# Lie groups and homogeneous spaces

Basefield $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$\mathfrak{g}$ Lie alg / $\mathbb{K}$, $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ Killing form,
$(X, Y) \longmapsto tr_{\mathbb{K}}(\,ad(X)\,ad(Y))$.

$\mathfrak{g}$ semisimple: dir sum of simple Lie algebras
$\quad$ L no ideals but $0$, $\mathfrak{g}$
$\quad$ non-abelian.

L.
$\Leftrightarrow$ B non-degenerate $\Rightarrow$ center$(\mathfrak{g}) = 0$.

(reductive: $\mathfrak{g} = \underset{\text{abelian}}{\mathfrak{z}} \oplus \underset{\text{semisimple}}{\mathfrak{g}_1}$ abelian ).

G connected Lie group over $\mathbb{K}$ with Lie alg $\mathfrak{g}$
Then

(find cover)

$$\begin{tikzcd} G \arrow[r, two heads, "Ad"] & Aut(\mathfrak{g})_e < GL(\mathfrak{g}) \\ \widetilde{Aut(\mathfrak{g})_e} \arrow[u, dashed] \arrow[ur] & \end{tikzcd}$$ (coverings)

1) $\mathbb{K} = \mathbb{C}$: covering finite, G linear algebraic
2) $\mathbb{K} = \mathbb{R}$, $B < 0$ $\widetilde{Aut(\mathfrak{g})_e}$ cpt.

Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ maximal subject to
- abelian
- $\forall_{X \in \mathfrak{h}}$ $ad(X)$ diagonalizable over $\overline{\mathbb{K}} = \mathbb{C}$.

$K = \mathbb{C}$: 1 conj class of CSA
$K = \mathbb{R}$: finitely many
$\Big\}$ $\underset{\text{indep}^t \text{ of } \mathfrak{h}}{\dim \mathfrak{h}} =: rk(\mathfrak{g})$.

$\mathbb{K} = \mathbb{C}$, $\mathfrak{g}$ cpx semisimple
cpx $\quad \mathfrak{t} \subset \mathfrak{g}$ Cartan

For $\lambda \in \mathfrak{t}^*$: $\quad \mathfrak{g}_\lambda := \{ X \in \mathfrak{g} \mid \forall_{H \in \mathfrak{t}} \ \mathrm{ad}\,H \cdot X = \lambda(H) X \}$

Roots: $\quad R = R(\mathfrak{g}, \mathfrak{t}) = \{ \alpha \in \mathfrak{t}^* \mid \alpha \neq 0,\ \mathfrak{g}_\alpha \neq 0 \}$

Root space deco: $\quad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \underbrace{\mathfrak{g}_\alpha}_{\text{1-dim!}}$

$R^+$ choice of positive system

$\mathfrak{n} := \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$, $\quad \mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ subalgebra

maximal solvable:
<u>fact</u>: $\mathfrak{b}$ max solvable, = Borel subalgebra.

<u>Lemma</u> $N_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{b}$ (Pf: exercise).

<u>Cor</u> $B = N_G(\mathfrak{b})$ closed subgroup with alg $\mathfrak{b}$.

<u>Example</u>

$G = SL(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid \mathrm{tr}\,A = 0 \}$

$\mathfrak{t} = \{$ diag matrices in $\mathfrak{sl}(n, \mathbb{C}) \}$

$\varepsilon_i \in \mathfrak{t}^*$: $H \mapsto H_{ii}$ $\qquad R = \{ \underbrace{\varepsilon_i - \varepsilon_j}_{\alpha_{ij} =} \mid i \neq j \}$

$\mathfrak{g}_{\alpha_{ij}} = \mathbb{C} E_{ij}$ $\qquad E_{ij} = \begin{array}{cc} & j \\ i & \begin{pmatrix} & 1 & \end{pmatrix} \end{array}$

$R^+ = \{ \alpha_{ij} \mid i < j \}$ positive system

$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$, $\quad \mathfrak{b} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \middle| \ \mathrm{tr} = 0 \right\}$

$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \middle| \ \det = 1 \right\}$.

