Solutions Exam Riemannian Geometry

January 6, 2009, 14:00 - 17:00 uur

Exercise 1. Consider the polar coordinate map $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\sigma(r, \varphi) = (r\cos\varphi, r\sin\varphi)$. You may use the obvious fact that the map σ defines a diffeomorphism from $S := (0, \infty) \times (\pi, 2\pi)$ onto the open set $U := \{(x, y) \in \mathbb{R}^2 \mid y < 0\}$ of $\mathbb{R}^2$.

We define the map $f : U \to \mathbb{R}^3$ by

$$f(\sigma(r, \varphi)) = \frac{r}{2}\,(\cos 2(\varphi - \pi)\,,\, \sin 2(\varphi - \pi)\,,\, \sqrt{3}).$$

(a) Show that f is an isometric embedding of U into $\mathbb{R}^3$, with image equal to $C \setminus L$, where C denotes the cone

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0,\ 3(x^2 + y^2) = z^2\}$$

and L the half line $L = \{(x, 0, x\sqrt{3}) \mid x > 0\}$. Conclusion: f describes folding U into the shape of a cone.

We equip the cone C with the restriction of the standard Euclidean metric of $\mathbb{R}^3$.

(b) Determine the Gauss curvature of C.

(c) Show that the curve

$$\gamma(t) = \frac{t}{2}(0, 1, \sqrt{3}), \qquad (t > 0),$$

defines a geodesic on C.

(d) Show that the circle consisting of the points $(x, y, z) \in C$ with $z = \sqrt{3}$ is not the image of a geodesic on C.

Solution

(a) Put $F = f \circ \sigma$. Then

$$\frac{\partial}{\partial r}F = \frac{1}{2}\,(\cos 2(\varphi - \pi)\,,\, \sin 2(\varphi - \pi)\,,\, \sqrt{3})$$

and

$$r^{-1}\frac{\partial}{\partial\varphi}F = (-\sin 2(\varphi - \pi), \cos 2(\varphi - \pi), 0).$$

These vectors are perpendicular and of unit length. On the other hand, by the chain rule, it follows that

$$\frac{\partial}{\partial r}F = df \cdot \frac{\partial\sigma}{\partial r}$$

and

$$r^{-1}\frac{\partial}{\partial\varphi}F = df \cdot r^{-1}\frac{\partial\sigma}{\partial\varphi},$$

where
$$\frac{\partial\sigma}{\partial r} = (\cos\varphi, \sin\varphi), \quad r^{-1}\frac{\partial\sigma}{\partial\varphi} = (-\sin\varphi, \cos\varphi)$$
are perpendicular and of unit length. It follows that df maps an orthonormal frame to an orthonormal set of vectors in $\mathbb{R}^3$. Hence f is an isometry. It is easily seen that f is injective and that $f(U) = C \setminus L$. Hence f is an isometric embedding of U onto $C \setminus L$.

(b) It follows that $C \setminus L$ is isometric to the flat manifold U, hence has Gauss curvature zero. By rotational symmetry it follows that C has zero curvature everywhere.

(c) The curve is the image under f of the straight line $c(t) = \sigma(t, 2\pi/3)$, which is a geodesic in U. It follows that γ is a geodesic in C.

(d) Let us parametrize the circle by $\rho(t) = (\cos t, \sin t, \sqrt{3})$. Then $\rho'(t) = (-\sin t, \cos t, 0)$ is of unit length, hence $\rho(t)$ is parametrized by arc length. Moreover, $\rho''(t) = -(\cos t, \sin t, 0)$ is not perpendicular to C. Hence, ρ is not a geodesic and the circle cannot be the image of a geodesic.

Exercise 2. In this exercise we assume that $p : E \to M$ is a smooth vector bundle of rank r over the base manifold M. We assume that E is equipped with an affine connection ∇. Let $f_1, \ldots, f_r$ be a local frame of E on an open subset $U \subset M$. Define the maps $A_i^j : \mathfrak{X}(U) \to C^\infty(U)$ by
$$\nabla_X f_i = \sum_{j=1}^r A_i^j(X) f_j.$$

(a) Show that each A_i^j defines a smooth one-form on U.

