Chapter 3

The rank 1 case

3.1 Preliminaries

In this chapter we assume that $\dim a = 1$; we then have $\# \Delta^{++} = 1$. We denote the element of $\Delta^{++}$ by $a$. The Weyl group $W$ consists of two elements, $I$ and $w$. The action of the latter on $a$ is given by $w(H) = -H (H \in a)$.

$G/P$ is a compact manifold of dimension $m = m(a) + m(2a)$. It has a Bruhat decomposition: $G/P$ is the disjoint union of the two Bruhat cells $\bar{N}P = \chi(\bar{N})$ and $\bar{w}P$ (recall that $\bar{w}$ is a fixed representative of $w$ in $M^*$, the normalizer of $a$ in $K$). Moreover, $\bar{N}P \cup \bar{w}P = G/P$ implies $\bar{w} \bar{N}P \cup P = G/P$, and so, $\chi_{\bar{w}}$ (for simplicity we shall write $\chi_{\bar{w}}$ from now on) maps $\bar{N}$ diffeomorphically onto the complement of $eP$ in $G/P$. From these facts it follows that $G/P$ is an $m$-sphere.

Let $R$ be a positive real number (in the next section we shall impose a condition on its magnitude). Define the compact ball $B_1$ in $a_{-a} \times a_{-2a}$ by

$$B_1 = \{(X,Y) \in a_{-a} \times a_{-2a}; (X,X) + (Y,Y) \leq R\}$$

(here $(\ , \ )$ denotes the inner product defined in section 1.2). The map $E: a_{-a} \times a_{-2a} \to \bar{N}$, $(X,Y) \to \exp(X+Y)$ is a diffeomorphism, hence $\chi E(B_1)$ is a compact $m$-dimensional submanifold of $G/P$ with boundary $\chi E(\partial B_1)$. Its complement $\chi E(B_1^\text{int})$ is an open subset of $G/P$ that does not contain $eP$ hence is contained in $\chi_{\bar{w}}(\bar{N})$. Similarly $\chi E(B_1^\text{int})$ is a compact subset of $\chi_{\bar{w}}(\bar{N})$. We obviously have
\[ C \chi E(B_i) = C \chi E(B_i^{\text{int}}) \]

and therefore \( C \chi E(B_i^{\text{int}}) \) is a compact \( m \)-dimensional submanifold of \( \chi_w(\overline{N}) \) with boundary \( \chi E(\partial B_i) \). Let \( B_w \) be the unique compact submanifold with boundary of \( \partial_\alpha \times \partial_{-\alpha} \) such that \( \chi_w E(B_w) = C \chi E(B_i^{\text{int}}) \). The disjoint union \( \chi E(B_i^{\text{int}}) \cup \chi_w E(B_w^{\text{int}}) \) has the set \( \chi E(\partial B_i) \) of measure zero as its complement in \( G/P \). Providing \( \chi E(B_i) \) and \( \chi_w E(B_w) \) with the orientations induced by the orientation of \( G/P \) we thus have:

\[
\phi_\lambda (x) = \int_{\chi E(B_i)} e^{(i\lambda - \rho) H(x,y)} \chi_w E(B_w(y)) + \int_{\chi_w E(B_w)} e^{(i\lambda - \rho) H(x,y)} \chi E(B_i(y)).
\]

(1)

In the next section we shall extend the manifolds \( \chi E(B_i) \) and \( \chi_w E(B_w) \) to cycles \( \gamma_i \) and \( \gamma_w \) in the complex flag manifold \( G_c/P_c \). Writing \( H_0 \) for the element of \( a \) with \( a(H_0) = 1 \), and writing \( a(\tau) (\tau \in \mathcal{O}) \) for \( \exp(\tau H_0) \), the cycles \( \Gamma_i: \partial([0, \pi] \times B_i) \rightarrow G_c/P_c \) and \( \Gamma_w: \partial([0, \pi] \times B_w) \rightarrow G_c/P_c \) will be defined by the following formulas:

\[
\Gamma_i(t,(X,Y)) = a(e^{-it}) \chi E(X,Y),
\]

\[
\Gamma_w(t,(X,Y)) = a(e^{-it}) \chi_w E(X,Y),
\]

(we use both notations \( \lambda \) and \( x \) for the left multiplication by an element \( x \in G_c \) in \( G_c/P_c \)). Observe that the cycles \( \Gamma_i, \Gamma_w \) are not smooth. We will show that the real branches of \( H(a,.) \) over \( \chi E(B_i) \) and \( \chi_w E(B_w) \) extend to branches \( H_i(a,.) \) and \( H_w(a,.) \) over the continuous maps \( \Gamma_i \) and \( \Gamma_w \) respectively. Consider the integrals of \( \exp[(i\lambda - \rho) H_s(a,.)] \) over \( \Gamma_s \) \((s = i,w)\). The contributions of these integrals over \( \Gamma_i \left([0, \pi] \times \partial B_i \right) \) and \( \Gamma_w \left([0, \pi] \times \partial B_w \right) \) cancel each other. On the other hand the contributions of the integrals over \( \Gamma_s \left([\pi] \times B_s \right) \) \((s = i,w)\) are equal to a factor \( \exp[2\pi \lambda(H_0)] \).
times the integrals over $\chi E(B_I)$ and $\chi_w E(B_w)$ in (1) respectively.
This finally leads to the formula:
\[
(e^{2\pi i (H_0)} - 1)\phi_\lambda(a) = \sum_{s=I, w} \int_{\Gamma_s} e^{(i\lambda - \rho)H_s(a,y)}\omega(y).
\] (2)

3.2 Construction of the cycles $\Gamma_I, \Gamma_w$

Let $H_0$ be the element of $a$ such that $\alpha(H_0) = 1$. The following formula for the map $H: \overline{N} \to a$ has been found independently by Helgason [2, p. 59] and Schiffmann [1, p. 24].

