2 = h_2 d$ for some $d \in \mathcal{M}$.

**Proof.** We may pass to $\text{Ad}(G_c) = (\text{Aut} a_c)^0$ and it suffices to prove that $\kappa_c$, $h_c$, $\nu_c$ map $\text{Ad}(G_c)^{F^{-1}}(0)$ into $\text{Ad}(G_c) \cap K_c)^0 = \text{Ad}(K_c)$, $\text{Ad}(G_c) \cap A_c = \text{Ad}(A_c)$ and $\text{Ad}(G_c) \cap N_c$ respectively. Now $\text{Ad}(G_c)^{F^{-1}}(0)$ is connected, and $\kappa$, $h$, $\nu$ map $\text{Ad}(G)$ into the Zariski closed subset $\text{Aut}(a_c)$ of $\text{GL}(a_c)$. It follows that $\kappa_c$, $h_c$, $\nu_c$ map $\text{Ad}(G_c)^{F^{-1}}(0)$ into $\text{Aut}(a_c)$. In view of Theorem 1.5 and the connectedness of $\text{Ad}(G_c)^{F^{-1}}(0)$ we obtain that $\kappa_c$, $h_c$, $\nu_c$ map $\text{Ad}(G_c)^{F^{-1}}(0)$ into $(\text{Ad}(G_c) \cap K_c)^0$, $\text{Ad}(G_c) \cap A_c$, $\text{Ad}(G_c) \cap N_c$ respectively. We complete the proof by the observation that $\text{Ad}(\mathcal{M}) = \text{Ad}(G_c) \cap \mathcal{M}$. 
Remark. As we will show in Chapter 4, the monodromy groups associated with $\kappa_c$ and $h_c$ are isomorphic to the full group $M$.

The map $\exp: a_c \to A_c$ is a covering. Therefore the inverse $\log: A \to a$ of $\exp: a \to A$ has a multi-valued analytic extension $A_c \to a_c$ (with respect to the base point I); we denote it by $\log_c$. Obviously $\exp \circ \log_c = \text{id}(A_c)$. It follows that the map $\exp: \text{ad}(a_c) \to \text{Ad}(A_c)$ is a covering, and since $\text{ad}$ and $\text{Ad}$ are diffeomorphisms such that $\exp \circ \text{ad} = \text{Ad} \circ \exp$, it follows that $\exp: a_c \to A_c$ is a covering. The inverse $\log: A \to a$ of $\exp: a \to A$ therefore has an extension to a multi-valued analytic map $A_c \to a_c$ (with respect to the base point e); it is denoted by $\log_c$. We obviously have $\exp \circ \log_c = \text{id}(A_c)$ and $\text{ad} \circ \log_c = \log_c \circ \text{Ad}$. Writing $H_c$ for the map $\log_c \circ h_c$ and $H_c$ for the map $\log_c \circ h_c$ we obtain the following theorem.

Theorem 1.7. The map $H_c: G_c \setminus F^{-1}(0) \to a_c$ is a multi-valued analytic extension of $H: G \to a$ (with respect to the base point I), $H_c: G_c \setminus S \to a$ is a multi-valued analytic extension of $H: G \to a$ (with base point e).

1.4 Some properties of the set $S$

Lemma 1.8.

\begin{align*}
GL(a_c) \setminus F^{-1}(0) &= k_c A_c N_c, \\
G_c \setminus S &= k_c A_c N_c.
\end{align*}

Proof. If $(k, a, n) \in k_c A_c \times N_c$, then

$$F(kan) = \prod_{k=1}^{n} D_k (n^a n) = \prod_{k=1}^{n} (a_{i1} \ldots a_{ik})^2 \neq 0.$$
This shows that $K_{C}A_{C}N_{C} \subseteq GL(\mathfrak{g}_{C}) \backslash F^{-1}(0)$. On the other hand the existence of the multi-valued analytic extensions $\kappa_{C}$, $h_{C}$, $\nu_{C}$ implies that $GL(\mathfrak{g}_{C}) \backslash F^{-1}(0) \subseteq K_{C}A_{C}N_{C}$. This proves (13). Formula (14) now follows from Proposition 1.2 and from $Ad(S) = Ad(G_{C}) \cap F^{-1}(0)$.

The following two lemmas will be useful in Chapter 3, where we will have to determine the set $S$ explicitly in the real rank 1 case, and in Chapter 4, where it will be necessary to compare $S$ with the analogous subsets of lower dimensional subgroups with compatible Iwasawa decompositions.

