Notes on tensors

E.P. van den Ban

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1 Linear spaces and their duals

In this section we will consider linear spaces defined over a fixed ground field k.

Is E a linear space over the ground field k, then we denote by E^* the space of linear maps $E \to k$. The linear space E^* is called the **linear dual** of E.

Let F be a second linear space over k; then we write $\mathrm{Hom}(E, F)$ for the space of linear maps $A : E \to F$. If $A \in \mathrm{Hom}(E, F)$, then A induces the map $A^* : F^* \to E^*$, defined by

$$A^*\eta = \eta \circ A \qquad (\eta \in F^*).$$

This map is linear again and thus belongs to $\mathrm{Hom}(F^*, E^*)$. It is called the adjoint or the transpose of A. Observe that the transpose of the identity $I_E : E \to E,\ x \mapsto x$ is given by

$$(I_E)^* = I_{E^*}. \tag{1}$$

Is $B : F \to G$ a second linear map then

$$(B \circ A)^* = A^* \circ B^*. \tag{2}$$

Remark 1.1 In view of the properties (1) and (2) the assignment $E \rightsquigarrow E^*$ is called a functor. The functor $E \rightsquigarrow E^*$ is said to be contravariant, because it reverses the directions of arrows representing maps: a linear map $A : E \to F$ gives rise to the linear map $A^* : E^* \leftarrow F^*$. The (identity) functor $E \rightsquigarrow E$, which assigns to a linear space E the same linear space E, preserves the directions of arrows, and is therefore called covariant. At a later stage we shall discuss the covariant functor $E \rightsquigarrow E^{**}$ which assigns the double dual to a space. For the precise definition of a functor we refer the reader to the section on category theory in [Lang1, Ch. I, §7][1].

If $A : E \to F$ is a linear isomorphism, then it follows from the functorial properties that dat $(A^{-1})^* \circ A^* = [A \circ A^{-1}]^* = I_{F^*}$, whereas $A^* \circ (A^{-1})^* = [A^{-1} \circ A]^* = I_{E^*}$, so that $A^* : F^* \to E^*$ is a linear isomorphism as well. Moreover, its inverse is given by:

$$(A^*)^{-1} = (A^{-1})^*.$$

We now assume that E is a finite dimensional space. Let $d = \dim E$ and let $e_1, \ldots, e_d$ be a basis of E. We define the linear functionals $e^1, \ldots, e^d \in E^*$ by

$$e^i(e_j) = \delta^i_j. \tag{3}$$

Here δ^i_j is the Kronecker symbol: it equals 1 if $i = j$, and 0 if $i \neq j$.

Lemma 1.2 *If E is finite dimensional and $e_1, \ldots, e_d$ a basis of E, then $e^1, \ldots, e^d$ is a basis of E^*. In particular, the space E^* has the same dimension as E.*

Proof. The collection of vectors $e^1, \ldots, e^d$ is linearly independent. Indeed, if $\sum_{i=1}^d \lambda_i e^i = 0$, then evaluation on e_j yields $\lambda_j = 0$, for all $1 \leq j \leq n$.

The collection $e^1, \ldots, e^d$ spans E^* op. For if $\xi \in E^*$, then

$$\xi = \sum_{i=1}^d \xi(e_i)\, e^i.$$

This is readily checked by evaluating the expressions on both sides of the equation in each of the basis vectors e_j. We conclude that the collection $e^1, \ldots, e^d$ is a basis of E^*. In particular, $\dim(E^*) = d = \dim E$. $\square$

The basis $e^1, \ldots, e^d$ of E^* defined above is called the basis of E^* **dual** to the basis $e_1, \ldots, e_d$ of E.

Remark 1.3 If $\xi \in E$, then we denote by $\operatorname{mat} \xi$ the matrix of ξ with respect to the basis $\{e_i\}$ of E. Note that this matrix is a row vector of length d. Note also that e^i is the element of E^* with matrix $(\delta^i_1, \ldots, \delta^i_d)$.

In the following we use the convention that k^d consists of column vectors. The reason for this is that a linear map $A : k^d \to k^p$ is given by the matrix multiplication by $\operatorname{mat} A$, the matrix of A with respect to the standard bases. In other words, $Ax = \operatorname{mat} A \cdot x$ for all $x \in k^d$. In this fashion we identify $\operatorname{Hom}(k^d, k^p)$ with the $p \times d$ matrices with coefficients from k. In particular, we identify $(k^d)^*$ with the $1 \times d$ matrices, i.e., with the row vectors of length d.

Remark 1.4 Is the linear space E equipped with a non-degenerate bilinear form $\langle \cdot\, , \, \cdot \rangle$, we can define a linear map $\mathbf{i} : E \to E^*$ by $\mathbf{i}\,(v)(w) = \langle v\, , \, w \rangle$, for $v, w \in E$. By non-degeneracy of the form it follows that $\mathbf{i}$ has a trivial kernel; therefore, the map is a linear isomorphism. It is

[1] S. Lang; Algebra, Addison-Wesley, 1965

easily seen that a basis $\{e_j\}$ satisfies the orthonormality relations $\langle e-_i\,, e_j\rangle = \delta_{ij}$ if and only if $\{\mathbf{i}\,(e_j)\}$ is the dual basis of E^*.

In case E is real and $\langle \cdot\,,\,\cdot\rangle$ a positive definite inner product, one often identifies the space E with its dual E^* by means of the map $\mathbf{i}$. Here one should keep in mind that the identification depends on the particular choice of inner product involved.

Remark 1.5 In analysis row vectors and column vectors appear as follows. If $c : I \to \mathbb{R}^n$ is a differentiable curve, then, for every $t \in I$, the velocity vector $c'(t)$ is a column vector in $\mathbb{R}^n$. Is a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ differentiable in the point a, then the derivative $df(a) = Df(a)$ is an element of $(R^n)^*$, and therefore represented by a row vector.

Let $\operatorname{grad} f(a)$ be the unique vector in $\mathbb{R}^n$ with $\mathbf{i}\,(\operatorname{grad} f(a)) = Df(a)$; here $\mathbf{i}$ is defined as above with respect to the standard inner product. Then $\operatorname{grad} f(a)$ is the column vector in $\mathbb{R}^n$ with components $\partial_i f(a)$, $1 \le i \le n$. For typographical reasons we denote the gradient often as a row vector. It would be better to use the notation $(v_1, \ldots, v_n)^{\mathrm{T}}$ for the column vector with components $v_1, \ldots, v_n$; here T stands for transposition.