**Lemma 3.1.** Let $\overline{n} = \exp(X+Y)$, $X \in \mathfrak{g}_{-a}$, $Y \in \mathfrak{g}_{-2a}$. Then:
\[
H(\overline{n}) = \frac{1}{2} \log[(1+c(X,X))^2 + 4c(Y,Y)]H_0
\] (3)
where $c^{-1} = 4(m(a) + m(2a))$.

We denote the holomorphic diffeomorphism $e^{-\mathfrak{g}_{-a},c} \times \mathfrak{g}_{-2a},c \to \overline{N}_c$, $(X,Y) \to \exp(X+Y)$ by $E$ too. Let $q$ be the polynomial function $e^{-\mathfrak{g}_{-a},c} \times \mathfrak{g}_{-2a},c \to \mathbb{C}$ defined by:
\[
q(X,Y) = (1 + c(X,X))^2 + 4c(Y,Y)
\] (4)
From (3) it follows that $\overline{N}_c \setminus E(q^{-1}(0))$ is the biggest open subset of $\overline{N}_c$ with the property that $H: \overline{N} \to a$ has a multi-valued analytic extension to it. Hence by Lemma 1.8 we see that $\overline{N}_c \cap S = E(q^{-1}(0))$.

**Example.** Let $G = SL(2, \mathbb{R})$, $K = SO(2, \mathbb{R})$. Define
\[
H_0 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
and let \( a = \mathbb{R}H_0 \). Let \( a \) be the root of the pair \((a,a)\) with \( a(H_0) = 1 \). Writing

\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

we have \( n = \mathbb{R}X, \ \overline{n} = \mathbb{R}Y \). We identify \( \overline{N}_C \) with \( \mathbb{C} \) (with the addition) via the group isomorphism

\[
\mathbb{C} \to \overline{N}_C, \ z \to \exp(zY) = \begin{pmatrix} 1 \\ z \\ 0 \end{pmatrix}.
\]

The manifold \( G_C/P_C \) is diffeomorphic to the Riemann sphere, the charts \( \chi^{-1} \) and \( \chi_{\overline{n}}^{-1} \) form an atlas for \( G_C/P_C \), and the transition map \( \chi_{\overline{n}}^{-1} \circ \chi : \mathbb{C}\{0\} \to \mathbb{C}\{0\} \) is given by

\[
\chi_{\overline{n}}^{-1} \circ \chi(z) = -\frac{1}{z} \quad (z \in \mathbb{C}\{0\}).
\]

Moreover, \( H : \overline{N} \to a \) is given by

\[
H(x) = \log(1 + x^2)H_0 \quad (x \in \mathbb{R}),
\]

and so \( S \cap \overline{N}_C = \{-i,+i\} \). Finally, we have

\[
\chi^*(\overline{w}) = \frac{1}{\overline{w}} \cdot \frac{dz}{1 + z^2}.
\]

In the picture on p. 3-5 we have indicated some concepts that will be introduced in the remainder of this chapter, specialized to the case of \( \text{SL}(2,\mathbb{R}) \). We hope it will help the reader to find his (her) way through this chapter.

Let us return to the general case. Fix \((X,Y) \in \mathfrak{a}_{-3} \times \mathfrak{a}_{-2}\), \((X,Y) \neq (0,0)\), and consider the polynomial function \( r : \mathbb{C} \to \mathbb{C} \) defined by:

\[
r(z) = q(zX, z^2Y).
\]

(5)
Figure 3.1.
Writing $a = c(X,X)$ and $b = 4c(Y,Y)$ we have $a \geq 0$, $b \geq 0$ and
\[
r(z) = (1 + az^2)^2 + bz^4.
\]
Let $\sqrt{a + i\sqrt{b}}$ denote the square root of $a + i\sqrt{b}$ that has its argument in the interval $[0, \frac{\pi}{4}]$, and let
\[
\zeta = i/\sqrt{a + i\sqrt{b}},
\]
then we obtain
\[
r(z) = (1 + \zeta^{-1}z)(1 - \zeta^{-1}z)(1 + \overline{\zeta}^{-1}z)(1 - \overline{\zeta}^{-1}z)
\]
(6)
(here $\overline{\zeta}$ denotes the complex conjugate of $\zeta$). It follows that $r$ has roots $\zeta$, $-\zeta$, $\overline{\zeta}$ and $-\overline{\zeta}$. Set $R_0 = (X,X) + (Y,Y)$. Observe that:
\[
|\zeta|^4 = (a^2 + b)^{-1} = (c^2 (X,X)^2 + 4c(Y,Y))^{-1}.
\]
Since $(X,X) \geq \frac{1}{2}R_0$ or $(Y,Y) \geq \frac{1}{2}R_0$ we obtain that:
\[
|\zeta|^4 \leq \max\left(\frac{R_0}{2}, \frac{2}{C^2 a R_0 C}\right).
\]
(7)
Furthermore $\arg(\zeta) = \frac{\pi}{2} - \frac{1}{2} \arg(a + i\sqrt{b})$, showing that:
\[
\frac{\pi}{4} \leq \arg(\zeta) \leq \frac{\pi}{2}.
\]
Moreover, $\arg(\zeta) = \frac{\pi}{4}$ iff $(X,X) = 0$ and $\arg(\zeta) = \frac{\pi}{2}$ iff $(Y,Y) = 0$.