**Lemma 1.9.** The map $h_{C}^{2}$ is a rational map $G_{C} \to A_{C}$. It is regular on $G_{C} \backslash S$. If $(x_{n})$ is any sequence in $G_{C} \backslash S$ converging to a point $x \in S$ then $\{h_{C}^{2}(x_{n}), n \in \mathbb{N}\}$ is not relatively compact in $A_{C}$.

**Proof.** The first statements follow readily from Theorem 1.4. As for the last statement we pass to $G_{C}$ by $Ad$. So let $(x_{n})$ be a sequence in $G_{C} \backslash F^{-1}(0)$ converging to $x \in F^{-1}(0)$. Let $k$ be the smallest element of $\{1, \ldots, n\}$ such that $D_{k}(g(x)) = 0$. By (9), (10) it follows that $(h_{C}^{2}(x_{n}))_{k} \to 0$, hence $\{h_{C}^{2}(x_{n}), n \in \mathbb{N}\}$ is not relatively compact in $A_{C}$. It is now easy to complete the proof.

**Lemma 1.10.** Let $L$ be a connected complex analytic submanifold of $G_{C}$, containing $e$. Then the set $L \backslash S$ is a connected dense open subset of $L$. Moreover, it can be characterized as follows. Fix a simply connected open neighbourhood $\mathcal{O}$ of $e$ in $G_{C}$, disjoint from $S$, and let $h_{0}$ denote the holomorphic extension of $h!(\mathcal{O} \cap S)$ to $\mathcal{O}$. Then
$\Lambda S$ is the biggest open subset $U$ of $L$ such that:

(i) $U$ is connected and contains $\emptyset \cap L$.

(ii) The restriction $h_0 |(\emptyset \cap L)$ of $h_0$ to $\emptyset \cap L$ has a multi-valued analytic extension to $U$ (with respect to the base point $e$).

**Proof.** The holomorphic function $F = F_{e}Ad$ is not identically zero on $L$. Therefore $\Lambda S = \Lambda F^{-1}(0)$ is a connected dense open subset of $L$.

If $U_1$ and $U_2$ are open subsets of $L$ such that (i) and (ii) are satisfied, then the same holds for $U_1 \cup U_2$. Therefore there exists a biggest open subset $U$ of $L$ such that (i) and (ii). Since $h_0 |(\Lambda S)$ extends $h_0 |(\emptyset \cap L)$, $\Lambda S$ is contained in $U$. Assume that $\Lambda S \neq U$. Then $S \cap U \neq \emptyset$. Now $S \cap U$ is the null set of $F$ in $U$, and $F$ is not identically $0$ on $U$. So $S \cap U$ is an analytic variety of positive codimension in $U$. Let $x$ be a smooth point of $S \cap U$ (for the existence of such a point cf. Griffiths-Harris [1, pp. 20, 21]). From the local structure of $S \cap U$ at $x$ it follows that there exists a point $x' \in \Lambda S$ and a continuous curve $c': [0,1] \to U$ such that $c'(0) = x', c'(1) = x$ and $c'(t) \notin S$ for $0 \leq t < 1$.

Hence, since $\Lambda S$ is connected and open in $L$, there exists a continuous curve $c: [0,1] \to L$, such that $c(0) = e$, $c(1) = x$ and $c(t) \notin S$ for $0 \leq t < 1$. Now $c([0,1]) \subset U$ and by definition of $U$ $h_0 |(\emptyset \cap L)$ has a continuation $h_1$ along $c$. It follows that $(h_1^2(c(t)); t \in [0,1])$ is a compact subset of $A_c$. On the other hand $h_1^2$ must be the restriction of $h_0^2$ over $c([0,1])$, hence $h_0^2(c([0,1]))$ is relatively compact, contradictory to the assertion of Lemma 1.9. We conclude that $\Lambda S = U$. 

Corollary 1.11. \( G_c \setminus S \) is the biggest open subset \( U \) of \( G_c \) such that

(i) \( U \) is connected and contains \( G \);

(ii) \( h: G \to A \) has a multi-valued analytic extension to \( U \).