<u>Lemma</u> All Borel subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{Aut}(\mathfrak{g})_e$ (hence $\mathrm{Ad}(G)$).

Let $\mathfrak{B} = \{ \mathfrak{b}' \subset \mathfrak{g} \mid \mathfrak{b}' \text{ Borel subalgebra} \}$

$$\subseteq Gr_k(\mathfrak{g}), \quad k = \dim \mathfrak{b} = \mathrm{rk}(\mathfrak{g}) + \# R^+$$

Then $\mathfrak{B}$ smooth proj variety (full flag mfd see exercise)

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[d] & \mathfrak{B}, \quad g \mapsto \mathrm{Ad}(g)\mathfrak{b} \\ G/N_G(\mathfrak{b}) \arrow[ur, "\simeq"'] \arrow[d, equal] & \\ G/B & \end{tikzcd}$$

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<u>Def</u> $H < G$ closed cpx subgroup called spherical iff $G/H$ has open $B$-orbit.
(<u>Lemma</u> then finitely many $B$-orbits, $\exists!$ open orbit)

<u>Note:</u> $BgH$ open $\Rightarrow$ $g^{-1}Bg\,H$ open $\Rightarrow$ $B'H$ open for a Borel subgp $B'$

<u>Note:</u> $G/H$ has open $B$-orbit $\Leftrightarrow$ $G/B$ has open $H$-orbit

<u>Examples</u>

1) $B$ is spherical, $G = \coprod_{w \in W} BwB$ (Bruhat decomposition)
$W = W(R) = N_G(\mathfrak{t}) / Z_G(\mathfrak{t})$.

dim of $BwB$ in $G/B$ :

$$\begin{tikzcd} B \arrow[r] \arrow[d] & G/B, \quad b \mapsto bwB \\ B/_{wBw^{-1}\cap B} \arrow[ur, hook] & \end{tikzcd}$$

$$\begin{aligned} \dim &= \dim(\mathfrak{b}/\mathfrak{b}\cap \mathrm{Ad}(w)\mathfrak{b}) \\ &= \dim(\mathfrak{n}/\mathfrak{n}\cap \mathrm{Ad}(w)\mathfrak{n}) \\ &= \dim(\mathfrak{n}\cap \mathrm{Ad}(w)\bar{\mathfrak{n}}) \\ &= \#\{\alpha\in R^+ \mid w^{-1}\alpha<0\} \end{aligned}$$

orbit open for longest Weyl group element. (exercises).

2) There is compact algebra $\mathfrak{u}\subset\mathfrak{g}$ s.t. $\mathfrak{g}=\mathfrak{u}\oplus i\mathfrak{u}$, CSA: $\mathfrak{t}_{\mathbb{C}}$ with $\mathfrak{t}\subset\mathfrak{u}$ max torus. $U$ connected subgp of $G$ ($/\mathbb{R}$)

**Fact:** $G = UB$, so $U$ is spherical, but <u>not</u> algebraic

3) $\sigma\in \mathrm{Aut}(G)$, $\sigma^2 = I$ : involution

$G^\sigma$ is spherical ($G/G^\sigma$ complex symmetric space)

4) Special case of 3): $G = {}'G\times {}'G$,

$\sigma:(x,y) = (y,x)$

$$\begin{tikzcd} {}'G\times{}'G \arrow[r] \arrow[d] & {}'G \quad (x,y)\mapsto xy^{-1} \\ G/G^\sigma \arrow[ur, "1-1"'] & \end{tikzcd}$$

See <u>exercises</u>

## Connection with rep thy

Def $G$ Lie grp $/ \mathbb{K}$, $H < G$ closed subgrp $/ \mathbb{K}$

$(\xi, V_\xi)$ continuous finite dimensional rep$^n$.

$$G \times V_\xi \circlearrowleft H, \quad (g, v) \cdot h = (gh, \xi(h)^{-1} v)$$

(proper & free, $G \to G/H$ principal fiber bundle)

$$\mathcal{V}_\xi = G \overset{H}{\times} V_\xi := G \times V_\xi / H$$

$pr_1 : G \times V_\xi \to G$ induces $G \overset{H}{\times} V_\xi \xrightarrow{p} G/H$

associated vector bundle : unique s.t.