Now fix $R$ (cf. Section 3.1) such that $|\zeta| < 1$ for all $(X,Y) \in \mathcal{O}_a \times \mathcal{O}_a$ with $(X,X) + (Y,Y) = R$. This is possible in view of (7). Recall that $B_{\mathbb{I}}$ denotes the compact ball $(X,X) + (Y,Y) \leq R$ in $\mathcal{O}_a \times \mathcal{O}_a$ (cf. Section 3.1), and that $\overline{N}$ and $G/P$ are oriented such that $\chi: \overline{N} \to G/P$ is orientation preserving (cf. Section 1.5). Give $\mathcal{O}_a \times \mathcal{O}_a$ the orientation that turns $E$ into an orientation preserving map and provide $B_{\mathbb{I}}$ with the induced orientation. Give $[0,\pi] \times B_{\mathbb{I}}$ the product orientation and orient $\partial([0,\pi] \times B_{\mathbb{I}})$ according to the outward normal. We
define the map $\gamma_I: \partial([0,\pi] \times \mathbb{B}_I) \to \bar{\mathbb{N}}_c$ by:

$$\gamma_I(t,(X,Y)) = a(e^{-it}) E(X,Y) a(e^{-it})^{-1}$$

(here $a(\tau) = \exp(\tau H_0)$ for $\tau \in \mathbb{C}$). It easily follows that:

$$\gamma_I(t,(X,Y)) = E(e^{itX}, e^{2itY}). \quad (8)$$

If $(X,Y) \in \partial B_I$ is fixed we have $q(e^{itX}, e^{2itY}) = r(e^{it}) \neq 0$
whence $\gamma_I(t,(X,Y)) \notin S$ for $(X,Y) \in \partial B_I$, $t \in [0,\pi]$. Since $\gamma_I$
maps $\{0\} \times B_I$ and $\{\pi\} \times B_I$ into $\bar{N}$ it follows that $\text{im}(\gamma_I) \subset \bar{\mathbb{N}}_c \setminus S$.

**Lemma 3.2.** The multi-valued analytic map $H: \bar{\mathbb{N}}_c \setminus S \to a_c$ has a
branch $H_I$ over $\gamma_I$ that equals the real branch $H_r$ over $\gamma_I \vert \{0\} \times B_I$.
Moreover

$$H_I(.) = H_r(.) + 2\pi i H_0 \quad \text{over} \quad \gamma_I \vert \{\pi\} \times B_I.$$

**Proof.** As we observed above, $\text{im}(\gamma_I) \subset \bar{\mathbb{N}}_c \setminus S$. Fix $(X,Y) \in \partial B_I$
and consider the curve $c: [0,\pi] \to \bar{\mathbb{N}}_c$, $t \to \gamma_I(t,(X,Y))$. From
(8), (5), (4) and (3) it follows that:

$$H(c(t)) = \frac{1}{2} \log(r(e^{it})) H_0.$$

From (6) it follows that the argument of $r(e^{it})$ increases
with $4\pi$ if $t$ increases from $0$ to $\pi$. This shows that the branch
of $H$ at $(-X,Y) = \gamma_I(\pi,(X,Y))$ obtained by continuation of the
real branch $H_r$ along $c$ is equal to $H_r + 2\pi i H_0$. From this it
follows that $H$ has a branch $H_I$ over $\gamma_I$ that restricts to the
branch $H_r$ over $\gamma_I \vert \{0\} \times B_I$. Over $\gamma_I \vert \{\pi\} \times B_I$ this branch is
equal to $H_r + 2\pi i H_0$, and at a point $\gamma_I(t_0,(X,Y))$ ($t_0 \in [0,\pi]$),
$(X,Y) \in \partial B_I$ it is equal to the branch of $H$ obtained by con-
tinuation of $H_r$ along the curve $t \to \gamma_I(t,(X,Y))$, $[0,t_0] \to \bar{\mathbb{N}}_c \setminus S$. 
Define the map $\Gamma_1: \partial([0,\pi] \times B_{\Gamma_1}) \to G_c/P_c$ by $\Gamma_1 = \chi_0 \gamma_1$. If $x \in G_c$ we write $Ad(x)$ for the conjugation $g \mapsto xg^{-1}, G_c \to G_c$; we write $Ad(x,g)$ for $Ad(x)(g) = xgx^{-1}$. With this notation we have $\lambda_a^* \chi = \chi_0 Ad(a)$ for $a \in A_c$, and hence indeed we have:

$$\Gamma_1(t,(X,Y)) = a(e^{-it}) \cdot \chi E(X,Y).$$

If $a \in A$, $\mu \in a_c^*$ let us write $a^\mu$ for $\exp(\mu(\log a))$.

**Lemma 3.3.** There exists a constant $C_1 > 0$ such that for all $a \in A$ with $a^a > C_1$ we have:

(i) $\text{im}(\Gamma_1) \cap (P \cup a^{-1} \cdot P) = \emptyset$;

(ii) the multi-valued analytic map $y \mapsto H(a,y)$ (cf. Theorem 1.20) has a branch $H_1(a,\cdot)$ over $\Gamma_1$ that restricts to the real branch $H_1'(a,\cdot)$ over $\Gamma_1\Gamma_1(\{0\} \times B_{\Gamma_1})$ and to the branch $H_1'(a,\cdot) - 2\pi i H_0$ over $\Gamma_1\Gamma_1(\{\pi\} \times B_{\Gamma_1})$.

**Proof.** If $a \in A$, $\bar{n} \in \bar{N}$ we have $H(a\chi(n)) = H(a\bar{n}) - H(\bar{n})$, hence:

$$H(a,\chi(n)) = H(a\bar{n}^{-1}) - H(\bar{n}) + \log a.$$ 

Select an open neigbourhood $V$ of $e$ in $\bar{N}_c$ such that the real branch $H_1$ of $H$: $\bar{N}_c \setminus S \to a_c$ extends holomorphically to $V$. In view of the compactness of $\text{im}(\gamma_1)$ there exists a constant $C_1 > 0$ such that $a\bar{n}^{-1} \in V$ for all $a \in A$ with $a^a > C_1$ and for all $\bar{n} \in \text{im}(\gamma_1)$. This shows that for $a \in A$ with $a^a > C_1$ we have $\text{im}(\gamma_1) \cap a^{-1} S a = \emptyset$. By Lemma 3.2 we have $\text{im}(\gamma_1) \cap S = \emptyset$. Hence (i). Moreover if $a \in A$, $a^a > C_1$ then the multi-valued analytic map $H(a,\cdot)$ has a branch $H_1(a,\cdot)$ over $\Gamma_1$, defined by:

$$H_1(a,\Gamma_1(y)) = H_1'(a\gamma_1(y)a^{-1}) - H_1'(\gamma_1(y)) + \log a \quad (9)$$
(here \( H_1(\cdot) \) denotes the branch given by Lemma 3.2). Consequently (ii) follows straightforwardly from the assertions of Lemma 3.2.