1.5 The manifold \( G_c/P_c \)

Consider the subalgebra \( \mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \) of \( \mathfrak{g} \); its normalizer \( P \) in \( G \) is equal to \( \text{MAN} \). The normalizer \( P_c \) of \( \mathfrak{p}_c \) in \( G_c \) is a parabolic subgroup of \( G_c \), hence connected (cf. Humphreys [1, p.143]).

Proposition 1.12. Let \( M_c \) denote the centralizer of \( \mathfrak{a} \) in \( K_c \). Then \( P_c = M_c A_c N_c \).

Proof. \( P_c \) is the normalizer of \( \mathfrak{p}_c \) in \( G_c \), hence \( P_c \supseteq M_c A_c N_c \). \( P_c \) has Lie algebra \( \mathfrak{p}_c = \mathfrak{m}_c + \mathfrak{a}_c + \mathfrak{n}_c \) and this is the Lie algebra of the closed subgroup \( M_c A_c N_c \) as well; hence \( M_c A_c N_c \) is an open subgroup of \( P_c \). Since \( P_c \) is connected this completes the proof.

By the Iwasawa decomposition we have \( K \cap P = M \), and therefore the map \( K/M \to G/P \) induced by the inclusion \( K \to G \) is injective. Again by the Iwasawa decomposition it follows that the map \( M/K \to G/P \) is a diffeomorphism.

Consider the inclusion \( K_c \to G_c \). If \((m,a,n) \in M_c \times A_c \times N_c \), then \( m, a, n \in K_c \), showing that \( \text{Ad}(an) \in (K_c \cap A_c N_c) \subseteq A_c \), whence \( n = e \). It follows that \( K_c \cap P_c = M_c \) and therefore the induced map \( K_c/M_c \to G_c/P_c \) is injective. We have \( K \cap M_c = M \) and \( G \cap P_c = P \), so we may identify \( K/M \) and \( G/P \) with submanifolds of \( K_c/M_c \) and \( G_c/P_c \).
via the maps induced by the inclusions $K \rightarrow K_c$ and $G \rightarrow G_c$ respectively. With these identifications the following diagram commutes.

\[
\begin{array}{ccc}
K/M & \xrightarrow{\pi} & G/P \\
\downarrow & & \downarrow \\
K_c/M_c & \rightarrow & G_c/P_c
\end{array}
\] (15)

From now on we shall identify via these maps.

Let $\pi$ denote the canonical projection $G_c \rightarrow G_c/P_c$, and let $P = \pi(S)$. In view of the right $P_c$-invariance of $S$ (cf. Lemma 1.8.) it follows that $S = \pi^{-1}(P)$.

**Proposition 1.13.** $(G_c/P_c) \setminus P = K_c/M_c$, $P$ is a closed left $K_c$-invariant subset of $G_c/P_c$, and $K_c/M_c$ is a dense open subset of $G_c/P_c$.

**Proof.** Since $S = \pi^{-1}(P)$ the identity follows from the fact that $K_cA_cN_c$ is the complement of $S$ in $G_c$. Moreover, since $K_cA_cN_c$ is a dense open subset of $G_c$, $K_c/M_c = \pi(K_cA_cN_c)$ is a dense open subset of $G_c/P_c$. Hence $P$ is closed; its left $K_c$-invariance follows immediately from the left $K_c$-invariance of $S$.

The complex analytic manifold $G_c/P_c$ is compact (it is even projective, cf. Humphreys [1, p. 135]). We define the map $\chi: \overline{N}_c \rightarrow G_c/P_c$ by

$$\chi(\overline{n}) = \overline{n}P_c.$$ 

As is well known, $\chi$ is a complex analytic diffeomorphism onto a dense open subset of $G_c/P_c$. 


If \( x \in G_C \), let \( \lambda_x \) denote the left multiplication by \( x \) in \( G_C/P_C \) and define \( \chi_x = \lambda_x \chi \). Thus \( \chi_x \) is a complex analytic diffeomorphism of \( \mathbb{C} \) onto a dense open subset of \( G_C/P_C \).

We write \( M^* \) for the normalizer of \( a \) in \( K \). The Weyl group \( W = M^*/M \) embeds naturally as a finite subset in \( K/M \subset G_C/P_C \). Let from now on for each \( w \in W \) a representative \( \overline{w} \in M^* \) be fixed. The acts \( \chi_\overline{w}(\mathbb{N}) \) (\( \overline{w} \in W \)) are dense open subsets of \( G/P \); they are independent of the particular choice of representatives and by the Bruhat decomposition of \( G/P \) they cover \( G/P \). We call the maps \( \chi^{-1}_\overline{w}: \overline{w} \mathbb{N} \to \mathbb{C} \) Bruhat charts of \( G_C/P_C \).