By dualizing a second time we obtain the double dual $E^{**} := (E^*)^*$ of a linear space E. There is a natural linear map $\iota : E \to E^{**}$; if $x \in E$, then $\iota(x)$ is given by

$$\iota(x)(\xi) = \xi(x) \qquad (\xi \in E^*).$$

Is $\iota(x) = 0$, then $\xi(x) = 0$ for all $\xi \in E^*$, from which we deduce that $x = 0$. It follows that ι is an injective linear map. It is called the natural embedding of E into E^{**}.

Lemma 1.6 *Let E be a finite dimensional linear space over the field k. Then the natural embedding $\iota : E \to E^{**}$ is a linear isomorphism onto.*

Proof. The linear map ι is injective. Since $\dim(E^{**}) = \dim E^* = \dim E$, the map ι is surjective as well. □

In the folllowing we shall identify the double dual E^{**} of a finite dimensional linear space E with E via the isomorphism ι.

Assume that E is finite dimensional with basis $e_1, \ldots, e_d$. Let $e^1, \ldots, e^d$ be the dual basis of E^*. Under the identification mentioned above, the associated dual basis in E^{**} corresponds precisely with the original basis $e_1, \ldots, e_d$. Indeed: $\iota(e_i)(e^j) = e^j(e_i) = \delta_i^j$ for all $1 \le i, j \le n$.

Let F be a second finite dimensional linear space, with basis $f_1, \ldots, f_p$. Let $f^1, \ldots, f^p$ be the corresponding dual basis of F^*. The matrix of a linear map $A : E \to F$ with respect to the chosen bases has entries $A^i_j \in k$ which are determined by the formula $A(e_j) = \sum_i A^i_j f_i$. By applying f^i to this basis, we see that

$$A^i_j = f^i(Ae_j). \tag{4}$$

The matrix of A with respect to the mentioned bases is denoted by $\operatorname{mat} A$, its transposed by $(\operatorname{mat} A)^{\mathrm{T}}$.

Lemma 1.7 $\operatorname{mat}(A^*) = (\operatorname{mat} A)^{\mathrm{T}}$

Proof. We have that

$$A^i_j = f^i(A(e_j)) = [A^*(f^i)](e_j) = e_j(A^*(f^i)) = (A^*)^j_i.$$

□

2 The tensor product of linear spaces

For k-linear spaces $E_1, \ldots, E_n$ and F we denote by

$$L^n(E_1, \ldots, E_n; F)$$

the linear space of n-multilinear maps $f : E_1 \times \cdots \times E_n \to F$ (n-multilinearity means that f is linear in each of its n variables).

Observed that in particular, for E a linear space, we have $E^* = L^1(E; k)$. Is E finite dimensional, then frowm the identification $E^{**} = E$ it follows that $E = L^1(E^*; k)$.

Definition 2.1 Let $E_1, \ldots, E_n$ be finite dimensional linear spaces over k ($n \geq 1$).

(a) We define $E_1 \otimes \cdots \otimes E_n := L^n(E_1^*, \ldots, E_n^*; k)$.

(b) For $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$ we define the element $x_1 \otimes \cdots \otimes x_n$ of $E_1 \otimes \cdots \otimes E_n$ by

$$[x_1 \otimes \cdots \otimes x_n](\xi_1, \ldots, \xi_n) := \xi_1(x_1) \cdots \xi_n(x_n). \tag{5}$$

Remark 2.2 Let $\iota_k : E_k \to E_k^{**}$ be the natural embedding, for $1 \leq k \leq n$. Then the expression on the right-hand side of (5) can be rewrtten as $\prod_{k=1}^n \iota_k(x_k)(\xi_k)$. In particular, the above definition for $n = 1$ is compatible with the identification $L^1(E_1^*; k) = E_1$.

Remark 2.3 One easily verifies that the map $\nu : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ given by $\nu(x_1, \ldots, x_n) = x_1 \otimes \cdots \otimes x_n$ is multilinear. In other words, the following rule of calculation applies to the tensorproduct, for $x_1 \in E_1, \ldots, x_n \in E_n$, and for $x_j' \in E_j, \lambda \in k$:

$$\begin{aligned} x_1 \otimes \cdots \otimes (x_j + \lambda x_j') \otimes \cdots \otimes x_n &= \\ &= \quad (x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n) \quad + \quad \lambda \, (x_1 \otimes \cdots \otimes x_j' \otimes \cdots \otimes x_n) \end{aligned}$$

In the sequel we will always assume that $E_1, \ldots, E_n$ are finite dimensional linear spaces over k.

Lemma 2.4 *For each $1 \leq k \leq n$ let $e_{k,1} \ldots e_{k,d_k}$ be a given basis of E_k. Let $\mathcal{I}$ be the collection of sequences $i = (i(1), \ldots, i(n))$ with $i(k) \in \{1, \ldots, d_k\}$, for all $1 \leq k \leq n$. Then the collection*

$$\{e_{1,i(1)} \otimes \cdots \otimes e_{n,i(n)} \mid i \in \mathcal{I}\}$$

is a basis of $E_1 \otimes \cdots \otimes E_n$.

Proof. For $1 \leq k \leq n$ we denote the dual basis of E_k^* by $e_k^1, \ldots, e_k^{d_k}$. For $i \in \mathcal{I}$ we denote byu $e_\cdot^i$ the sequence of vectors $(e_1^{i(1)}, \ldots, e_n^{i(d_k)})$. Let $t \in E_1 \otimes \cdots \otimes E_n$, then it follows from the multi linearity of t that $t = 0$ if and only if $t(e_\cdot^i) = 0$ for all $i \in \mathcal{I}$.

For $i \in \mathcal{I}$ we denote by $(\otimes e)_i$ the vector $e_{1,i(1)} \otimes \cdots \otimes e_{n,i(n)}$ from $E_1 \otimes \cdots \otimes E_n$. If also $j \in \mathcal{I}$, then $(\otimes e)_i(e_\cdot^j)$ equals 1 for $i = j$, and 0 for $i \neq j$. Let now a scalar $\lambda^i \in k$ be given for each $i \in \mathcal{I}$. Moreover, assume that $\sum_{i \in \mathcal{I}} \lambda^i (\otimes e)_i = 0$. By evaluating the expressions on both sides in the sequence $e^j_\cdot$ we infer that $\lambda^j = 0$ for all $j \in \mathcal{I}$. We conclude that the vectors $(\otimes e)_i$, $i \in \mathcal{I}$ are all linearly independent.