We end this section with a proof of (2). First we have to discuss briefly integrals of the type occurring in (2). So let \( B \) be an oriented \( m \)-dimensional \( C^\infty \) manifold with boundary, diffeomorphic to a compact ball in \( \mathbb{R}^m \). Let \( a, b \in \mathbb{R}, \ a < b, \) and set \( I = [a,b] \). Give \( C = I \times B \) the product orientation and orient

\[
\partial C = (\{a\} \times B) \cup (\{b\} \times B) \cup (I \times \partial B)
\]

according to the outward normal vector. Now let \( X \) be a \( C^\infty \) manifold. A \( C^\infty \) map \( \gamma: \partial C \to X \) will be called a cylinder cycle of dimension \( m \) in \( X \). As in Chapter 2 we define the linear form \( \int_X : \Omega^m(X) \to \mathbb{C} \) by

\[
\int_X a = \int_{\partial C} \gamma^*(a)
\]

\[
= \int_{\{a\} \times B} \gamma^*(a) + \int_{\{b\} \times B} \gamma^*(a) + \int_{I \times \partial B} \gamma^*(a).
\]

We define integrals of type (2.1) in a similar way.

Recall the definition of the compact submanifold \( A_{\alpha} \) of \( A_{-\alpha} \times A_{-2\alpha} \) with boundary (cf. Section 3.1), and provide \( B_w \) with the orientation of the ambient space. Give \( [0,\pi] \times B_w \) the product orientation and orient \( \partial([0,\pi] \times B_w) \) according to the outward normal. Let \( \Gamma_w \) be the cylinder cycle \( \partial([0,\pi] \times B_w) \to G_c/P_c \) defined by

\[
\Gamma_w(t, (X,Y)) = a(e^{-it}) \cdot x_w f(X,Y).
\]  

(10)

Lemma 3.4. Let \( C_I \) be as in Lemma 3.3. Then for all \( a \in A \) with \( a^\alpha > C_I \) we have:
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(i) \( \text{im}(\Gamma_w) \cap (P \cup a^{-1}P) = \emptyset; \)

(ii) the multi-valued analytic map \( H(a,.) \) has a branch \( H_r(a,.) \) over \( \Gamma_w \) that restricts to the real branch \( H_r(a,.) \) over \( \Gamma_w \backslash \{0\} \times B_w \) and to the branch \( H_r(a,.) - 2\pi i H_0 \) over \( \Gamma_w \backslash \{\pi\} \times B_w \).

Proof. Let \( B = \chi E(\mathcal{B}_1) \). \( B \) is a compact m-dimensional submanifold of \( G/P \) with boundary \( \partial B = \chi E(\partial \mathcal{B}_1) = \chi_w E(\partial B_w) \) (cf. Section 3.1).

Fix a point \( y \in \partial B \) and write \( y = \chi E(X,Y) = \chi_w E(X_w,Y_w) \) with \( (X,Y) \in \partial \mathcal{B}_1 \), \( (X_w,Y_w) \in \partial B_w \). Then we have:

\[
\Gamma_I(t,(X,Y)) = a(e^{-it}).y = \Gamma_w(t,(X_w,Y_w)).
\]

This shows that \( \Gamma_w(\{0,\pi\} \times \partial B_w) = \Gamma_I(\{0,\pi\} \times \partial \mathcal{B}_1) \subset \text{im}(\Gamma_I) \). Since both \( \Gamma_w(\{0\} \times B_w) \) and \( \Gamma_w(\{\pi\} \times B_w) \) are contained in \( G/P \) whereas \( \lambda(a)^{-1} \) leaves \( G/P \) invariant, (i) follows.

Consider the curve \( c: [0,\pi] \to G/P_c, t \to \Gamma_w(t,(X_w,Y_w)) \). By (11) we have

\[
c(t) = \Gamma_I(t,(X,Y)),
\]

hence, by Lemma 3.3, the branch of \( H(a,.) \) obtained by continuation of the real branch \( H_r(a,.) \) along \( c \) is equal to \( H_r(a,.) - 2\pi i H_0 \).

This proves (ii). Observe that at a point \( \Gamma_w(t,(X,Y)) \) \( (t \in [0,\pi], \quad (X,Y) \in \partial B \) the branch \( H_w(a,.) \) equals the branch \( H_r(a,.) \) given by Lemma 3.3.

**Theorem 3.5.** Let \( C,I,H_r(.),H_w(.) \) be as in the Lemmas 3.3 and 3.4. If \( \lambda \in a^*_C \) and if \( a \in A \) is such that \( a^\alpha > C_I \), then:

\[
(e^{2\pi i \lambda(H_0)} - 1) \phi_\lambda(a) = \sum_{s=I,W} \int_{\Gamma_s} e^{i(s-\rho)H_s(a,y)} \omega(y). \tag{12}
\]
Proof. We prove formula (12) by decomposing the integrals at its right hand side in integrals over $\Gamma_0((0,\pi)\times B_0)$, $\Gamma_0((\pi)\times B_0)$ and $\Gamma_0((0,\pi)\times B_0)$ ($s = 1, \omega$).