**Lemma 1.14.** There exists a polynomial function \( p: \overline{w} \mathbb{C} \to \mathbb{C} \) such that for any \( k \in K \) we have:

\[
\chi^{-1}_k(p) = S \cap \overline{w} \mathbb{C} = \exp(p^{-1}(0)).
\]

**Proof.** In view of the left \( K \)-invariance of \( P \) we have

\[
\chi^{-1}_k(p) = \chi^{-1}(p) \cap \overline{w} \mathbb{C} = S \cap \overline{w} \mathbb{C}.
\]

The function

\[
p = \int_0^1 \text{Ad}^\circ (\exp_{\mathbb{C}}(\overline{w} t)) - \exp_{\mathbb{C}}(\overline{w})
\]

is polynomial and we have \( S \cap \overline{w} \mathbb{C} = \exp(p^{-1}(0)) \).

At this stage we shall rewrite the integral in formula (1) as an oriented integral of a differential form over \( K/M \). First we recall a lemma.

**Lemma 1.15.** Let \( G \) be a real (complex) analytic Lie group and let \( H \) be a closed real (complex) analytic subgroup. Let \( m \) be the dimension of the real (complex) analytic manifold \( G/H \). Then the following conditions are equivalent.

(i) \( G/H \) has a non-zero \( G \)-invariant real analytic (holomorphic)
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differential $m$-form $\tilde{\omega}$.

(ii) $\det(\text{Ad}_G(h)) = \det(\text{Ad}_H(h))$ for any $h \in H$.

If these conditions are satisfied, then $\tilde{\omega}$ is unique up to a real (complex) non-zero factor.

For a proof of the real analytic part of this lemma, we refer the reader to Helgason [1, p. 386]. The complex analytic part is proved analogously. Recall that for $h \in H$ we have $\text{Ad}_H(h) = \text{Ad}_G(h)|\mathfrak{n}$, where $\mathfrak{n}$ denotes the Lie algebra of $H$.

Lemma 1.16. The manifold $K/M$ has a non-zero $K$-invariant differential $m$-form $\tilde{\omega}$ (here $m = \dim \mathfrak{n} = \dim (K/M)$).

Proof. To begin with, $M = MN^0$ (cf. Warner [1, pp. 28,29]). This shows that for each $m \in M$ we have $\det (\text{Ad}(m)|_m) = 1$ and hence $\det (\text{Ad}_M(m)) = 1$. $K$ is a connected compact group and therefore $\det(\text{Ad}_K(k)) = 1$ for $k \in K$. By Lemma 1.15 this completes the proof.

Corollary 1.17. $K/M$ has a $K$-invariant orientation.

From now on let us fix an orientation of $K/M$, and an orientation of $\mathcal{N}$ such that $\chi|_{\mathcal{N}}$ is orientation preserving. Let $\bar{\omega}$ denote the $K$-invariant differential $m$-form on $K/M$ such that:

$$\int_{K/M} \bar{\omega} = 1$$

(throughout this thesis we assume all integrals over forms to be oriented).

Now consider the map $G \times K \to \mathfrak{a}$, $(x,k) \to H(xk)$. Since $M$ centralizes $A$ and normalizes $\mathcal{N}$ this map is right $M$-invariant in its second variable. We denote the induced map $G \times K/M \to \mathfrak{a}$ by
H(.,.). It is given by \( H(x,kM) = H(xk) \). We may rewrite formula (1) as:

\[
\phi_{\lambda}(x) = \int_{K/M} e^{(i\lambda - \rho)H(x,y)} \overline{\omega}(y) \quad (x \in G).
\]

In the remainder of this section we shall study complex analytic extensions of \( \overline{\omega} \) and \( H(.,.) \).

**Lemma 1.18.** The form \( \overline{\omega} \) has a unique extension to a holomorphic \( m \)-form \( \overline{\omega}_C \) on \( K_C/M_C \). \( \overline{\omega}_C \) is left \( K_C \)-invariant.

**Proof.** \( P_C = M_C A^*_C C \subseteq M_C A_C \times N_C \) is connected, and therefore \( M_C A_C \) is a connected subgroup of \( G_C \).