Finally, let $t \in E_1 \otimes \cdots \otimes E_n$. Write $t^i = t(e^i_\cdot)$, for $i \in \mathcal{I}$. Then

$$t = \sum_{i \in \mathcal{I}} t^i (\otimes e)_i.$$

This follows by evaluation of the expressions on either side of this equality on each sequence of vectors $e^j_\cdot$, $j \in \mathcal{I}$. We conclude that the vectors $(\otimes e)_i$, $i \in \mathcal{I}$, span the space $E_1 \otimes \cdots \otimes E_n$. $\square$

Theorem 2.5 *Let $E_1, \ldots, E_n$ be a collection of finite dimensional linear spaces. Then the multilinear map $\nu : E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ satisfies the following* **universal property***.*

For every linear space F and each multilinear map $f : E_1 \times \cdots \times E_n \to F$ there exists precisely one linear map $\bar{f} : E_1 \otimes \cdots \otimes E_n \to F$ so that the following diagram commutes:

$$\begin{array}{ccc} E_1 \times \cdots \times E_n & \xrightarrow{f} & F \\ \nu \downarrow & \nearrow \bar{f} & \\ E_1 \otimes \cdots \otimes E_n & & \end{array}$$

Proof. In the following we will use the notation from Lemma 2.4 and its proof. Then $\{(\otimes e)_i \mid i \in \mathcal{I}\}$ is a basis of $E_1 \otimes \cdots \otimes E_n$. Let $f \in L^n(E_1, \ldots, E_n; F)$. Then there is precisely one linear map $\bar{f} : E_1 \otimes \cdots \otimes E_n \to F$ with $\bar{f}((\otimes e)_i) = f(e_{1,i(1)}, \ldots, e_{n,i(n)})$ for all $i \in \mathcal{I}$. One now readily checks that $\bar{f} \circ \nu = f$ on each sequence $(e_{1,i(1)}, \ldots, e_{n,e(n)})$, $i \in \mathcal{I}$. By multilinearity of the expressions on both sides it follows that $\bar{f} \circ \nu = f$. De uniqueness of $\bar{f}$ is a straightforward consequence of Lemma 2.4. $\square$

Remark 2.6 The assignment $\varphi \mapsto \varphi \circ \nu$ defines a linear mapping

$$\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n; F) \longrightarrow L^n(E_1, \ldots, E_n; F).$$

The existence part of the universal property expresses precisely that the above map is surjective. The uniqueness part expresses that the above map is injective. Hence, the natural map defined above is a linear isomorphism.

Example 2.7 Is X a set, then by $\mathcal{F}(X)$ we denote the linear space of all functions $X \to k$. Is $a \in X$ then we write e_a for the element of $\mathcal{F}(X)$ given by $e_a(x) = 0$ if $x \neq a$ and by $e_a(a) = 1$. Is X finite then the e_a, $a \in X$, form a basis for $\mathcal{F}(X)$. The dual basis is given by e^a, $a \in X$, with $e^a : f \mapsto f(a)$.

Let Y be a second finite set. We will show that the tensor product $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ is isomorphic to $\mathcal{F}(X \times Y)$ in a natural fashion. For this we consider the bilinear map $\varphi : \mathcal{F}(X) \times \mathcal{F}(Y) \to \mathcal{F}(X \times Y)$ given by $\varphi(f, g)(x, y) = f(x)g(y)$. By the universal property the map φ factors to a linear map

$$\bar{\varphi} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \times Y).$$

This mapping maps the basis $\{e_a \otimes e_b \mid a \in X, b \in Y\}$ to the basis $\{e_{(a,b)} \mid (a,b) \in X \times Y\}$ of $\mathcal{F}(X \times Y)$, hence is a linear isomorphism.

In what follows we shall identify $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ with $\mathcal{F}(X \times Y)$ via the natural isomorphism $\bar{\varphi}$. In particular, we denote, for $f \in \mathcal{F}(X)$ and $g \in \mathcal{F}(Y)$, the function $(x,y) \mapsto f(x)g(y)$ by $f \otimes g$.

The above situation arises e.g. by combining two quantum mechanical systems A en B with finitely many quantum states. Let the states of A be labeled by the set X and those of B by the set Y. Then $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are the state spaces of A and B. The state space of the combined system is $\mathcal{F}(X \times Y) = \mathcal{F}(X) \otimes \mathcal{F}(Y)$.

The universal property in Theorem 2.5 provides motivation for the following definition.

Definition 2.8 A **tensor product** of the linear spaces $E_1, \ldots, E_n$ is defined to be a linear space G, together with a linear map $g \in L^n(E_1, \ldots, E_n; G)$ such that the following universal property holds. *For each linear space F and every multilinear mapping $f : E_1 \times \cdots \times E_n \to F$ there exists precisely one linear mapping $\bar{f} : G \to F$ so tha the following diagram commutes:*

$$\begin{array}{ccc} E_1 \times \cdots \times E_n & \xrightarrow{f} & F \\ {\scriptstyle g}\downarrow & \nearrow {\scriptstyle \bar{f}} & \\ G & & \end{array}$$

Remark 2.9 After this definition, we may reformulate Theorem 2.5 as follows: *Het paar $E_1 \otimes \cdots \otimes E_n, \nu$ is a tensor product of the linear spaces $E_1, \ldots, E_n$.*

The next lemma asserts that all tensor products of $E_1, \ldots, E_n$ are naturally isomorphic with $E_1 \otimes \cdots \otimes E_n$. Accordingly, we will speak of *the* tensor product $E_1 \otimes \cdots \otimes E_n$.

Lemma 2.10 *Let G, g and G', g' be two tensor products of $E_1, \ldots, E_n$. Then there exists a unique linear map $h : G \to G'$ such that the following diagram commutes*

$$E_1 \times \cdots \times E_n \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} G' \\ \uparrow \\ G \end{array} . \qquad (6)$$

The map h is a linear isomorphism.