First observe that the integrals over $\Gamma_0((0,\pi)\times B_0)$ ($s = 1, \omega$) cancel each other. This is seen as follows. As is easily verified, the maps $\Gamma_0((0,\pi)\times B_0)$ are embeddings into $G_{2}/P_{0}$. Their images are the same $m$-dimensional $C^{\infty}$ submanifold $\Sigma$ of $G_{2}/P_{0}$. As we saw in the proof of Lemma 3.4 the branches $H_{1}(a,\cdot)$ and $H_{1}(\cdot,\cdot)$ coincide over $\Sigma$. However, the orientation of $\Sigma$ induced by $\Gamma_1$ is the opposite of the one induced by $\Gamma_0$, and so indeed the integrals cancel.

Next consider $\partial((0,\pi)\times B_0) = ((0)\times B_0) \cup ((\pi)\times B_0) \cup (0,\pi)\times B_0)$ with the orientation corresponding to the outward normal. Since $\partial((0)\times B_0) \to B_0$, $(0,y) \to y$ is orientation reversing whereas $\chi_{0}\mathcal{E}$ is orientation preserving, it follows that $\Gamma_0((0)\times B_0)$ is an orientation reversing diffeomorphism of $(0)\times B_0$ onto $B = \chi_{0}\mathcal{E}(B_0)$. Consequently:

$$
\int_{\Gamma_0((0)\times B_0)} \frac{(i\lambda - \rho)H_{1}(a,y)}{\omega(y)} = -\int_{\chi_{0}\mathcal{E}(B_0)} \frac{(i\lambda - \rho)H_{1}(a,y)}{\omega(y)}. 
$$

On the other hand if $(X,Y) \in B_0$ then $\Gamma_0((\pi,0)\times B_0) = \chi_{0}\mathcal{E}(0,\pi,Y)$. The map $(\pi)\times B_0 \to B_0$, $(\pi,y) \to y$ is orientation preserving whereas $B_0 \to B_0$, $(X,Y) \to (-X,Y)$ changes the orientation by $(-1)^{m(a)}$. Hence $\Gamma_0((\pi)\times B_0)$ is a diffeomorphism onto $\chi_{0}\mathcal{E}(B_0)$, changing the orientation by $(-1)^{m(a)}$. Since $H_{1}(a,\cdot)$ equals $H_{0}(a,\cdot) - 2\pi i H_{0}$ over $\Gamma_0((\pi)\times B_0)$ whereas $(i\lambda - \rho)(-2\pi i H_{0}) = 2\pi \lambda(H_{0}) + 
+ \pi i(m(a) + 2m(2a))$ we obtain:

$$
\int_{\Gamma_0((\pi)\times B_0)} \frac{(i\lambda - \rho)H_{1}(a,y)}{\omega(y)} = \frac{2\pi \lambda(H_{0})}{\chi_{0}\mathcal{E}(B_0)} \int_{\chi_{0}\mathcal{E}(B_0)} \frac{(i\lambda - \rho)H_{0}(a,y)}{\omega(y)}. 
$$
Similarly (13) and (14) hold with $I$ replaced by $w$. Combining all these results with (1) we obtain (12).

3.3 Asymptotic behaviour of the integrals over $\Gamma_I, \Gamma_w$

In this section we shall study the asymptotic behaviour of the integrals in (11) when $a^\alpha \to +\infty$. We will obtain two theorems (3.6 and 3.10) that will enable us to compare formula (11) with Harish-Chandra's asymptotic expansion for $\Phi(a)$.

Let us first consider the integral

$$\int e^{i\lambda \cdot \rho} H_I(a, y) \overline{\omega(y)}$$

(15)

(we assume that $a \in A$, $a^\alpha > C_I$, $C_I$ as in Lemma 3.3). Consider formula (9) for $H_I(a, \cdot)$. Since

$$\chi(\overline{\omega}) = e^{-2\rho H(\cdot)}$$

(cf. (21), Section 1.5), it follows that the integral (15) is equal to

$$\int e^{i\lambda \cdot \rho} (i\lambda \cdot \rho) H_I(a \overline{a}^{-1}, \overline{n})$$

$$\overline{\omega(n)}$$

(16)

If $z \in \mathcal{C}$, $\overline{n} = \exp(X + Y)$, $X \in \mathfrak{g}_{-\alpha, c}$, $Y \in \mathfrak{g}_{-2\alpha, c}$, we write $z \overline{n} = \exp(zX + z^2Y)$. With this notation we have

$$a \overline{a}^{-1} = a^{-\alpha} \overline{n}$$

(a $\in A$, $\overline{n} \in \overline{N}_c$).

Recall that the real branch $H_\rho(\cdot)$ of $H$ extends holomorphically to the neighbourhood $V$ of $e$ in $\overline{N}_c$ (cf. the proof of Lemma 3.3). Select an open neighbourhood $U_I$ of $\mathcal{C}$ in $\mathcal{C}$ such that $z \overline{n} \in V$ for $z \in U_I$, $\overline{n} \in \text{im}(\gamma_I)$, and such that $a^\alpha > C_I$ for $a \in A$ with $a^{-\alpha} \in U_I$. 
Then:

$$\psi_I(\lambda, z) = \int e^{(i\lambda - \rho)H_I(z, \bar{n})} e^{-i\lambda \rho H_I(\bar{n})} \gamma_{I_I}$$

is well defined for $z \in U_I$, and the map $\psi_I: a_c^* \times U_I \to \mathbb{C}$ is holomorphic in both variables. We have proved the following theorem.

