The Harish-Chandra transform for Whittaker functions

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Whittaker functions

Setting

- G real reductive group
- K maximal compact, $G = KAN_0$ Iwasawa decomposition
- $\chi : N_0 \to \mathrm{U}(1)$ unitary character, regular (!)

$$\text{i.e.: } \forall \alpha \in \Sigma(\mathfrak{n}_0, \mathfrak{a}) \text{ simple: } d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0.$$

Whittaker functions

$$\mathcal{M}(G/N_0, \chi) := \{f : G \xrightarrow{\text{meas}} \mathbb{C} \mid f(xn) = \chi(n)^{-1} f(x) \quad (x \in G, n \in N_0)\}$$

$$L^2(G/N_0, \chi) := \{f \in \mathcal{M}(G/N_0, \chi) \mid |f| \in L^2(G/N_0)\}$$

- Left reg^r rep^n: $L = \mathrm{Ind}_{N_0}^{G}(\chi)$ is unitary

Whittaker Plancherel formula

Abstractly

- $\mathrm{Ind}_{N_0}^{G}(\chi) = \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$.

Concrete realization

- Harish-Chandra, Announcement 1982.

  Details in Collected Papers Vol 5 (posthumous), 141- 307,
  eds. R. Gangolli, V.S. Varadarajan, Springer 2018.
  Final step not clear.

- Pres^n Bestwig 2020 ($\sim$) : final step by Paley-Wiener technique
- Today: final step by using cusp forms & HC transform
- Important ref: Wallach, RRG II: discrete part, cusp forms, and functional equation and holomorphic dependence of Whittaker vectors

Discrete part of decomposition

Discrete part

$\pi \in \widehat{G}$ (unitary dual) is said to appear discretely in $L^2(G/N_0, \chi)$ if it can be realized as a closed subrepresentation.
The closed span of such π is denoted $L^2_d(G/N_0, \chi)$.

Theorem (HC, W)

If $\pi \in \widehat{G}$ appears in $L^2_d(G/N_0, \chi)$, then it appears in $L^2_d(G)$, i.e., it belongs to the discrete series of G.

Spherical functions

Let (τ, V_τ) be a finite dimensional unitary representation of K.

$$L^2_d(\tau, G/N_0, \chi) := (L^2_d(G/N_0, \chi) \otimes V_\tau)^K$$

$$\hookrightarrow \{f \in \mathcal{M}(G, V_\tau) \mid f(kxn) = \chi(n)^{-1}\tau(k)f(x)\}$$

This space will be characterized as a space of cusp forms

Whittaker functions of Schwartz type

- Define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{N_0})$.

Definition (Schwartz space)

$\mathcal{C}(G/N_0, \chi)$: the space of $f \in C^\infty(G/N_0, \chi)$ s.t. $\forall u \in U(\mathfrak{g}), N \in \mathbb{N}$,

$$|L_u f(kan)| \leq C_{u,N}\,(1 + |\log(a)|)^{-N} a^{-\rho} \qquad (kan \in KAN_0).$$

Property: $C_c^\infty(G/N_0, \chi) \subset \mathcal{C}(G/N_0, \chi) \subset L^2(G/N_0, \chi)$.

- $P_0 := Z_K(A)A\,N_0$, minimal psg.
- $\mathcal{P}_{st}$: (finite) set of psg's $P < G$ with $P \supset P_0$ (standard psg's).
- For $P \in \mathcal{P}_{st}$, Langlands deco: $P = M_P A_P N_P$, $\quad M_{1P} := M_P A_P$.

Lemma (HC, W)

If $f \in \mathcal{C}(G/N_0; \chi)$ and $P \in \mathcal{P}_{st}$ then $\int_{\bar{N}_P} |f(\bar{n})|\, d\bar{n} < \infty$.

The map $f \mapsto \int_{\bar{N}_P} |f(\bar{n})|\, d\bar{n}$ is continuous.

Cusp forms

Definition (Space of cusp forms)

${}^\circ\mathcal{C}(G/N_0,\chi)$:= space of $f \in \mathcal{C}(G/N_0,\chi)$ s.t. $\forall P \in \mathcal{P}_{st}$

$$\int_{\bar{N}_P} f(x\bar{n})\, d\bar{n} = 0, \qquad (x \in G).$$

For (τ, V_τ) a finite dimensional unitary representation of K,

$$ {}^\circ\mathcal{C}(\tau, G/N_0,\chi) := ({}^\circ\mathcal{C}(G/N_0,\chi) \otimes V_\tau)^K. $$

Thm (HC,W) $\quad {}^\circ\mathcal{C}(\tau, G/N_0,\chi) = L^2_d(\tau, G/N_0,\chi).$

The space is finite dimensional.