$$\begin{tikzcd} G \times V_\xi \arrow[r, "\tilde{\pi}"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\circlearrowleft"] & G \overset{H}{\times} V_\xi \arrow[d, "p"] \\ G \arrow[r, "\pi"] & G/H \end{tikzcd} \quad \text{pull-back}$$

$$\begin{aligned} \Gamma(\mathcal{V}_\xi) &:= \mathcal{O}(G/H, \mathcal{V}_\xi) && \text{if } \mathbb{K} = \mathbb{C} \\ &= C^\infty(G/H, \mathcal{V}_\xi) && \text{if } \mathbb{K} = \mathbb{R}. \end{aligned}$$

left $G$-action of $G$ on $G \times V_\xi$ induces

$\tilde{l} : G \times \mathcal{V}_\xi \to \mathcal{V}_\xi$ s.t. $\forall g \in G$

$$\begin{tikzcd} \mathcal{V}_\xi \arrow[r, "\tilde{l}_g"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{V}_\xi \arrow[d] \\ G/H \arrow[r, "l_g"] & G/H \end{tikzcd} \quad \text{VB isomorphism.}$$

$\mathcal{V}_\xi$ : equivariant vector bundle.

$\rightsquigarrow$ natural representation of

G in $\Gamma(G/H, \mathcal{V}_\xi)$, denoted $\underbrace{\mathrm{ind}_H^G}_{\text{induced rep}^n}(\xi) = \pi_\xi$

Given by: $(\pi_\xi(g)s)(x) = \tilde{\ell}_g\big(\underbrace{s(g^{-1}x)}_{\in \mathcal{V}_{\xi, g^{-1}x}}\big)$.

<u>Remark:</u>

$\pi^* : \Gamma(G/H, \mathcal{V}_\xi) \hookrightarrow \Gamma(G, G\times V_\xi) \simeq$

$\simeq \mathcal{O}(G, V_\xi)$

has image

$$\mathcal{O}(G, V_\xi)^H := \{ f : G \xrightarrow{\mathcal{O}} V_\xi \mid f(gh) = \xi(h)^{-1} f(g) \}$$

In this picture $\pi_\xi = \mathrm{ind}_H^G(\xi)$ is given by left regular rep$^n$: $\pi_\xi(g)\varphi(x) = \varphi(g^{-1}x)$.

<u>Rep theory</u> for of complex semisimple.

$\Lambda \subset \mathfrak{h}^*$ weight lattice, $\Lambda^+$ dominant wts

$\lambda \in \Lambda^+ \longrightarrow$ unique f.d. irreducible rep$^n$ $(\delta_\lambda, V_\lambda)$ of highest weight $\lambda$.

<u>Lemma</u> $\delta_\lambda$ lifts to a rep$^n$ of $G \iff$

$\iff \lambda$ lifts to $\mathbb{T} = \exp \mathfrak{t} < G$, i.e.

$\exists\ \xi_\lambda : \mathbb{T} \to \mathbb{C}^*$; $\lambda = d\xi(e)$.

<u>Notation</u> $\Lambda^+(T) = \{\lambda \in \Lambda^+ \mid \lambda \text{ lifts to } T\}$

$\Lambda^+(T) \ni \lambda \overset{1-1}{\longmapsto} \tilde{\delta}_\lambda \in \text{irreps}(G)$.

<u>Borel-Weil</u>:

Given $\lambda \in \Lambda(T)$, let $\mathbb{C}_\lambda$ be $\mathfrak{t}$-module of weight $\lambda$, extend to $\mathfrak{b}$-module. Lift to $B$-module. i.e. $b \cdot z = \xi_\lambda(b) z$, $\xi_\lambda(tn) = \xi_\lambda(t)$.

Define $\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda$.

$$\mathcal{O}(G/B, \mathcal{L}_\lambda) \cong \{\varphi \in \mathcal{O}(G) \mid \varphi(gb) = \xi_\lambda(b)^{-1}\varphi(g)\}$$

<u>Thm</u> Let $\lambda \in \Lambda(T)$. (Borel-Weil)

(a) if $\lambda \notin \Lambda^+(T)$ then $\mathcal{O}(G/B, \mathcal{L}_{-\lambda}) = 0$

(b) if $\lambda \in \Lambda^+(T)$ then $\mathcal{O}(G/B, \mathcal{L}_{-\lambda}) \simeq \delta_\lambda^\vee$.