If \( m \in M \), then \( \det(\text{Ad}(m)|_{m_C}) = \det(\text{Ad}_M(m)) = 1 \) (here the first determinant is the (complex multilinear) determinant of the complex linear map \( \text{Ad}(m)|_{m_C} \) whereas the second is the determinant of the real linear map \( \text{Ad}_M(m): m \to m \); cf. also the proof of Lemma 1.16). Hence if \( l \in MA \), then \( \det(\text{Ad}(l)|_{m_C}) = 1 \). Since \( M_C A_C \) is connected it follows by analytic continuation that \( \det(\text{Ad}(l)|_{m_C}) = 1 \) for \( l \in M_C A_C \); in particular this holds for \( l \in M_C \).

If \( k \in K \), then \( \det(\text{Ad}(k)|_{1_C}) = \det(\text{Ad}_K(k)) = 1 \). By analytic continuation we have \( \det(\text{Ad}(k)|_{1_C}) = 1 \) for \( k \in K_C \), so in particular this holds for \( k \in M_C \).

By Lemma 1.15 there exists a non-zero left \( K_C \)-invariant holomorphic \( m \)-form \( \overline{\omega}_C \) on \( K_C/M_C \). Its pull back to \( K/M \) under the natural embedding is a non-zero \( K \)-invariant complex valued \( m \)-form on \( K/M \) hence equal to \( C \overline{\omega} \) for some \( C \in \mathbb{C} \), \( C \neq 0 \). The form \( \overline{\omega}_C = C^{-1} \overline{\omega}_C \) satisfies all requirements.
By the same argument as in Harish-Chandra [2, p. 287] there exists a unique invariant differential m-form \( \Omega \) on \( \mathbb{N} \) such that

\[
(\chi|_{\mathbb{N}})^* (\bar{\omega})_{\mathbb{N}} = e^{-2\rho H(\bar{n})} \Omega(\bar{n}) \tag{16}
\]

(here \( (\chi|_{\mathbb{N}})^* (\bar{\omega}) \) denotes the pull-back of \( \bar{\omega} \) under \( \chi|_{\mathbb{N}} \)).

Since \( \chi(\mathbb{N}) \) is a dense open subset in \( K/M \), its complement is of measure zero and hence

\[
\int_{\mathbb{N}} e^{-2\rho H(\bar{n})} \Omega(\bar{n}) = \int_{K/M} \bar{\omega} = 1,
\]

the integrals being absolutely convergent. Let \( \Omega_c \) denote the extension of \( \Omega \) to a holomorphic differential m-form on \( \mathbb{N}_c \). We have the following holomorphic version of (16).

**Lemma 1.19.** The function \( \mathbb{N} \to \exp(-2\rho H_c(\bar{n})), \mathbb{N}_c \setminus S \to \mathbb{C} \) is single valued. Moreover if \( k \in K_c \), then

\[
(\chi_k|_{\mathbb{N}_c \setminus S})^* (\bar{\omega}_c) = e^{-2\rho H_c(\cdot)} \Omega_c \tag{17}
\]

**Proof.** In view of the left \( K_c \)-invariance of \( \omega_c \) it suffices to prove (17) for \( k = e \). But then (17) follows from (16) by analytic continuation. By abuse of language we write \( (\chi_k)^* (\bar{\omega}_c) \) for the left hand side of (17). Since \( \bar{\omega}_c \) and \( \Omega_c \) are nowhere vanishing holomorphic m-forms on \( \mathbb{N}_c \setminus S \) we obtain that the map \( \mathbb{N} \to \exp(-2\rho H_c(\bar{n})) \) is single valued.

**Remark 1.** The first assertion of Lemma 1.19 can also be proved by the following direct argument. If \( H \in a_c \) is such that \( \exp H \in K \), then \( \exp ad H \) normalizes \( g, f, a, n \) and we have \( 1 = \det(\phi|_g) = \det(\phi| f) = \det(\phi| a) \). So, by the direct sum decomposition \( g = f + a + n \), we obtain that \( \det(\phi| n) = 1 \). On the other hand \( \det((\exp ad(H))| n) = \exp(2\rho(H)) \). It follows that
\[ e^{2\rho(H)} = 1 \text{ for } H \in \mathfrak{a}_C \text{ with } \exp H \in M. \]

In view of Theorem 1.6 this implies that \( \overline{n} \to \exp(-2\rho H_c(\overline{n})) \) is single valued.