Proof. The existence and uniqueness of h follow from the universal property of G, g (we have $h = \bar{g}'$). From the universal property of G', g' we deduce the existence of a unique liner map $h' : G' \to G$ so that the following diagram commutes:

$$E_1 \times \cdots \times E_n \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} G \\ \uparrow \\ G' \end{array} . \qquad (7)$$

From the commutativity of the diagrams (6) and (7) it follows that the following diagram is commutative if the vertical map is taken to be the composition $h' \circ h$. On the other hand, the diagram is also commutative if the vertical map is taken to be the identity I_G :

$$E_1 \times \cdots \times E_n \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} G \\ \uparrow \\ G \end{array} .$$

The uniqueness part of the universal property tells us that $h' \circ h = I_G$. In a similar fashion we conclude that $h \circ h' = I_{G'}$. Thus, h is an isomorphism. □

We observe that Definition 2.8 and Lemma 2.10 are valid in case the spaces $E_1, \ldots, E_n$ are not necessarily finite dimensional. However, in this case $L^n(E_1, \ldots, E_n; k)$ is not a tensor product anymore. This is the reason that we have chosen for the above definition in terms of the universal property. The existence of the product is then a problem which can be solved in a different manner. The above lemma guarantees the uniqueness up to natural linear isomorphism. This approach also works in the more general context of tensor products of modules $E_1, \ldots, E_n$ for a commutative ring with unit. For details we refer the reader to [Lang1, Ch. XVI].

We end this section with a useful lemma.

Lemma 2.11 *Let $E_1, \ldots, E_n$ be a collection of finite dimensional linear spaces over k. Then there exists a natural isomorphism*

$$(E_1 \otimes \cdots \otimes E_n)^* \simeq E_1^* \otimes \cdots \otimes E_n^*.$$

Proof. We have $(E_1 \otimes \cdots \otimes E_n)^* = \mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, k) \simeq L^n(E_1, \ldots, E_n; k)$, see Remark 2.6. By Lemma 1.6 the latter space is naturally isomorphic to $L^n(E_1^{**}, \ldots, E_n^{**}; k) = E_1^* \otimes \cdots \otimes E_n^*$. □

3 Tensors and components

We assume that E is a finite dimensional linear space over k, with basis $e_1, \ldots, e_d$. The dual basis of E^* is denoted by $e^1, \ldots, e^d$. If $x \in E$, we write

$$x = \sum_{i=1}^{d} x^i e_i;$$

the coefficients in this expression are given by $x^i = e^i(x)$. If $\xi \in E^*$, we write

$$\xi = \sum_{i=1}^{d} \xi_i e^i;$$

the coefficients are now given by $\xi_i = \xi(e_i)$.

We now discuss a useful example of a tensor product. Let F be a second finite dimensional linear space over the field k, with basis $f_1, \ldots, f_p$. We use the notation $\mathrm{Hom}(E, F)$ for the space of linear maps $E \to F$.

Lemma 3.1 *There exists a unique linear map* $\varphi : F \otimes E^* \to \mathrm{Hom}(E, F)$ *such that for* $\xi \in E^*, y \in F$ *the element* $\varphi(y \otimes \xi) \in \mathrm{Hom}(E, F)$ *is given by*

$$\varphi(y \otimes \xi) : E \to F,\ x \mapsto \xi(x)y.$$

The linear map φ *is an isomorphism.*

Proof. Let $f \in L^2(F, E^*; \mathrm{Hom}(E, F))$ be defined by $f(y, \xi) : x \mapsto \xi(x)y, E \to F$ for $\xi \in E^*, y \in F$. By the universal property of the tensor product, the map factors to a unique linear map $\bar{f} : F \otimes E^* \to \mathrm{Hom}(E, F)$. This is the uniquely determined map φ.

For $1 \leq i \leq p$ and $1 \leq j \leq d$ we define the linear map $L^i_j : E \to F$ by $L^j_i = \varphi(f_i \otimes e^j)$. Thus, L^j_i is given by

$$L^j_i(x) = x^j f_i.$$

If $A : E \to F$ is a linear map, then its matrix is equals $(A^i_j)_{i,j}$, where $A^i_j = A(e_j)^i$. It is now readily checked that

$$A = \sum_{\substack{1 \leq i \leq d \\ 1 \leq j \leq p}} A^i_j L^j_i.$$

Indeed, when evaluated in e_j, the i-th component of both the expression on the right-hand side and that on the right-hand side becomes equal to A^i_j. From this it follows that the L^j_i constitute a basis of $\mathrm{Hom}(E, F)$. Thus, φ maps a basis to a basis, and is therefore a linear isomorphism. □

In the sequel we shall use the isomorphism φ from the above lemma as identifying map. After this identication, $F \otimes E^* = \mathrm{Hom}(E, F)$. The element $f_i \otimes e^j$ van $F \otimes E^*$ is identified with the map $L^j_i \in \mathrm{Hom}(E, F)$ from the above proof. For a linear map $A \in \mathrm{Hom}(E, F)$ we thus obtain the tensor notation

$$A = \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq d}} A^i_j\, f_i \otimes e^j,$$

with $(A^i_j)_{i,j}$ the matrix of A relative to the bases $e_1, \dots, e_d$ of E and $f_1, \dots, f_p$ of F. The matrix of A appears here as a collection of components for the tensor $A \in F \otimes E^*$ with respect to the basis $f_i \otimes e^j$ of $F \otimes E^*$.

We use the notation $\mathrm{End}(E) = \mathrm{Hom}(E, E)$ for the space of linear endomorphisms of E. With the above identification, applied to $F = E$, we see that

$$E \otimes E^* = \mathrm{End}(E). \tag{8}$$

In view of the universal property, the natural bilinear map $c : E \times E^* \to k$ given by $c(x, \xi) = \xi(x)$ factors to a unique linear map $C : E \otimes E^* \to k$. This linear map is called **contraction**.

Lemma 3.2 *Via the identification (8) the contraction* $C : E \otimes E^* \to k$ *corresponds to the trace* $\mathrm{tr} : \mathrm{End}(E) \to k,\ A \mapsto \mathrm{tr}\,(A)$.

Proof. Relative to the basis $e_1, \dots, e_d$ of E we write $A = \sum_{1 \leq i,j \leq d} A^i_j e_i \otimes e^j$. Then

$$C(A) = \sum_{1 \leq i,j \leq d} A^i_j C(e_i \otimes e^j) = \sum_{1 \leq i,j \leq d} A^i_j \delta^j_i = \sum_{1 \leq i \leq d} A^i_i = \mathrm{tr}\,(A).$$

□

For obvious reasons the next lemma is known as associativity of the tensor product.

Lemma 3.3 *Let E, F, G be finite dimensional linear spaces over k. Then there exists a natural isomorphism*

$$(E \otimes F) \otimes G \simeq E \otimes F \otimes G;$$

it is given by $(x \otimes y) \otimes z \mapsto x \otimes y \otimes z$. Similarly, there is a natural isomorphism $E \otimes (F \otimes G) \simeq E \otimes F \otimes G$.