<u>Proof.</u> $G/B$ cpt, cpx $\Rightarrow \dim \mathcal{O}(G/B, \mathcal{L}_{-\lambda}) < \infty$.

$\mathcal{O}(G/B, \mathcal{L}_{-\lambda}) \cong \delta_1 \oplus \dots \oplus \delta_n$ (deco in irreps).

$n = \dim \mathcal{O}(G/B, \mathcal{L}_{-\lambda})^{\bar{N}} \leftarrow$ negative roots

$\bar{N}$ open orbit, therefore $n \leq 1$.

So, $\mathcal{O}(G/B, \mathcal{L}_{-\lambda}) = 0$ or irreducible.

$ev_1 \in \mathcal{O}(G/B, \mathcal{L}_{-\lambda})^* : f \longmapsto f(e)$

$(b \cdot ev_1) f = ev_1 L_b^{-1} f := f(b) = \xi_\lambda(b)^{-1} f(e)$

$= \xi_\lambda(b)\, ev_1(f)$. So $ev_1$ highest weight vector of weight $\lambda$.

We see that $\mathcal{O}(G/B, \mathcal{L}_{-\lambda})^* \simeq \delta_\lambda$. $\boxtimes$

Exercise $U(\mathfrak{g}) \to \mathcal{O}(G/B, \mathcal{L}_{-\lambda})^*$ gives iso.

Lemma Let $H$ be spherical. Then
$\forall \delta$ irrep of $G$: $\dim V_\delta^H \leq 1$.

Proof choose Borel s.t. $HB$ open in $G/B$.
Sufficient: $\forall_{\lambda \in \Lambda(T)}$ $\dim \mathcal{O}(G/B, \mathcal{L}_{-\lambda})^H \leq 1$.
If $\varphi \in \mathcal{O}(G/B, \mathcal{L}_{-\lambda})^H = (\mathcal{O}(G) \otimes \mathbb{C}_{-\lambda})^{B, H}$
and $\varphi(e) = 0$ then $\varphi = 0$ on $HB \Rightarrow \varphi \equiv 0$.
So $ev_1 : \mathcal{O}(G/B, \mathcal{L}_{-\lambda})^H \to \mathbb{C}$ injective $\boxtimes$

Remark if $H < G$ algebraic & $\forall_{\delta \in \widehat{G}}$
$\dim V_\delta^H \leq 1$ then $H$ spherical (Thm).
Classification M. Brion

The real setting

$G$ real connected, semisimple.

Example $SL(n, \mathbb{R})$.

Cartan involution of $\mathfrak{g}$: $\theta \in \mathrm{Aut}(\mathfrak{g})$, $\theta^2 = 1$
$B < 0$ on $\mathfrak{g}^\theta$, $B > 0$ on $\underbrace{\ker(\theta + I)}$.
$\mathfrak{g}^\theta = \mathfrak{k}$, $\ker(\theta+I) =: \mathfrak{s}$

$$\mathfrak{g} = \underbrace{\overset{}{\underset{+1}{\mathfrak{k}}} \oplus \underset{-1}{\mathfrak{s}}}_{\theta}$$

$B < 0$ on $\mathfrak{k}$, $B > 0$ on $\mathfrak{s}$
$[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$, $\mathrm{Ad}(K)$ ... $\mathfrak{s}$
$[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$

Let $K = \langle \exp \mathfrak{k} \rangle_{gr}$. Then $\mathrm{Ad}(K) : \mathfrak{s} \to \mathfrak{s}$ and

<u>Theorem</u> (Cartan deco)

$K \times \mathfrak{s} \longrightarrow G$, $(k, X) \longmapsto k \exp X$ is diffeo.

<u>Cor</u> 1) $\exists! \tilde{\theta}$ invol of $G$ : $d\tilde{\theta}(e) = \theta$

2) $K = G^{\theta}$

3) $\# Z(G) < \infty \Rightarrow K$ compact. | Write $\tilde{\theta} = \theta$.