**Remark 2.** If \( \mathfrak{g} \) is a real rank 1 algebra (i.e. \( \dim \mathfrak{a} = 1 \)) and if we use the notations of Chapter 3, the function \( \exp(-2\rho H) \) is given by:

\[ e^{-2\rho(H)(\overline{n})} = \left(1 + c(X,X)\right)^2 + 4c(Y,Y)\left(\frac{1}{2}m(a) - m(2a)\right) \]

(where \( \overline{n} \) is \( \exp(X + Y) \), \( X \in \mathfrak{a}_{-a} \), \( Y \in \mathfrak{a}_{-2a} \)). Hence it follows from Lemma 1.19 that we must have \( m(2a) = 0 \) or else \( m(a) \) even. This is in agreement with Proposition 2.3 of Araki [1].

We now turn our attention to the map \( H(\ldots) : G \times (K/M) \to \mathfrak{a} \). In view of the identification of \( K/M \) with \( G/P \) we may consider it as a map \( G \times (G/P) \to \mathfrak{a} \). As such it is given by

\[ H(x,yP) = H(xk(y)) \quad (x,y \in G). \quad (18) \]

**Theorem 1.20.** Let

\[ P_2 = \{(x,y) \in G \times (G_c/P_c) ; y \in P \text{ or } \lambda_x(y) \in P\}, \]

then \( (G \times (G_c/P_c))\backslash P_2 \) is a connected dense open subset of \( G \times (G_c/P_c) \). The map \( H(\ldots) : G \times (G/P) \to \mathfrak{a} \) has an extension to a multi-valued analytic map \( H_c(\ldots) : (G \times (G_c/P_c))\backslash P_2 \to \mathfrak{a} \) (with respect to the base point \( (e,P_C) \)).

**Proof.** Starting point of our proof is formula (18). We first consider the map \( G \times G \to \mathfrak{a} \), \((x,y) \to H(xk(y))\). Writing \( \phi \) for the map \( G \times G \to G \), \((x,y) \to xk(y)\), this map is equal to \( H \circ \phi \). Let
\[ S_2 = \{(x,y) \in G_c \times G_c; y \in S \text{ or } xy \in S\}. \]

Then \( \phi \) has the multi-valued extension \( \phi_c: (G_c \times G_c) \setminus S_2 \to G_c \)
defined by \( \phi_c(x,y) = x\kappa_c(y) \). If \( (x,y) \notin S_2 \) we have \( xy \notin S \),
and since \( x\kappa_c(y) \equiv xy \mod P_c \) it follows that \( x\kappa_c(y) \notin S \).
Therefore the image of \( \phi_c \) is contained in \( G_c \setminus S \) and \( \mathcal{H}_c \circ \phi_c \) is
a well defined multi-valued analytic map \( (G_c \times G_c) \setminus S_2 \to a_c \); it
is the extension of \( \mathcal{H}_c \cdot \phi_c \). \( S_2 \) is the null set of the analytic
function \( G_c \times G_c \to \mathcal{F}, (x,y) \mapsto F(y).F(xy) \), hence its complement
is a connected dense open subset of \( G_c \times G_c \). Let \( \pi_2 \) denote the
map \( G_c \times G_c \to G_c \times (G_c/P_c), (x,y) \mapsto (x,yP_c) \). The set \( S_2 \) is right
\( P_c \)-invariant in the second coordinate and hence the complement of \( P_2 = \pi_2(S_2) \) is a connected dense open subset of \( G_c \times (G_c/P_c) \).

The map \( \mathcal{H}_c \circ \phi_c: (G_c \times G_c) \setminus S_2 \to a_c \) is right \( P_c \)-invariant in
its second variable and therefore there exists a unique multi-
valued analytic map \( \mathcal{H}_c(\ldots): (G_c \times (G_c/P_c)) \setminus P_2 \to a_c \) such that:

\[ \mathcal{H}_c(\ldots) \circ \pi_2 = \mathcal{H}_c \circ \phi_c. \]

In view of formula (18), \( \mathcal{H}_c(\ldots) \) is the multi-valued analytic
extension of \( \mathcal{H}(\ldots) \).

We write \( h_c(\ldots) \) for the multi-valued analytic map
\( \exp \circ \mathcal{H}_c(\ldots): (G_c \times (G_c/P_c)) \setminus P_2 \to A_c \). It is the multi-valued
analytic extension of \( h(\ldots) = \exp \circ \mathcal{H}(\ldots) \).