We also assume E to be equipped with a positive definite inner product structure g, i.e., each fiber E_x is equipped with a positive definite inner product g_x, depending smoothly on $x \in M$. Moreover, we assume the given frame to be orthonormal with respect to the metric g, i.e., $g(f_i, f_j) = \delta_{ij}$, for all $1 \leq i, j \leq r$.

(b) Give a definition of flatness of the metric g by requiring a suitable formula for $Xg(s,t)$, for all smooth sections $s, t \in \Gamma(M, E)$ and all smooth vector fields $X \in \mathfrak{X}(M)$.

(c) Assume that g is flat. Show that $A_i^j = -A_j^i$ for all $1 \leq i, j \leq r$.

(d) Conversely, assume that $A_i^j = -A_j^i$ for all $1 \leq i, j \leq r$. Show that g is flat on U.

Solution

(a) Let $\varphi \in C^\infty(U)$. Then $\nabla_{\varphi X} f_i = \varphi \nabla_X f_i$ implies that $A_i^j(\varphi X) = \varphi A_i^j(X)$. This implies that the map A_i^j is $C^\infty(U)$-linear, hence a one-form.

(b) Generalizing the definition of flatness for the metric of a Riemannian manifold, we say that g is flat if and only if
$$Xg(s,t) = g(\nabla_X s, t) + g(s, \nabla_X t),$$
for all $s, t \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

(c) From the above with $s = f_i$ and $t = f_j$ it follows that on U,

$$0 = g(\nabla_X f_i, f_j) + g(f_i, \nabla_X f_j) = A_i^j(X) + A_j^i(X).$$

(d) Conversely, assume the anti-symmetry. Then it follows as above that

$$g(\nabla_X f_i, f_j) + g(f_i, \nabla_X f_j) = A_i^j(X) + A_j^i(X) = 0 = Xg(f_i, f_j).$$

This establishes the formula for flatness with $s = f_i$ and $t = g_j$. A general section of E over U may be expressed as $s = s^i f_i$ (Einstein convention) so that

$$\begin{aligned} Xg(s, f_j) &= X[s^i g(f_i, f_j)] \\ &= X(s^i)g(f_i, f_j) + s^i Xg(f_i, f_j) \\ &= X(s^i)g(f_i, f_j) + s^i g(\nabla_X f_i, f_j) + s^i g(f_i, \nabla_X f_j) \\ &= g(\nabla_X (s^i f_i, f_j) + g(s^i f_i, \nabla_X f_j) \\ &= g(\nabla_X s, f_j) + g(s, \nabla_X f_j). \end{aligned}$$

In the same fashion one obtains the desired formula for $Xg(s, t)$, with s, t general sections of E over U.

Exercise 3. Let (M, J) be an almost complex manifold. Thus, $J^2 = -\mathbb{I}$ on the tangent space T_pM at any point $p \in M$.

(a) Show that the eigenvalues of J on T_pM are $\pm i$.

(b) Show that these eigenvalues $+i$ and $-i$ have equal multiplicities.

Recall that a complex vector field Z on M is said to be holomorphic if $JZ = iZ$.

(c) Let θ be a one-form on M. Show that $\theta(Z) = 0$ on any holomorphic vector field Z if and only if θ is of type $(0, 1)$, i.e. $P^+\theta = 0$ and $P^-\theta = \theta$ with $P^\pm = \frac{1}{2}(\mathbb{I} \mp iJ)$.

Solution

(a) The eigenvalue equation $JX = \lambda X$, for X in the complexified tangent space and $\lambda \in \mathbb{C}$, implies $J^2X = \lambda^2 X = -X$. Hence $\lambda^2 = -1$ and so $\lambda = \pm i$.