**Theorem 3.6.** There exists an open neighbourhood $U_I$ of $0$ in $\mathbb{C}$ such that the map $\psi_I: a_c^* \times U_I \to \mathbb{C}$ defined by (17) is holomorphic, and if $a \in A$ is such that $a^{-\alpha} \in U_I$, then $a^{\alpha} > C_I$ and:

$$\int e^{(i\lambda - \rho)H_I(a, y)} e^{-i\lambda \rho \bar{\omega}(y)} = a \psi_I(\lambda, a^{-\alpha}).$$

Next we turn our attention to the integral

$$\int e^{(i\lambda - \rho)H_I(a, y)} \gamma_{I_I}.$$

(19)

If $a \in A$, $\bar{n} \in \bar{N}$ then $H(a \bar{\omega} k(\bar{n})) = H(a^{-1} k(\bar{n})) = H(a^{-1} \bar{n}) - H(\bar{n})$,

whence

$$H(a, \chi_I(\bar{n})) = H(a^{-1} \bar{n}) - H(\bar{n}) - \log a.$$  

(20)

Define the cylinder cycle $\gamma_I: \partial([0, \pi] \times B_w) \to \bar{N}_c$ by

$$\chi_I(t, (X, Y)) = \exp(e^{-it}X + e^{-2it}Y).$$

(21)

We obviously have $\Gamma_I = \chi_I^* \gamma_I$. It follows that the multi-valued analytic map $\phi(a): \bar{N}_c \setminus (S \cup aSa^{-1}) \to a_c$ defined by

$$\phi(a, \bar{n}) = H(a^{-1} \bar{n}a) - H(\bar{n})$$

has a branch $\phi_I(a)$ over $\gamma_I$ such that the integral (19) is equal to:
If $a \in A$ then the map $Ad(a) : \overline{N}_c \rightarrow \overline{N}_c$, $\overline{n} \mapsto Ad(a, \overline{n}) = a\overline{n}a^{-1}$ is a holomorphic diffeomorphism, and we have

$$Ad(a)^* \Omega = a^{-2D} \Omega.$$  

Taking formula (21), Section 1.5 into account it follows that the

the multi-valued analytic map $\psi(a) = Ad(a)^* \phi(a)$ has a branch $\psi_w(a)$ over $Ad(a^{-1})w$ such that the integral (22) is equal to

$$a^{-i\lambda - \rho} \int \exp \left( i\lambda - \rho \right) \psi_w(a, \overline{n}) - 2\rho H(a\overline{n}a^{-1})$$

Taking formula (21), Section 1.5 into account it follows that the

the multi-valued analytic map $\psi(a) = Ad(a)^* \phi(a)$ has a branch $\psi_w(a)$ over $Ad(a^{-1})w$ such that the integral (22) is equal to

$$a^{-i\lambda - \rho} \int \exp \left( i\lambda - \rho \right) \psi_w(a, \overline{n}) - 2\rho H(a\overline{n}a^{-1})$$

Since $\psi(a, \overline{n}) = H(\overline{n}) - H(a\overline{n}a^{-1})$ we can treat the integral (23) as in the proof of Theorem 3.6, if we manage to replace the cycle $Ad(a^{-1})w$ by a fixed cycle without changing the value of the integral. We will do this by means of a homotopy $Ad(a(t)^{-1})w$. The singular set of the integrand of (23) is equal to 

$(S \cup Ad(a^{-1})S) \cap \overline{N}_c$, so the image of our homotopy has to be disjoint from this set.

**Proposition 3.7.** If $b \in A$ then the following statements are equivalent.

(i) $\text{im}(\gamma_w) \cap Ad(b)S = \emptyset$;

(ii) $\text{im}(\gamma_I) \cap Ad(b^{-1})S = \emptyset$.

**Proof.** Since $\lambda(b^{-1}) \circ \chi_w = \chi_w \circ Ad(b)$ it follows that (i) is equivalent to $\text{im}(\gamma_w) \cap \lambda(b^{-1})P = \emptyset$. Now $\lambda(b^{-1})$ leaves $G/P$ invariant, and since $(G/P) \cap P = \emptyset$ whereas $\text{im}(\gamma_w) \setminus (G/P) = \text{im}(\gamma_I) \setminus (G/P)$ it follows that (i) is equivalent to $\text{im}(\gamma_I) \cap \lambda(b^{-1})P = \emptyset$. The assertion now follows from the fact that $\lambda(b^{-1}) \circ \chi_I = \chi_I \circ Ad(b^{-1})$. 
Proposition 3.8. If $b \in A$ is such that $b^\alpha \leq 1$ or $b^\alpha > C_I$ then $\text{im}(\gamma_w) \cap \text{Ad}(b)S = \emptyset$.

Proof. By the preceding proposition it suffices to show that in both cases $\text{im}(\gamma_I) \cap \text{Ad}(b^{-1})S = \emptyset$.

First let $b \in A$, $b^\alpha \leq 1$. Fix $(X,Y) \in 2B_I$ and let $t \in [0,\pi]$. Then

$$\text{Ad}(b)\gamma_I(t,(X,Y)) = \exp(e^{it}b^{-\alpha}X + e^{2it}b^{-2\alpha}Y)$$

and since $(b^{-\alpha}X,b^{-\alpha}X) + (b^{-2\alpha}Y,b^{-2\alpha}Y) \geq (X,X) + (Y,Y) = R$ it follows that $\text{Ad}(b)\gamma_I(t,(X,Y)) \notin S$ (cf. Section 3.2). We now easily obtain that $\text{im}(\gamma_I) \cap \text{Ad}(b^{-1})S = \emptyset$.

Next let $b \in A$, $b^\alpha > C_I$. By Lemma 3.2 we have that $\text{im}(\gamma_I) \cap \text{Ad}(b^{-1})S = \emptyset$.