<u>Example</u>

$\theta : \mathfrak{g} \to \mathfrak{g}$, $x \longmapsto -x^T$

$G = SL(n, \mathbb{R})$, $\theta : G \to G$, $x \longmapsto (x^{-1})^T$

$K = SO(n)$, $\mathfrak{s} = \mathfrak{s}_n = \{ X \in M_n(\mathbb{R}) \mid \mathrm{tr}\, X = 0,\ X^T = X \}$

$SO(n) \times \mathfrak{s}_n \longrightarrow SL(n, \mathbb{R})$ is diffeomorphism (see exercises).

<u>$G/K$ as Riemannian symm space</u>

Select $\beta = \langle \cdot, \cdot \rangle$ on $\mathfrak{s}$, $> 0$, $\mathrm{Ad}(K)$-inv't.

e.g. $\beta = B$ (Killing)

Identify $\mathfrak{s} \simeq \mathfrak{g}/\mathfrak{k} \simeq T_{[e]} G/K$.

$$g_{[x]} = d\ell_x(\overset{*}{e})^{-1 *} \beta \in \otimes^2 T^*_{[x]} G/K$$

defines $G$-invariant metric on $G/K$.

<u>Thm:</u> The Riemannian $\mathrm{Exp}_{[e]} : \mathfrak{s} \to G/K$ is given by $X \longmapsto \exp X \cdot [e]$.

Cor: $\operatorname{Exp}_{[e]} : \mathfrak{s} \to G/K$ diffeomorphism.

Let $\bar{\theta} : G/K \to G/K$ be induced by $\theta$.

Lemma The following diagram commutes

$$\begin{array}{ccc} G/K & \xrightarrow{\bar{\theta}} & G/K \\ {\scriptstyle \operatorname{Exp}_{[e]}} \uparrow & \circlearrowleft & \uparrow {\scriptstyle \operatorname{Exp}_{[e]}} \\ \mathfrak{s} & \xrightarrow{-I} & \mathfrak{s} \end{array}$$

Hence $\bar{\theta} = S_{[e]}$

geodesic reflection at $[e]$.

Lemma $\theta$ is isometry.

Cor $\forall_{a \in G/K}$ $S_a$ extends to isometry

i.e. $G/K$ globally Riemannian symm space

( $\overset{\text{E. Cartan}}{\iff}$ $G/K$ geodesically complete & $\nabla R = 0$ ).

Curvature: $R_e : \mathfrak{s} \times \mathfrak{s} \to O(\mathfrak{s})$

$(X, Y) \longmapsto \operatorname{ad}([X,Y])|_{\mathfrak{s}}$.

Sectional curvature $X, Y \in \mathfrak{s}$ orthonormal

$$\begin{aligned} K_e(X,Y) &= B(R(X,Y)X, Y) \\ &= B([[X,Y],X], Y) \\ &= B([X,Y],[X,Y]) \leq 0. \end{aligned}$$

Holonomy group subgroup of $O(\mathfrak{s})$ generated by Lie algebra $\operatorname{ad}([\mathfrak{s},\mathfrak{s}])|_{\mathfrak{s}}$.

More generally, if $\sigma : G \to G$ is an involution, $(G^\sigma)_e < H < G^\sigma$ then $G/H$ is pseudo Riemannian globally symmetric space. ~~Let~~

Remark: $\langle\cdot,\cdot\rangle : (X,Y) \mapsto -B(X,\theta Y)$ is positive inner product on $\mathfrak{g}$. $K$-invariant.

If $X \in \mathfrak{s}$ then $(\mathrm{ad}\, X)^T = \mathrm{ad}\, X \longrightarrow$ diagonalisable

Fix $\mathfrak{a} \subset \mathfrak{s}$ maximal abelian subspace

L: All such are $\mathrm{Ad}(K)$-conjugate

For $\lambda \in \mathfrak{a}^*$, $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \forall_{H \in \mathfrak{a}} \; [H,X] = \lambda(H)X\}$

$\Sigma = \Sigma(\mathfrak{g},\mathfrak{a})$ roots $:= \{\alpha \in \mathfrak{a}^* \mid \alpha \neq 0,\ \mathfrak{g}_\alpha \neq 0\}$.