(b) This follows from the fact that the eigenvectors of J come in pairs: if Z is a vector field with eigenvalue $+i$, satisfying $JZ = iZ$, then its complex conjugate $\bar{Z}$ is an eigenvector with eigenvalue $-i$, $J\bar{Z} = -i\bar{Z}$. This can be most easily seen by decomposing the complexified vector fields in terms of real vector fields $Z = X + iY, \bar{Z} = X - iY$, and then use $JX = -Y, JY = X$.

(c) First we decompose $\theta = \theta^{(1,0)} + \theta^{(0,1)}$. Then we compute, with $Z = Z^\mu \partial_\mu$ in local coordinates,

$$\theta(Z) = P^+\theta(Z) + P^-\theta(Z) = P_\mu^{+\nu}\theta_\nu Z^\mu + P_\mu^{-\nu}\theta_\nu Z^\mu \ . \tag{1}$$

For any holomorphic vector field Z we have that $P^+Z = Z$ and $P^-Z = 0$, and so $\theta(Z) = \theta^{(1,0)}(Z)$. If now $\theta(Z) = 0$, $\forall Z$, then it follows that $\theta^{(1,0)} = 0$, hence θ is of type $(0, 1)$. Conversely, if θ is of type $(0, 1)$, $\theta = \theta^{(0,1)}$ and $\theta^{(1,0)} = 0$, then the above calculation shows that $\theta(Z) = 0$ on any holomorphic vector field Z.

Exercise 4. Let M be a hyperkähler manifold, endowed with three integrable complex structures $\vec{J} = (J_1, J_2, J_3) = (I, J, K)$ satisfying the quaternionic algebra relations

$$I^2 = J^2 = K^2 = IJK = -\mathbb{I} \ . \tag{2}$$

A hyperkähler manifold admits, by definition, a triplet of closed fundamental two-forms $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$

$$\vec{\omega}(X, Y) \equiv g(\vec{J}X, Y) \ , \tag{3}$$

where g is a hyper-Hermitian metric. Now define

$$\omega_+ \equiv \frac{1}{2}(\omega_2 - i\omega_3) \ , \qquad \omega_- \equiv \frac{1}{2}(\omega_2 + i\omega_3) \ . \tag{4}$$

(a) Show that $\omega_+(IX, Y) = i\omega_+(X, Y)$ and $\omega_-(IX, Y) = -i\omega_-(X, Y)$ on any two vector fields X and Y.

(b) Use property (a) to show that ω_+ is of type $(2, 0)$ and ω_- of type $(0, 2)$, with respect to the projection operators $P^\pm = \frac{1}{2}(\mathbb{I} \mp iI)$.

(c) Give, up to an overall normalization, the hyperkähler two-forms $\omega_\pm$ and ω_1 on $\mathbb{R}^4 \cong \mathbb{C}^2$ in terms of the complex Cartesian coordinates z^1 and z^2.

Solution

(a) On any two vector fields X and Y, we have

$$\omega_+(X, Y) = \frac{1}{2}(g(JX, Y) - ig(KX, Y)) \ . \tag{5}$$

We then compute

$$\omega_+(IX, Y) = \frac{1}{2}(g(KX, Y) + ig(JX, Y)) = i\omega_+(X, Y) \ , \tag{6}$$

and similarly $\omega_-(IX, Y) = -i\omega_-(X, Y)$.

(b) This follows from the properties

$$\omega_+(P^-X, Y) = \omega_+(X, P^-Y) = 0 \ , \qquad \omega_+(P^+X, Y) = \omega_+(X, P^+Y) = \omega_+(X, Y) \ . \tag{7}$$

These properties imply $\omega_+(P^-X, P^-Y) = \omega_+(P^+X, P^-Y) = \omega_+(P^-X, P^+Y) = 0$ and $\omega_+(P^+X, P^+Y) = \omega_+(X, Y)$, i.e. ω_+ is $(2, 0)$. Similarly ω_- is $(0, 2)$, as follows from complex conjugation.

(c) The two-forms can be constructed as

$$\omega_+ = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \ , \quad \omega_- = \overline{\omega_+} \ , \quad \omega_1 = i\Big(\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 + \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2\Big) \ , \tag{8}$$

up to an overall normalization.