Corollary 3.9. Let $a,b \in A$ be such that $C_I < b^\alpha < a^\alpha$. If $x \in A$ is such that $b^\alpha \leq x^\alpha \leq a^\alpha$, then:

$$\text{im}(\text{Ad}(x^{-1})\gamma_w) \cap (S \cup \text{Ad}(a^{-1})S) = \emptyset.$$ 

Proof. Observe that $x^\alpha > C_I$, that $(xa^{-1})^\alpha \leq 1$ and apply Proposition 3.8.

Let us return to the integral (23). Select a $b \in A$ such that $b > C_I$. From now on we assume that $a \in A$, $a^\alpha > b^\alpha$. Consider the homotopy $\text{Ad}(a(t)^{-1})\gamma_w$ ($a(\log b) \leq t \leq a(\log a)$). In view of Corollary 3.9 its image is disjoint from $S \cup \text{Ad}(a^{-1})S$ and so the branch $\psi_w(a)$ extends to a branch over this homotopy; we denote it by $\psi_w(a)$ too. The integral
does not depend on $t$ (this might be proved by approximating
$\gamma_W$ by a sequence of homotopic smooth cycles; cf. also de Rham
[1, §14]). It follows that the integral (19) is equal to (23)
with $Ad(a^{-1}) \circ \gamma_W$ replaced by $Ad(b^{-1}) \circ \gamma_W$.

Now select a constant $C_W > b^\alpha$ such that $a^{-1} \in V$ if $a \in A$,
$a^\alpha > C_W$ and $\bar{n} \in \text{im}(Ad(b^{-1}) \circ \gamma_W)$. If $a \in A$, $a^\alpha > C_W$ then $\psi(a)$ has
the branch $\psi_W(a)$ over $Ad(b^{-1}) \circ \gamma_W$ and so $H$ has the branch $H_W$
defined by

$$H_W = \psi_W(a) + H_0 \circ Ad(a)$$

over $Ad(b^{-1}) \circ \gamma_W$. Select an open neighbourhood $U_W$ of $0$ in $C$
such that $z, \bar{n} \in V$ if $z \in U_W$, $\bar{n} \in \text{im}(Ad(b^{-1}) \circ \gamma_W)$ and such that
$a^{-\alpha} \in U_W$ if $a^\alpha > C_W$. Then for $\lambda \in a_0^\ast$, $z \in U_W$ the integral

$$\psi_W(\lambda, z) = \int_{Ad(b^{-1}) \circ \gamma_W} e^{(i\lambda - \rho)H_W(\bar{n}) - (i\lambda + \rho)H_0(z, \bar{n})} \Omega(\bar{n})$$

(24)
is well defined, and the function $\psi_W: a_0^\ast \times U_W \to C$ is holomorphic.
Moreover, if $a^{-\alpha} \in U_W$ then the integral (19) is equal to

$$a^{-i\lambda - \rho} \psi_W(\lambda, a^{-\alpha})$$

We have proved the following theorem.

**Theorem 3.10.** There exists an open neighbourhood $U_W$ of $0$ in $C$
and a holomorphic function $\psi_W: a_0^\ast \times U_W \to C$ (cf. formula (24)),
such that for $a \in A$ with $a^{-\alpha} \in U_W$ we have $a^\alpha > C_W$ and:

$$f \frac{(i\lambda - \rho)H_W(a, y)}{\bar{\omega}(y)} \omega(y) = a^{-i\lambda - \rho} \psi_W(\lambda, a^{-\alpha}).$$

(25)
3.4 Harish-Chandra's formula

We define the holomorphic function \( d : \mathfrak{a}_c^* \to \mathfrak{c} \) by

\[
d(\lambda) = e^{2\pi i \lambda(H_0)} - 1.
\]

With this notation we have the following easy consequence of Theorems 3.5, 3.6, 3.10.

**Corollary 3.11.** Let \( \lambda \in \mathfrak{a}_c^* \), \( a \in A \), \( a^{-\alpha} \in U_I \cap U_w \). Then

\[
d(\lambda) \phi_\lambda(a) = \sum_{s=1, w} a^{i s \lambda - \rho} \psi_s(\lambda, a^{-\alpha}).
\]

Since \( \psi_s \) \((s=I, w)\) is holomorphic on \( \mathfrak{a}_c^* \times U_s \) it has a power series expansion

\[
\psi_s(\lambda, z) = \sum_{n=0}^{\infty} b_{s, n}(\lambda) z^n,
\]

valid for \( z \) is some neighbourhood of zero. The functions

\[
b_{s, n}(\lambda) = \left\{ (d/dz)^n \psi_s(\lambda, z) \right\}_{z=0}
\]

depend holomorphically on \( \lambda \). It follows that

\[
\phi_\lambda(a) = \sum_{s=1, w} a^{i s \lambda - \rho} \sum_{n=0}^{\infty} d(\lambda)^{-1} b_{s, n}(\lambda) a^{-n \alpha}
\]

for \( \lambda \in \mathfrak{a}_c^* \), \( \lambda(H_0) \in \mathbb{Z} e_i \). Obviously this series is an asymptotic expansion for \( \phi_\lambda(a) \) \((a^\alpha \to \infty)\), and it necessarily corresponds to Harish-Chandra's asymptotic expansion for \( \phi_\lambda \) \(\text{cf. [2]}\), see also Section 4.4 for a more detailed discussion of this expansion.

Following Harish-Chandra, we define the c-function to be the coefficient of the principal power \( a^{i \lambda - \rho} \). Thus

\[
c(\lambda) = d(\lambda)^{-1} \psi_I(\lambda, 0),
\]

and we obtain the following theorem.
Theorem 3.12. For all $\lambda \in a^*_C$ with $\lambda(H_0) \notin \mathbb{Z}$ we have

$$c(\lambda) = d(\lambda)^{-1} \int_{Y_1} e^{-i(\lambda + \rho)H_I(n)} \Omega(n)$$

(26)

It is now possible to derive the usual integral formula for the $c$-function (cf. Harish-Chandra [2, Theorem 4, p. 291]). This is the subject of the following lemma.