Harish-Chandra descent transform

For $P \in \mathcal{P}_{st}$ define $d_P : P \to \mathbb{R}^+$ by $d_P(p) := |\det \mathrm{Ad}(p)|_{\mathfrak{n}_P}|^{1/2}$.

Definition (HC transform)

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$ define $f^{(\bar{P})} : M_{1P} \to V_\tau$ by

$$f^{(\bar{P})}(m) := d_P(m) \int_{\bar{N}_P} f(m\bar{n})\, d\bar{n}.$$

Property $\quad f^{(\bar{P})} \in C^\infty(\tau_P, M_{1P}/M_{1P} \cap N_0, \chi_P),$

where $\tau_P := \tau|_{M_{1P}\cap K}, \quad \chi_P := \chi|_{M_{1P}\cap N_0}$.

Thm (HC, W)

For $a \in A_P$ define $R_a(f^{(\bar{P})}) : m \mapsto f^{(\bar{P})}(ma)$. Then

$$R_a(f^{(\bar{P})}) \in \mathcal{C}(\tau_P, M_P/M_P \cap N_0, \chi_P).$$

Role of the Harish-Chandra transform

- For $P \in \mathcal{P}_{st}$ put ${}^\circ\mathcal{C}_{P,\tau} := {}^\circ\mathcal{C}(\tau_P, M_P/M_P \cap N_0, \chi_P)$.

Def (HC)

Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$. Then

$$f^{(\bar{P})} \sim 0 \quad :\Longleftrightarrow \quad R_a(f^{(\bar{P})}) \perp {}^\circ\mathcal{C}_{P,\tau} \ (\forall a \in A_P).$$

More explicitly, the assertion on the right means that for all $a \in A_P$ and all $\psi \in {}^\circ\mathcal{C}_{P,\tau}$,

$$\int_{M_P/M_P \cap N_0} \langle\, f^{(\bar{P})}(ma), \psi(m)\,\rangle_{V_\tau}\, dm = 0.$$

Thm (HC's completeness theorem)

Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$. If $f^{(\bar{P})} \sim 0$ for each $P \in \mathcal{P}_{st}$ then $f = 0$.

Proof of HC's completeness

Thm (HC's completeness theorem)

Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$. If $f^{(\bar{P})} \sim 0$ for each $P \in \mathcal{P}_{st}$ then $f = 0$.

Sketch of proof

1. Transitivity of descent. Let $P, Q \in \mathcal{P}_{st}$ with $P \subset Q$. Then ${}^*P := P \cap M_{1Q}$ is a parabolic subgroup of M_{1Q}. For $f \in \mathcal{C}(\tau, G/N_0, \chi)$, by Fubini:

$$(f^{(\bar{Q})})^{({}^*\bar{P})} = f^{(\bar{P})}.$$

2. Apply induction on $\mathrm{rk}_{\mathbb{R}} G = \dim \mathfrak{a}$. Assume $f^{(\bar{P})} \sim 0$ for all $P \in \mathcal{P}_{st}$. If $Q \in \mathcal{P}_{st}, Q \neq G$ then $\mathrm{rk}_{\mathbb{R}} M_Q < \mathrm{rk}_{\mathbb{R}} G$ hence by (1) and induction hypothesis

$$f^{(\bar{Q})} = 0.$$

   Hence $f \in {}^\circ\mathcal{C}(\tau, G/N_0, \chi) = {}^\circ\mathcal{C}_{G,\tau}$.

3. Now $f^{(\bar{G})} \sim 0$ implies $f \perp f$ hence $f = 0$.

Parabolic induction and Whittaker integrals

Let $P = M_P A_P N_P \in \mathcal{P}_{st}$ and $\psi \in {}^\circ\mathcal{C}_{P,\tau}$. For $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ define $\psi_\lambda : G \to V_\tau$ by

$$\psi_\lambda(kma\bar{n}) = a^{\lambda+\rho_P}\tau(k)\psi(m).$$

For $\mathrm{Re}(\lambda) >_P 0$, the integral

$$\mathrm{Wh}(P, \psi, \lambda, x) := \int_{N_P} \chi(n)\psi_\lambda(xn)\, dn \qquad (x \in G)$$

is abs^y conv^t and defines a function $\mathrm{Wh}(P, \psi, \lambda) \in C^\infty(\tau, G/N_0, \chi)$ which depends holomorphically on λ in the indicated region.