Remark: $\dim \mathfrak{g}_\alpha > 1$ possible

$\{\alpha, 2\alpha\} \subset \Sigma$ possible

$\Sigma_0 = \{\alpha \in \Sigma \mid \alpha/2 \notin \Sigma\}$ genuine roots

Root space deco: $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$

$\theta = -1$ on $\mathfrak{a}$, $\mathfrak{g}_0$ $\theta$-stable, $= \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{a} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$.

fix $\Sigma^+$ positive system

Lemma: $\mathfrak{s} = \mathrm{Ad}(K)\overline{\mathfrak{a}^+}$ ← unique $\qquad \overline{A^+} := \exp \overline{\mathfrak{a}^+}$

Cor: $G = K\overline{A^+}K$ ← unique $\qquad$ Pf: $x \in G$: $x = k \exp X$

$= k \exp \mathrm{Ad}(k')H$

$= kk' \exp H (k')^{-1}$ ∎.

<u>Def</u> $\mathfrak{n} := \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n} = \mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$

<u>Example</u>

$\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{s} = \mathfrak{s}_n = \{X\in M_n(\mathbb{R}) \mid X^T = X\}$

$\mathfrak{a} = \left\{ \begin{pmatrix} * & \\ & \ddots \end{pmatrix} \mid tr = 0\right\}$

$\mathfrak{n} = \left\{\begin{pmatrix}0 & * \\ 0 & 0\end{pmatrix}\right\}$

$\mathfrak{p} = \left\{\begin{pmatrix} * & * \\ 0 & *\end{pmatrix} \mid tr=0\right\}$

Note: $\mathfrak{p}_\mathbb{C}$ is Borel in $\mathfrak{g}_\mathbb{C}$.

<u>Def</u> parabolic subalgebra of $\mathfrak{g}$: subalgebra $\mathfrak{q}\subset\mathfrak{g}$ s.t. $\mathfrak{q}_\mathbb{C} \supset$ a Borel of $\mathfrak{g}_\mathbb{C}$.

<u>Lemma</u> $\mathfrak{p}$ is a minimal p.s.a of $\mathfrak{g}$
(all such are $G$-conjugate).

<u>Pf</u>: fix max torus $\mathfrak{t}\subset\mathfrak{m}$. Then $\mathfrak{j} = \mathfrak{t}\oplus\mathfrak{a}$ is C.S.A of $\mathfrak{g}$. Then $\Sigma \overset{?}{=} R|_\mathfrak{a} \setminus \{0\}$.
$(R = R(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}))$. Fix $R^+$ s.t. $\Sigma^+ = R^+|_\mathfrak{a}\setminus\{0\}$.
$\rightsquigarrow \mathfrak{b}$ and

$$\mathfrak{p}_\mathbb{C} = \mathfrak{z}_\mathfrak{g}(\mathfrak{a})_\mathbb{C} \oplus \bigoplus_{\substack{\alpha\in R^+\\ \alpha|_\mathfrak{a}\neq0}} \mathfrak{g}_{\mathbb{C}\alpha}$$

$$\supseteq \mathfrak{j}_\mathbb{C} \oplus \bigoplus_{\substack{\alpha\in R\\ \alpha|_\mathfrak{a}=0}} \mathfrak{g}_{\mathbb{C}\alpha} \oplus \bigoplus_{\substack{\alpha\in R^+\\ \alpha|_\mathfrak{a}\neq0}} \mathfrak{g}_{\mathbb{C}\alpha} \supseteq \mathfrak{b}_\mathbb{C}.$$

☒

<u>Lemma</u> $\mathfrak{a} < \mathfrak{g}$ psa $\Rightarrow \mathcal{N}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a}$

<u>Def</u> Associated p.s.gp: $Q := N_G(\mathfrak{a})$

Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ as before

<u>Lemma</u> $P = \underset{\uparrow \text{ Langlands decomp}}{MAN} \simeq M \times A \times N$

where: $M = Z_K(\mathfrak{a})$, $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$

( $\exp : \mathfrak{a} \xrightarrow{\simeq} A$, $\mathfrak{n} \xrightarrow{\simeq} N$ ).