Lemma 3.13. Let $\lambda \in a^*_C$, $\text{Im}(\lambda(H_0)) < 0$. Then:

$$\int_{Y_1} e^{-i(\lambda + \rho)H_I(n)} \Omega(n) = d(\lambda) \int_{\overline{N}} e^{-i(\lambda + \rho)H(n)} \, d\overline{n}$$

(27)

where $d\overline{n}$ denotes the Haar measure of $\overline{N}$ normalized by $\int_{\overline{N}} \exp(-2\rho H(n)) \, d\overline{n} = 1$. The integral on the right hand side of (27) is absolutely convergent and by (26) it equals $c(\lambda)$.

Proof. The argument follows the familiar pattern of estimation of contour integrals in the elementary theory of functions of one complex variable.

Define the $C^\infty$ map $\phi : [0, \infty) \times \overline{N}_C \rightarrow \overline{N}_C$ by

$$\phi(t, \overline{n}) = Ad(a(-t), \overline{n}).$$

Consider the homotopy $\phi_t \gamma_I$ (here $\phi_t(\cdot) = \phi(t, \cdot)$). First observe that the image of $\phi_t \gamma_I$ is disjoint from $S$ for $t \geq 0$ (this follows from the proof of Proposition 3.8). It follows that the branch $H_I$ extends to a branch over $\phi_t \gamma_I$; we denote it by $H_I$ as well. As in the proof of Theorem 3.10 the value of the integral

$$\int_{\phi_t \gamma_I} e^{-i(\lambda + \rho)H_I(n)} \Omega(n)$$

(28)

is independent of $t$. 

Now consider the integral

\[ I(t) = \int e^{-\langle i\lambda + \rho \rangle H_t(\vec{n})} \Phi_t \circ \gamma_I([0,\pi] \times \mathbb{R}_+) \Omega(\vec{n}) \].

Since \( Ad(a(-t))^* \Omega = \exp(2\rho(H_0) t) \Omega \), the \( m \)-dimensional Euclidean measure of \( \Phi_t \circ \gamma_I([0,\pi] \times \mathbb{R}_+) \) is \( O(\exp 2\rho(H_0) t) \) for \( t \to +\infty \). On the other hand, writing \( \vec{n} = \vec{n}(t,\tau,X,Y) (\tau \in [0,\pi], (X,Y) \in \mathbb{R}_+) \), we have

\[ e^{-\langle i\lambda + \rho \rangle H_I(\vec{n})} \langle \text{Re}\lambda \rangle \langle \text{Im}\gamma_I(\vec{n}) \rangle \langle \text{Im}\lambda - \rho \rangle \langle \text{Re}\gamma_I(\vec{n}) \rangle \leq e^{2\pi|\text{Re}\lambda(H_0)| (\text{Im}\lambda - \rho)(\text{Re}\gamma_I(\vec{n}))} \]

Moreover, writing \( a = c(X,X), b = 4c(Y,Y) \), we have

\[ \alpha(\text{Re}\gamma_I(\vec{n})) = \frac{1}{2} \log |1 + ae^{2t-2it}|^2 + be^{4t-4it} | \]

\[ = 2t + O(1) \quad \text{as} \quad t \to \infty, \]

uniformly in \((\tau,(X,Y)) \in [0,\pi] \times \mathbb{R}_+ \). Hence

\[ |I(t)| = O(e^{2\rho(H_0)t} 2(\text{Im}\lambda - \rho)(H_0)t) \]

showing that:

\[ \lim_{t \to +\infty} I(t) = 0. \quad \text{(29)} \]

By the above estimates it also follows that the integral on the right hand side of (27) converges absolutely if \( \text{Im} \lambda(H_0) < 0 \), and therefore, by dominated convergence,

\[ \lim_{t \to +\infty} \int e^{-\langle i\lambda + \rho \rangle H_t(\vec{n})} \Phi_t \circ \gamma_I(\vec{n}) = \int e^{-\langle i\lambda + \rho \rangle \Omega(\vec{n})} \Phi_t \circ \gamma_I(\vec{n}) \]

Moreover by an argument similar to the one in the proof of Theorem 3.5 it follows that:
\[
\begin{align*}
\int_{\gamma_\lambda} e^{-i\lambda \rho H_I(n)} \Omega(n) = d(\lambda), \\
\int_{\Phi_I(B_\lambda)} e^{-i\lambda + \rho \hat{H}(n)} \phi_I(n) \quad (31)
\end{align*}
\]

where \( \mathcal{B} = \{0 \times B_\lambda\} \cup \{\pi \times B_\lambda\} \). Hence using that the integral (28) is independent of \( t \) and decomposing it in a sum of \( I(t) \) and the integral on the left hand side of (31), we obtain (27) by application of (29) and (31).

By formula (26) it follows that the \( c \)-function is meromorphic; its poles are contained in the set \( \{ \lambda \in a_c^*; \lambda(H_0) \in \mathbb{Z}i \} \) and are all at most of first order. This agrees with the formula (cf. Harish-Chandra [2], p.303)

\[
c(\lambda) = \frac{c_0 \cdot \Gamma(i\lambda_0)}{\Gamma(\frac{1}{4}(\frac{1}{2}m(\alpha) + 1 + i\lambda_0)) \Gamma(\frac{1}{4}(\frac{1}{2}m(\alpha) + m(2\alpha) + i\lambda_0))}
\]

where we have written \( \lambda_0 = \lambda(H_0) \), where \( \Gamma \) denotes the classical Gamma function having no zeros and having poles at 0, -1, -2, \ldots, and where \( c_0 \) is some non-zero constant.

**Remark.** By an explicit computation formula (32) can be deduced from (3) and the integral on the right hand side of (27). This has been done by Helgason (cf. [2]) and Schiffmann (cf. [1]).