Proof. Define the map $f : E \times F \times G \to (E \otimes F) \otimes G$ by $f(x, y, z) = (x \otimes y) \otimes z$. The map f is multi-linear en factors therefore to a linear map $\bar{f} : E \otimes F \otimes G \to (E \otimes F) \otimes G$. Observe that $\bar{f}(x \otimes y \otimes z) = (x \otimes y) \otimes z$. Let $(e_i), (f_j)$ and (g_k) be bases of E, F and G, respectively. Then $\bar{f}$ maps the basis $e_i \otimes f_j \otimes g_k$ of $E \otimes F \otimes G$ to the basis $(e_i \otimes f_j) \otimes g_k$ of $(E \otimes F) \otimes G$. This implies that $\bar{f}$ is a linear isomorphism. The inverse of $\bar{f}$ fulfilles the requirements of the above lemma. The last part of the lemma is proved in a similar way. □

Remark 3.4 In the sequel we shall identify via the natural ismorphisms described above. In other words, we shall no longer distinguish between $E \otimes F \otimes G$, $(E \otimes F) \otimes G$ or $E \otimes (F \otimes G)$.

Example 3.5 In this example we show that the composition of two linear maps can be decribed by means of a contraction. In a natural way this leads to the well known formula for multiplication of matrices.

Let E, F, G be finite dimensional linear spaces over k, and $A : E \to F$ and $B : F \to G$ linear maps. We view A as a tensor in $F \otimes E^*$ and B as a tensor in $G \otimes F^*$. The tensor $B \otimes A$ is thus contained in $G \otimes F^* \otimes F \otimes E$ (here we use the associativity of the tensor product).

The multilinear map $G \times F^* \times F \times E^* \to G \otimes E$, $(g, f^*, f, e^*) \mapsto f^*(f)\, g \otimes e^*$ factors to a unique linear map $C_2^3 : G \otimes F^* \otimes F \otimes E \to G \otimes E$ which for obvious reasons is called the contraction with respect to the second and the third component. We will show that

$$B \circ A = C_2^3(B \otimes A). \tag{9}$$

The maps $\varphi : (B, A) \mapsto B \circ A$ and $\psi : (A, B) \mapsto C_2^3(B \otimes A)$ are both bilinear as maps from $G \otimes F^* \times F \otimes E^*$ to $G \otimes E^*$. It is therefore sufficient to check the equality φ en ψ on elements of the form $A = f \otimes e^*$ en $B = g \otimes f^*$. Now for elements of this form, we have, for $e \in E$,

$$\varphi(B, A)(e) = [g \otimes f^*](e^*(e) f) = e^*(e) f^*(f) g = f^*(f)[g \otimes e^*](e) = \psi(A, B)(e).$$

This establishes (9).

From the formula we deduce that the following holds for the components with respect to the bases of E, F, G and the corresponding dual bases of E^*, F^*. We have

$$(B \circ A)_i^k = [C_2^3(A \otimes B)]_i^k = \sum_j (B \otimes A)_{ji}^{kj} = \sum_j B_j^k A_i^j.$$

This is the well known formula which expresses that the matrix of the composition $B \circ A$ can be obtained by multiplication of the matrices of A and B.

In the following we use, for $r \geq 1$, the notation

$$\otimes^r E := \overbrace{E \otimes \cdots \otimes E}^{r}$$

for the r-fold tensor product of E with itself. With $\otimes^1 E$ we mean the space $L^1(E^*; k) = E^{**}$, see Definition 2.1. In view of Lemma 1.6 this space is naturally isomorphic with E; thus, $\otimes^1 E = E$. Finally, we agree that $\otimes^0 E := k$. The space $\otimes^r E$ is called the space of contravariant tensors of degree r on E. Unfortunately, this traditional termonology is opposite to what one would expect from the similar terminology for functors. Later on we will see that $E \rightsquigarrow \otimes^r E$ defines a covariatn functor.

Let $s \in \mathbb{N}$; then $\otimes^s E^*$ is called the space of covariant tensors of degree s on E. Later on we will see that $E \rightsquigarrow \otimes^s E^*$ defines a contravariant functor.

For $r, s \in \mathbb{N}$ we define the space

$$\mathcal{T}^r_s E := \otimes^r E \otimes \otimes^s E^* = \overbrace{E \otimes \cdots \otimes E}^{r} \otimes \overbrace{E^* \otimes \cdots \otimes E^*}^{s}.$$

The elements of this space are called: (mixed) tensors on E of contravariance degee r and covariance degree s.

Notice that from the definitions it follows that

$$\mathcal{T}^r_s E = L^{r+s}(\overbrace{E^*, \ldots, E^*}^{r}, \overbrace{E, \ldots, E}^{s}; k) \tag{10}$$

We finally observe that from (8) it follows that $\mathcal{T}^1_1 E = \mathrm{End}(E)$.

Lemma 2.4 implies that a basis of $\mathcal{T}^r_s := \mathcal{T}^r_s E$ is given by the vectors

$$e_{i(1)} \otimes \cdots \otimes e_{i(r)} \otimes e^{j(1)} \otimes \cdots \otimes e^{j(s)},$$

with $i \in \mathcal{I}_r$ en $j \in \mathcal{I}_s$. Here we have used the notation $\mathcal{I}_r := \{1, \ldots, d\}^r$. By convention, one uses the following component notation for the elements of the space $\mathcal{T}^r_s$:

$$T = \sum_{\substack{i \in \mathcal{I}_r \\ j \in \mathcal{I}_s}} T^{i(1)\ldots i(r)}_{j(1)\ldots j(s)} \, e_{i(1)} \otimes \cdots \otimes e_{i(r)} \otimes e^{j(1)} \otimes \cdots \otimes e^{j(s)}.$$

By evaluation on suitable sequences of dual basis vectors one readily sees that the components in this expression are given by

$$T^{i(1)\ldots i(r)}_{j(1)\ldots j(s)} = T(e^{i(1)}, \ldots, e^{i(r)}, e_{j(1)}, \ldots, e_{j(s)}).$$

Definition 3.6 Let $1 \leq k \leq r$ and $1 \leq l \leq s$. Then we define the **contraction** C^k_l to be the linear map $\mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1}$ given by

$$C^k_l : x_1 \otimes \cdots \otimes x_r \otimes \xi^1 \otimes \cdots \otimes \xi^s \mapsto \xi^l(x_k)\, x_1 \otimes \cdots \otimes \hat{x}_k \otimes \cdots \otimes x_r \otimes \xi^1 \otimes \cdots \otimes \hat{\xi}^l \otimes \cdots \otimes \xi^s.$$

Here the symbol $\widehat{\ }$ on top of an element indicates that the element is left out.