<u>Lemma</u>$^{(*)}$ $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (Iwasawa)

<u>Thm</u> $G = KP = KAN \simeq K \times A \times N$

<u>Cor</u> $K/M \simeq G/P$ (compact)

all min parabs of $\mathfrak{g}$ are $K$-conjugate.

<u>Example</u> $G = SL(n, \mathbb{R})$, $K = SO(n)$,

$A = \left\{ \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_n} \end{pmatrix} \middle| \sum t_j = 0 \right\}$, $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$.

$M = \left\{ \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} \middle| \det = 1 \right\}$, $P = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

$P = MAN$, $G = KAN$.

<u>Def.</u> $H < G$ closed subgroup called (real)

<u>spherical</u> if $G/H$ has open $P$-orbit.

$\left( \Leftrightarrow \exists_{x \in G}\ x^{-1} P x H \text{ open} \Leftrightarrow \exists_{x \in G}\ \mathrm{Ad}(x)\mathfrak{p} + \mathfrak{h} = \mathfrak{g}. \right.$

<u>Lemma</u> $K$ is spherical.

<u>Pf.</u> Show that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (see exercises).

<u>Examples</u>

1) $P$ is spherical (Bruhat $G = \coprod_{w \in W(\Sigma)} PwP$)

$(W(\Sigma) \simeq N_K(\mathfrak{a}) / Z_K(\mathfrak{a}))$

2) $K$ is spherical

3) $\sigma : G \to G$ involution, then $H = (G^\sigma)_e$ called <u>symmetric</u>. Is spherical.
Special case: diag spherical in $G \times G$

4) $N$ is spherical.

<u>Thm</u> (Krötz - Schlichtkrull) Assume $H < G$ closed, connected. Then

$$H \text{ spherical} \iff \#(P \backslash G / H) < \infty$$

$\uparrow$ conjecture Matsuki

<u>Recall</u> from complex case

$$\mathbb{C}[G/H] \simeq \bigoplus_{\delta \in \widehat{G}} V_\delta \otimes (V_\delta^*)^H$$

$$(V_\delta^*)^H \simeq \mathcal{O}(G/B, \mathcal{L}_\delta)^H$$

## Importance of $P$ for rep thy.

**Lemma** $G$ simple not compact, $(\pi, \mathcal{H})$ unitary rep. Then $\dim \mathcal{H} < \infty \Rightarrow \pi = 1$.

**Pf.** Assume $\dim \mathcal{H} < \infty$. Then $G \xrightarrow{\pi} U(\mathcal{H})$ maps onto simple subgp $\Rightarrow \pi(G)$ compact $\Rightarrow B_{\pi_*(G)} < 0 \Rightarrow \pi_* = 0 \Rightarrow \pi = 1$. ☒

Must consider $\infty$ dim reps.

**Def** $(\pi, V)$ continuous rep of $G$ in Banach space. Then $V^\infty := \{ v \in V \mid x \mapsto \pi(x) v,\ G \xrightarrow{C^\infty} V \}$

**Note:** $G \curvearrowright V^\infty$. $\pi^\infty : G \times V^\infty \to V^\infty$.

**Thm** (Harish-Chandra, Casselman). Let $(\pi, V)$ be irrep in Banach space. Then $\exists\ (\omega, V_\omega)$ fin. dim irrep of $P$ s.t.

$$V^\infty \overset{G}{\hookrightarrow} C^\infty \operatorname{ind}_P^G(\omega) = \Gamma^\infty(G/P, \mathcal{V}_\omega).$$

In general, $\Gamma^\infty(G/P, \mathcal{V}_\omega)^H = 0$. Look at $\Gamma^{-\infty}(\ \ )^H$!

**Thm** (vdB) if $H$ symmetric then

$$\dim \Gamma^{-\infty}(G/P, \mathcal{V}_\omega)^H < \infty.$$

($\Rightarrow$ finite multiplicities in Planch formula).