Remark. In this chapter we have used the subscript \( c \) to
distinguish between a real analytic function (or differential
form) and its holomorphic or multi-valued analytic extension.
In order to keep our notations simple we shall omit the sub-
script \( c \) in the remainder of this thesis.
Appendix to Chapter 1

Multi-valued analytic maps

Let \( X \) be a connected complex analytic manifold. A covering \( p: E \to X \) together with points \( e \in E, \ a \in X \) such that \( p(e) = a \) is called a covering with base points of \( X \). We write \( p: (E,e) \to (X,a) \) for such a covering. Let a point \( a \in X \) be fixed from now on.

Fix a universal covering \( \pi: (\tilde{X},a) \to (X,a) \) with base points of \( X \). An analytic map \( f \) of \( \tilde{X} \) into a complex analytic manifold \( Z \) is called a multi-valued analytic map of \( \tilde{X} \) into \( Z \). Let us denote the germ of an analytic map \( F \) at a point \( z \) by \( F_z \). Then \( f_0 = f_{\pi_0}(\pi_0)^{-1} \) is the germ of an analytic map at \( a \), and the multi-valued analytic map \( f \) is said to be the multi-valued analytic extension of \( f_0 \). If we work with multi-valued maps defined on a complex analytic manifold this will always be done with respect to a fixed point, called the base point of the function. Thus we may speak of multi-valued analytic extensions.

We prefer to use the terminology of multi-valued analytic maps rather than to introduce universal covering spaces with base points. On the one hand the notations remain simple this way, on the other hand it is not sufficient to work merely with universal covering spaces if one wishes to integrate a branch of a multi-valued analytic function over a smooth cycle; for such purposes the so-called covering space associated with the multi-valued analytic function (the analogon of the Riemann surface) must be introduced.

In the remainder of this appendix we fix some more terminology.

**Branch at a point.** If \( x \in X, \ \xi \in \pi^{-1}(x) \) then the germ \( f_1 = f_\xi(\pi_\xi)^{-1} \) is called a branch of \( f \) at \( x \). Let \( y \in X \) and let \( k: [0,1] \to X \) be a continuous curve with \( k(0) = x, \ k(1) = y \). Then
k has a unique lifting to a curve \( \tilde{k}: [0,1] \to \tilde{X} \) with \( \pi, \tilde{k} = k \), \( \tilde{k}(0) = \xi \). Let \( \eta = \tilde{k}(1) \), then \( f_{\eta}(\pi_{\eta})^{-1} \) is called the branch of \( f \) at \( y \) obtained by continuation of \( f_{\tilde{1}} \) along \( k \).

**Composition.** If \( Z, Z' \) are connected complex analytic manifolds, \( g: Z \to Z' \) an analytic map and \( f: X \to Z \) a multi-valued analytic map, \( g \circ f \) is a well defined multi-valued analytic map \( X \to Z' \). If \( g \) is also a multi-valued analytic map (with base point \( c = f(a) \)) we define \( g \circ f \) as follows. \( f \) is actually an analytic map \( \tilde{X} \to Z \).

Let \( p: (\tilde{Z}, \gamma) \to (Z, c) \) be a universal covering with base points. Since \( \tilde{X} \) is simply connected, there exists a unique analytic map \( \tilde{f}: \tilde{X} \to \tilde{Z} \) such that \( f = p \circ \tilde{f} \) and \( \tilde{f}(a) = \gamma \), called the lifting of \( f \).

The map \( g \circ \tilde{f}: \tilde{X} \to Z' \) is a multi-valued analytic map on \( X \), it is called the composition of \( g \) and \( f \) and denoted by \( g \circ f \). In particular, if \( F: X \to Z \) is an analytic map, then we identify \( F \) with the multi-valued analytic map \( F \circ \pi: \tilde{X} \to Z \) on \( \tilde{X} \). Thus if \( g: Z \to Z' \) is a multi-valued analytic map with base point \( c = F(a) \), the composition \( g \circ F \) is defined in the above sense. It is a multi-valued analytic map \( X \to Z' \) with base point \( a \).