Observe that the map C^k_l is well defined by the universal property of the tensor product. Indeed, C^k_l is the mapping obtained from factoring the multi-linear map

$$(x_1, \dots x_r, \xi^1, \dots \xi^s) \mapsto \xi^l(x_k)\, x_1 \otimes \cdots \otimes \hat{x}_k \otimes \cdots \otimes x_r \otimes \xi^1 \otimes \cdots \otimes \hat{\xi}^l \otimes \cdots \otimes \xi^s.$$

The contraction C^k_l is nothing but the earlier defined contraction C applied to the k-th contravariant and the l-de covariant component of the tensor product. By means of Lemma 3.2 we now see that the contraction on the tensor components acts as the trace with respect to the k-th contravariant and the l-th covariant component. If $T \in \mathcal{T}^r_s$, then the components of $C^k_l(T)$ are given in terms of those of T by the formula

$$C^k_l(T)^{i(1),\dots,\widehat{i(k)}\dots i(r)}_{j(1),\dots,\widehat{j(l)}\dots j(s)} = \sum_{\nu=1}^{d} T^{i(1),\dots,i(k-1),\nu,i(k+1)\dots i(r)}_{j(1),\dots,j(l-1),\nu,j(l+1),\dots j(s)}.$$

4 Transformation of tensors under maps

Assume that E and F are two finite dimensional linear spaces over k. We will describe how a linear map $A : E \to F$ induces a linear map on tensors.

Let $r \in \mathbb{N}$. The map $(x_1, \dots, x_r) \mapsto Ax_1 \otimes \cdots \otimes Ax_r$, $E \times \cdots \times E \to \otimes^r F$ is multilinear, and in view of the universal property induces a linear map $\otimes^r E \to \otimes^r F$ which we denote by $\otimes^r A$. Then:

$$\otimes^r A\,(x_1 \otimes \cdots \otimes x_r) = Ax_1 \otimes \cdots \otimes Ax_r.$$

In view of (5) this means that

$$[\otimes^r A]S = S \circ (\overbrace{A^* \times \cdots \times A^*}^{r}).$$

Observe that $\otimes^r I_E$ equals the identity on $\otimes^r E$. If G is a third finite dimensional linear space, and $B : F \to G$ a linear map, then

$$\otimes^r (B \circ A) = \otimes^r B \circ \otimes^r A.$$

In view of these two properties, the assignment $E \rightsquigarrow \otimes^r E$ is called a covariant functor.

Lemma 4.1 *If $A : E \to F$ is a linear isomorphism, then $\otimes^r A : \otimes^r E \to \otimes^r F$ is a linear isomorphism too.*

Proof. In view of the functorial properties, we have

$$\otimes^r (A^{-1}) \circ \otimes^r A = \otimes^r (A^{-1} \circ A) = \otimes^r (I_E) = I_{\otimes^r E}.$$

In reversed order, the composition gives the identity on $\otimes^r F$. This implies that $\otimes^r A$ is a linear isomorphism with inverse $\otimes^r A^{-1}$. □

We shall now describe the mapping $\otimes^r A$ in components. Let $e_1, \ldots, e_p$ be a basis of E, and $f_1, \ldots, f_q$ a basis of F. Let $\{e^i\}$ and $\{f^j\}$ be the corresponding dual bases of E^* and F^*, respectively. If $T \in \otimes^r E$, then the components of $S = [\otimes^r A](T)$ are given by

$$\begin{aligned} S^{i(1)\ldots i(r)} &= S(f^{i(1)}, \ldots, f^{i(r)}) \\ &= T(A^* f^{i(1)}, \ldots, A^* f^{i(r)}) \end{aligned}$$

In view of Lemma 1.7 the matrix coefficients of A^* are given by

$$(A^*)^i_j = A^j_i.$$

If we apply this to the above, we find

$$\begin{aligned} S^{i(1)\ldots i(r)} &= T(\sum_{k=1}^{d} A^{i(1)}_k e^k, \ldots, \sum_{k=1}^{d} A^{i(r)}_k e^k) \\ &= \sum_{k \in \mathcal{I}_r} A^{i(1)}_{k(1)} \cdots, A^{i(r)}_{k(r)}\, T(e^{k(1)}, \ldots, e^{k(r)}). \\ &= \sum_{k \in \mathcal{I}_r} A^{i(1)}_{k(1)} \cdots, A^{i(r)}_{k(r)}\, T^{k(1)\ldots k(r)}. \end{aligned} \qquad (11)$$

Let $s \in \mathbb{N}$. A linear map $A : E \to F$ induces the linear map $A^* : F^* \to E^*$. The map A^* induces a linear map $\otimes^s A^* : \otimes^s F^* \to \otimes^s E^*$. In view of (5) this implies that

$$[\otimes^r A^*]S = S \circ (\overbrace{A \times \cdots \times A}^{r}).$$

Note that $\otimes^s I_E^* = I_{\otimes^s E^*}$. If $B : F \to G$ is another linear map, then

$$\otimes^s (B \circ A)^* = \otimes^s A^* \circ \otimes^s B^*.$$

In view of these properties, the assignment $E \rightsquigarrow \otimes^s E^*$ is called a contravariant functor (we repeat that the elements of the obtained tensor space are called covariant).

If $T \in \otimes^s F^*$, then the tensor $S = [\otimes^s A^*](T)$ belongs to $\otimes^s E^*$. By combining the above with (11) we find that the components of S are given by the formula:

$$S_{j(1)\ldots j(s)} = \sum_{l \in \mathcal{I}_s} A^{l(1)}_{j(1)} \cdots A^{l(s)}_{j(s)}\, T_{l(1)\ldots l(s)} \qquad (12)$$

Finally, we will discuss how mixed tensors transform under linear mappings. In view of the mix covariant and contravariant character this only mekes sense for linear isomorphisms.