Remark

The above Whittaker integral is essentially a finite sum of generalized matrix coefficients (defined by Jacquet integrals) of $\mathrm{Ind}_P^G(\sigma \otimes -\lambda \otimes 1)$, with $\sigma \in \widehat{M}_{P,\mathrm{ds}}$ appearing in ${}^\circ\mathcal{C}_{P,\tau}$. (Analogue of Eisenstein integral.)

Holomorphic extension

Theorem (W)

$\mathrm{Wh}(P, \psi, \lambda)$, *initially defined for* $\mathrm{Re}\lambda >_P 0$, *extends to entire holom^c function of* $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ *with values in* $C^\infty(\tau, G/N_0, \chi)$.

Remark: HC: there exists a merom^c extension, regular on $i\mathfrak{a}^*_P$.

Theorem ($\sim$): Uniformly tempered estimates

Let $\varepsilon > 0$ *be suff^{t}ly small. If* $u \in U(\mathfrak{g})$ *then* $\exists C, N, r > 0$ *s.t.*

$$|\mathrm{Wh}(P, \psi, \lambda, u; ka)| \leq C(1 + |\lambda|)^N (1 + |\log a|)^N e^{r|\mathrm{Re}\lambda||\log a|} a^{-\rho},$$

for all $k \in K$, $a \in A$, $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ *with* $|\mathrm{Re}\lambda| < \varepsilon$.

- Bernstein-Sato type functional equation for Jacquet integrals.
- Uniformly moderate estimates.
- Wallach's method of improving estimates along max psg's, with parameters.

Fourier transform

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$, $P \in \mathcal{P}_{st}$, $\lambda \in i\mathfrak{a}_P^*$, the Fourier transform $\mathcal{F}_P f(\lambda) \in {}^\circ\mathcal{C}_{P,\tau}$ is defined by

$$\langle \mathcal{F}_P f(\lambda), \psi \rangle := \int_{G/N_0} \langle f(x), \mathrm{Wh}(P, \psi, \lambda, x) \rangle_{V_\tau} \, dx, \quad (\psi \in {}^\circ\mathcal{C}_{P,\tau}).$$

Theorem (∼)

$$\mathcal{F}_P : \mathcal{C}(\tau, G/N_0, \chi) \to \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau},$$

continuous linearly.

Remark: HC proves this for $\mathcal{F}_P$ restricted to $C_c^\infty(\tau, G/N_0, \chi)$.

Proof

this follows from the uniformly tempered estimates.

Remark: $\mathcal{F}_G = \text{orth}^l \text{ proj}^n \quad \mathcal{C}(\tau, G/N_0, \chi) \to {}^\circ\mathcal{C}(\tau, G/N_0, \chi)$.

Relation with the Harish-Chandra transform

Let $\mathcal{F}_e$ denote the Euclidean Fourier transform $\mathcal{S}(A_P) \to \mathcal{S}(i\mathfrak{a}_P^*)$.

Thm ($\sim$)

If $f \in \mathcal{C}(\tau, G/N_0, \chi)$ and $\psi \in {}^\circ\mathcal{C}_{P,\tau}$, define

$$f_\psi^{(\bar{P})} : a \mapsto \int_{M_P/M_P\cap N_0} \langle\, f^{(\bar{P})}(ma), \psi(m)\,\rangle_{V_\tau}\, dm.$$

Then $f_\psi^{(\bar{P})}$ belongs to $\mathcal{S}(i\mathfrak{a}_P^*)$ and

$$\mathcal{F}_e(f_\psi^{(\bar{P})})(\lambda) = \langle\, \mathcal{F}_P f(\lambda), \psi\,\rangle, \qquad (\lambda \in i\mathfrak{a}_P^*).$$

Proof

1. First for $\operatorname{supp} f$ compact modulo N_0 by using explicit calculation, holomorphy of $\mathcal{F}_P f$ and application of Cauchy's theorem.
2. Use density of $C_c^\infty(\tau, G/N_0, \chi)$ in $\mathcal{C}(\tau, G/N_0, \chi)$, combined with continuity on $\mathcal{C}(\tau, G/N_0, \chi)$ of the HC transform $f \mapsto R_a(f^{(\bar{P})})$ (due to HC) and the Fourier transform $\mathcal{F}_P$ (by uniform temperedness).