**Restriction.** Let \( f: X \to Z \) be a multi-valued analytic map with base point \( a \). If \( Y \) is a connected complex analytic submanifold of \( X \) containing \( a \), then the inclusion \( i: Y \to X \) is an analytic map with \( i(a) = a \). The multi-valued analytic map \( f \circ i \) (with base point \( a \)) is called the restriction of \( f \) to \( Y \). It is denoted by \( f \mid Y \).

**Single valued maps.** If \( F: X \to Z \) is an analytic map, then the analytic map \( G = F \circ \pi: \tilde{X} \to Z \) satisfies \( G_{\xi}(\pi_{\xi})^{-1} = F_{\xi} \circ G_{\eta}(\pi_{\eta})^{-1} \) for all \( \xi, \eta \in \pi^{-1}(x) \). Conversely if \( G: \tilde{X} \to Z \) is an analytic map such that \( G_{\xi}(\pi_{\xi})^{-1} = G_{\eta}(\pi_{\eta})^{-1} \) for all \( \xi, \eta \in \tilde{X} \) with
\( \pi(\xi) = \pi(\eta) \), then there exists an analytic map \( F: X \to Z \) such that \( G = F \circ \pi \). Such a map \( G \) will be called single valued on \( X \).

**Definition of \( \hat{X}_f \) (analogon of the Riemann surface).** Let \( f: \mathcal{X} \to Z \) be an analytic map. The relation \( ~ \) in \( X \) defined by:

\[
\xi \sim \eta \text{ iff } \pi(\xi) = \pi(\eta) \text{ and } f_{\xi}(\pi(\xi))^{-1} = f_{\eta}(\pi(\eta))^{-1}
\]

is an equivalence relation. Let \( \hat{X}_f \) be the set of equivalence classes, \( p_f: \hat{X} \to \hat{X}_f \) the canonical projection, and let \( \pi_f \) be the map \( \hat{X}_f \to X \) that makes the diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{p_f} & \hat{X}_f \\
\downarrow{\pi} & & \downarrow{\pi_f} \\
X & & 
\end{array}
\]

commutative. \( \hat{X}_f \) has a unique structure of complex analytic manifold that makes \( \pi_f: \hat{X}_f \to X \) an analytic covering. With this structure \( p_f: \hat{X} \to \hat{X}_f \) is an analytic covering as well. The covering \( \pi_f: (\hat{X}_f, p_f(\alpha)) \to (X, a) \) with base points is called the covering associated with the multi-valued analytic map \( f \). The analytic map \( \hat{X}_f \to Z \), \( p_f(\xi) \to f(\xi) \) is denoted by \( \hat{f} \).

**Branch over a continuous map.** With the notations of the preceding alinea let \( T \) be a topological space and let \( \tau: T \to X \) be a continuous map. The multi-valued analytic map \( f: X \to Z \) is said to have a branch over \( \tau \) iff there exists a continuous map \( \tau: T \to X_f \) such that \( \pi_f \circ \tau = \tau \). The germ \( f_\tau \) of \( \hat{f} \) at \( \text{im}(\tau) \) is called a branch of \( f \) over \( \tau \). Now let \( T' \) be a subspace of \( T \) and let \( f_{\tau'} \) be a branch of \( f \) over \( \tau' = \tau|_{T'} \). So \( f_{\tau'} \) is the germ of \( \hat{f} \) at \( \text{im}(\tau') \) where \( \hat{\tau'}: T' \to \hat{X}_f \) is a continuous map with \( \pi_f \circ \hat{\tau}' = \tau' \). If \( \hat{\tau} \) is a lifting
of \tau \text{ such that } \tilde{\tau}! \tilde{T}' = \tilde{\tau}' \text{ then the germ } f_{\tilde{\tau}} \text{ of } \tilde{f} \text{ at im}(\tilde{\tau}) \text{ is said to be the branch of } f \text{ over } \tau \text{ that extends } f_{\tilde{T}_1} \text{ (or restricts to } f_{\tilde{T}_2} \text{ over } \tau'\). Finally if } S \text{ is a subset of } X, \text{ then } f \text{ is said to have a branch over } S \text{ if it has a branch over the inclusion } \sigma: S \to X; \text{ a branch of } f \text{ over } \sigma \text{ is also called a branch of } f \text{ over } S. \text{ If } S' \text{ is a subset of } S, \text{ a branch of } f \text{ over } S \text{ is said to extend a branch over } S' \text{ if the corresponding branch over } \sigma \text{ extends the corresponding branch over } \sigma! S'.\)