Let A be a linear isomorphism from E onto F, with inverse $A^{-1} : F \to E$. Then $A^* : F^* \to E^*$ is a linear isomorphism from F^* onto E^*, with inverse $(A^*)^{-1} = (A^{-1})^*$. In view of Lemma 4.1 we now see that a linear isomorphism $A : E \to F$ induces a linear isomorphism

$$A_* := [\otimes^r A] \otimes [\otimes^s A^{-1*}] : \mathcal{T}^r_s E \to \mathcal{T}^r_s F.$$

If $T \in \mathcal{T}^r_s E$ we see, by combining (11) and (12), that the components of the tensor $A_* T$ are given by:

$$[A_* T]^{i(1)\ldots i(r)}_{j(1)\ldots j(s)} = \sum_{\substack{i \in \mathcal{I}_r \\ j \in \mathcal{I}_s}} A^{i(1)}_{k(1)} \cdots A^{i(r)}_{k(r)} [A^{-1}]^{l(1)}_{j(1)} \cdots [A^{-1}]^{l(s)}_{j(s)}\, T^{k(1)\ldots k(r)}_{l(1)\ldots l(s)}. \qquad (13)$$

5 Tensor bundles and tensor fields

In this section we assume that all linear spaces are defined over the base field $k = \mathbb{R}$.

Let M be a smooth manifold (with smooth we always mean C^∞). If $\pi : \mathcal{V} \to M$ is a smooth vector bundle, then we write $\mathcal{V}_p$ for the fiber $\pi^{-1}(p)$ above a point $p \in M$. If $\Omega \subset M$ is an open subset, we write $\mathcal{V}_\Omega$ for the pre-image $\pi^{-1}(\Omega)$; this is the disjoint union of the fibers $\mathcal{V}|_p$, for $p \in \Omega$. We write $\Gamma(\Omega, \mathcal{V})$ for the linear space of smooth sections $\Omega \to \mathcal{V}$. The space $\Gamma(M, \mathcal{V})$ is briefly denoted by $\Gamma(\mathcal{V})$.

By a **frame** of $\mathcal{V}$ on an open subset $\Omega \subset M$ we mean an ordered m-tuple of sections $(\sigma_1, \dots, \sigma_m)$ in $\Gamma(\Omega, \mathcal{V})$ such that the collection $\sigma_1(p), \dots, \sigma_m(p)$ is a basis of the fiber $\mathcal{V}_p$, for each $p \in \Omega$. The next result is an immediate consequence of the definition of a vector bundle.

Lemma 5.1 *Let $\pi : \mathcal{V} \to M$ be a smooth vector bundle of rank r. Let $\Omega \subset M$ be open. Then the following assertions are equivalent.*

(a) *The vector bundle $\mathcal{V}$ is trivial over Ω.*

(b) *The vector bundle $\mathcal{V}$ has a frame $(\sigma_1, \dots, \sigma_r)$ over Ω.*

Proof. '$(a) \Rightarrow (b)$': Lett $\mathcal{V}$ be trivial over Ω. Then there exists a trivialization $\tau : \mathcal{V}_\Omega \to \Omega \times V$, with V a linear space of dimension r. Choose a basis $e_1, \dots, e_r$ of V, and define $\bar{\sigma}_j : \Omega \to \Omega \times V$ by $\bar{\sigma}_j(p) = (p, e_j)$. Then $\sigma_j := \tau^{-1} \circ \bar{\sigma}_j$ defines a frame of $\mathcal{V}$ over Ω.

'$(b) \Rightarrow (a)$': let $\sigma_1, \dots, \sigma_r$ be a frame for $\mathcal{V}$ over Ω. Then

$$\rho : (p, a) \mapsto (p, a_1\sigma_1(p) + \cdots + a_r\sigma_r(p))$$

defines a diffeomorphism $\Omega \times \mathbb{R}^r \to \mathcal{V}_\Omega$, which is linear on each of the fibers. The inverse $\sigma = \rho^{-1}$ is a trivialization of $\mathcal{V}$ over Ω. □

Let $\mathcal{V}$ be a given vector bundle over M and let $r, s \geq 0$. Then in a natural way we can define a vector bundle $\mathcal{T}_s^r \mathcal{V}$ whose fiber in a point $p \in M$ equals the tensor product $\mathcal{T}_s^r(\mathcal{V}_p)$ (see (10). As a set $\mathcal{T}_s^r \mathcal{V}$ equals the disjoint union of the linear spaces $\mathcal{T}_s^r(\mathcal{V}_p)$:

$$\mathcal{T}_s^r \mathcal{V} = \{(p, t) \mid p \in M,\ t \in \mathcal{T}_s^r \mathcal{V}_p\}.$$

In particular, $\mathcal{T}_1^0 \mathcal{V}$ is equal to the dual bundle $\mathcal{V}^*$.

If $\tau : \mathcal{V}_\Omega \to \Omega \times V, (p, v) \mapsto (p, \tau_p(v))$ is a trivialization of the vector bundle $\mathcal{V}$ over an open subset Ω, then we define the mapping $\mathcal{T}_s^r \tau : (\mathcal{T}_s^r \mathcal{V})_\Omega \to \Omega \times \mathcal{T}_s^r V$ by

$$\mathcal{T}_s^r \tau(p, t) = (p, (\tau_p)_*(t)).$$

One readily verifies that $\mathcal{T}_s^r \mathcal{V}$ has a unique structure of vector bundle, for which the $\mathcal{T}_s^r \tau$ are local trivializations. For details we refer the reader to [Lang2, § 3.4].[2]

Is $(\sigma_1, \dots, \sigma_r)$ a frame of $\mathcal{V}$ on an openb subset $\Omega \subset M$, then we define the sections $\sigma^1, \dots, \sigma^r$ of the dual bundle $\mathcal{V}^*$ by

$$\sigma^i(p)(\sigma_j(p)) = \delta_j^i \qquad (p \in \Omega,\ 1 \leq i, j \leq r).$$

[2] S. Lang. Differential manifolds. Addisson-Wesley, 1972

From this we see that $\{\sigma^i(p) \mid 1 \leq i \leq r\}$ is the basis of $\mathcal{V}_p^*$, dual to the basis $\{\sigma_i(p) \mid 1 \leq i \leq r\}$ of $\mathcal{V}_p$. In particular, $(\sigma^1, \ldots, \sigma^r)$ is a frame of the dual bundle $\mathcal{V}^*$ over Ω. This frame is said to be the dual of $(\sigma_1, \ldots, \sigma_r)$.

If $i \in \mathcal{I}_r$ and $j \in \mathcal{I}_s$, we define the section $\sigma_i^j \in \Gamma(\Omega, \mathcal{T}_s^r\mathcal{V})$ by

$$\sigma_i^j(p) = \sigma_{i(1)}(p) \otimes \cdots \otimes \sigma_{i(r)}(p) \otimes \sigma^{j(1)}(p) \otimes \cdots \otimes \sigma^{j(s)}(p). \tag{14}$$

From Lemma 2.4 it now follows that the elements (14) form a basis of $\mathcal{T}_s^r\mathcal{V}_p$, for every $p \in \Omega$. The elements σ_i^j, $i \in \mathcal{I}_r, j \in \mathcal{I}_s$, therefore constitute a frame for the bundle $\mathcal{T}_s^r\mathcal{V}$ on Ω.

An arbitrary element $T \in \Gamma(\Omega, \mathcal{T}_s^r\mathcal{V})$ now has a unique decomposition of the form

$$T = \sum_{i,j} T^{i(1)\ldots i(r)}_{j(1)\ldots j(s)} \, \sigma_{i(1)} \otimes \cdots \otimes \sigma_{i(r)} \otimes \sigma^{j(1)} \otimes \cdots \otimes \sigma^{j(s)},$$

with components $T^{i(1)\ldots i(r)}_{j(1)\ldots j(s)} \in C^\infty(\Omega)$.

In particular, the above definitions apply to $\mathcal{V} = TM$, de tangent bundle of M. The bundle $\mathcal{T}_s^r TM$ is called the tensor bundle of type (r, s) on M. Its sections are called tensor fields, or briefly tensors, of type (r, s) on M.

We finish this section with a description of tensors by means of components, as is customary in physics.

If $x = (x^1, \ldots, x^n) : \Omega_x \to \mathbb{R}^n$ is a local coordinate chart for M, then the vector fields $\frac{\partial}{\partial x^i}$, $1 \leq i \leq n$, form a frame of the tangent bundle TM, defined on Ω_x. Moreover, the one forms dx^i, $1 \leq i \leq n$, form a frame for the cotangent bundle een frame van de coraakbundel T^*M, also defined on Ω_x. Finally,

$$dx^i(p)\left(\frac{\partial}{\partial x^j}\right)_p = \delta^i_j,$$

so that the frame $(dx^i \mid 1 \leq i \leq n)$ is dual with respect to the frame $(\frac{\partial}{\partial x^i} \mid 1 \leq i \leq n)$. Every tensor $T \in \Gamma(\mathcal{T}_s^r TM)$ therefore has a unique expression of the following form on Ω_x,

$$T = \sum_{i,j} {}_xT^{i(1)\ldots i(r)}_{j(1)\ldots j(s)} \, \frac{\partial}{\partial x^{i(1)}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i(r)}} \otimes dx^{j(1)} \otimes \cdots \otimes dx^{j(s)}. \tag{15}$$

Here the coefficients ${}_xT^{i(1)\ldots i(r)}_{j(1)\ldots j(s)}$ belong to $C^\infty(\Omega_x)$. If $y = (y^1, \ldots, y^n) : \Omega_y \to \mathbb{R}^n$ is a second local coordinate chart of M, then T has on Ω_y a representation with components ${}_yT^{k(1)\ldots k(r)}_{l(1)\ldots l(s)}$. In the following passage we discuss the connection between the components of T relative to x and to y on the intersection of the coordinate neighborhoods Ω_x and Ω_y. Let $\{e_i \mid 1 \leq i \leq n\}$ be the standard basis of $\mathbb{R}^n$ and let $\{e^i \mid 1 \leq i \leq n\}$ be the dual basis of $(\mathbb{R}^n)^*$. Let p be a point in the intersection of Ω_x and Ω_y. Then the map $P := (Dx)_p : T_pM \to \mathbb{R}^n$ is a linear isomorphism which sends the basis $(\frac{\partial}{\partial x^i})_p$ to the standard basis e_i. The map $P_* = P^{*-1}$ sends the corresponding dual basis $dx^i(p)$ to the standard dual basis e^i. The map P_* sends the tensor $T(p)$ to

$${}_xT(p) := \sum_{i,j} {}_xT^{i(1)\ldots i(r)}_{j(1)\ldots j(s)}(p) \; e_{i(1)} \otimes \cdots \otimes e_{i(r)} \otimes e^{j(1)} \otimes \cdots \otimes e^{j(s)}.$$

Let $Q = (Dy)_p$. Then we have, analogously to the above, with ${}_yT(p) = Q_*(T(p))$, that

$${}_yT(p) = \sum_{k,l} {}_yT^{k(1)\ldots k(r)}_{l(1)\ldots l(s)}(p) \; e_{k(1)} \otimes \cdots \otimes e_{k(r)} \otimes e^{l(1)} \otimes \cdots \otimes e^{l(s)}.$$

On the other hand,

$$ {}_yT(p) = A_*[{}_xT(p)], \tag{16} $$

where $A = Q{\circ}P^{-1} = (Dy)_p{\circ}(Dx)_p^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. The components of A with respect to the standard basis of $\mathbb{R}^n$ are given by

$$ A^i_j = e^i(Ae_j) = [Q^*(e^i)](P^{-1}e_j) = (dy^i)_p \left(\frac{\partial}{\partial x^j}\right)_p = \frac{\partial y^i}{\partial x^j}(p). \tag{17} $$

Likewise, the components of the inverse matrix A^{-1} are given by:

$$ [A^{-1}]^i_j = \frac{\partial x^i}{\partial y^j}(p). \tag{18} $$

Combining (16), (17) and (18) with (13) we find the following transformation rule on $\Omega_x \cap \Omega_y$:

$$ {}_yT^{i(1)\dots i(r)}_{j(1)\dots j(s)} = \sum_{k,l} \frac{\partial y^{i(1)}}{\partial x^{k(1)}} \cdots \frac{\partial y^{i(r)}}{\partial x^{k(r)}} \frac{\partial x^{l(1)}}{\partial y^{j(1)}} \cdots \frac{\partial x^{l(s)}}{\partial y^{j(s)}} \, {}_xT^{k(1)\dots k(r)}_{l(1)\dots l(s)}. \tag{19} $$

Conversely, let $\mathcal{A}$ be an atlas of M, and assume that for every $x \in \mathcal{A}$ a collection C^∞-functions ${}_xT^{i(1)\dots i(r)}_{j(1)\dots j(s)} \in C^\infty(\Omega_x)$ is given, for $i \in \mathcal{O}_r$ en $j \in \mathcal{I}_s$, such that for all $x, y \in \mathcal{A}$ the transformation rules (19) are valid on the intersection of Ω_x and Ω_y. Then there exists a unique $T \in \Gamma(\mathcal{T}^r_s TM)$ such that in every chart $x \in \mathcal{A}$ the formula (15) holds. This is the way in which tensors are usually described in physics.