Injectivity of the Fourier transform

Thm ($\sim$)

If $f \in \mathcal{C}(\tau, G/N_0, \chi)$ *and* $\psi \in {}^\circ\mathcal{C}_{P,\tau}$, *define*

$$f_\psi^{(\bar{P})} : a \mapsto \int_{M_P/M_P \cap N_0} \langle\, f^{(\bar{P})}(ma), \psi(m)\,\rangle_{V_\tau}\, dm.$$

Then $f_\psi^{(\bar{P})}$ *belongs to* $\mathcal{S}(A_P)$ *and*

$$\mathcal{F}_e(f_\psi^{(\bar{P})})(\lambda) = \langle\, \mathcal{F}_P f(\lambda), \psi\,\rangle, \qquad (\lambda \in i\mathfrak{a}_P^*).$$

Corollary (injectivity FT)

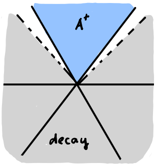
Let $f \in \mathcal{C}(\tau, G/N_0, \chi)$*. If* $\mathcal{F}_P(f) = 0$ *for all* $P \in \mathcal{P}_{st}$ *then* $f = 0$.

Proof

1. $\mathcal{F}_P f = 0$ implies that $f_\psi^{(\bar{P})} = 0$ for all $\psi \in {}^\circ\mathcal{C}_{P,\tau}$. Hence $f^{(\bar{P})} \sim 0$.
2. $f = 0$ by HC's completeness thm.

C-function, Normalized Whittaker integral

- $\mathrm{Wh}(P, \psi, \lambda)$ is finite under $\mathfrak{Z} := \mathrm{center}(U(\mathfrak{g}))$,
- top order asymptotic behavior of $\exp^l$ type along $\mathrm{cl}(A^+)$,
- very rapid decay outside $\mathrm{cl}(A^+)$.



Lemma

Let $P \in \mathcal{P}_{st}$. For $\psi \in {}^\circ\mathcal{C}_{P,\tau}$, $\mathrm{Re}\lambda \in \mathfrak{a}_P^{*+}$, $m \in M_P$, $a \to \infty$ in A_P^+,

$$\mathrm{Wh}(P, \psi, \lambda)(ma) \sim a^{\lambda - \rho_P}[C_P(\lambda)\psi](m),$$

with $C_P(\lambda) \in \mathrm{End}({}^\circ\mathcal{C}_{P,\tau})$, merom^c in $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$ (reg^r for $\mathrm{Re}\lambda \in \mathfrak{a}_P^{*+}$).

Definition (HC) $\mathrm{Wh}^\circ(P, \psi, \lambda) := \mathrm{Wh}(P, C_P(\lambda)^{-1}\psi, \lambda)$ (mero^c in λ)

- $P \sim Q :\iff \exists w \in W(\mathfrak{a}) : \; w(\mathfrak{a}_P) = \mathfrak{a}_Q$ (associated).
- $W(\mathfrak{a}_Q|\mathfrak{a}_P) := \{s \in \mathrm{Hom}(\mathfrak{a}_P, \mathfrak{a}_Q) \mid \exists w \in W(\mathfrak{a}) : \; s = w|_{\mathfrak{a}_P}\}$.

Functional equations, Maass-Selberg relations

Lemma (Functional equations: HC)

Let $P, Q \in \mathcal{P}_{st}$, $P \sim Q$. Then for all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$,

$$\mathrm{Wh}^\circ(Q, C^\circ_{Q|P}(s,\lambda)\psi, s\lambda) = \mathrm{Wh}^\circ(P, \psi, \lambda), \quad (\lambda \in \mathfrak{a}^*_{P\mathbb{C}}),$$

with $C^\circ_{Q|P}(s,\lambda) \in \mathrm{Hom}({}^\circ\mathcal{C}_{P,\tau}, {}^\circ\mathcal{C}_{Q,\tau})$ a uniquely determined merom^c function of $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$.

Thm (Maass-Selberg relations, HC)

For all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$, $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$C^\circ_{Q|P}(s, -\bar{\lambda})^* \circ C^\circ_{Q|P}(s,\lambda) = I_{{}^\circ\mathcal{C}_{P,\tau}}.$$

In particular, for $\lambda \in i\mathfrak{a}^*_P$, the map $C^\circ_{Q|P}(s,\lambda)$ is unitary.

Theorem (HC) $\lambda \mapsto \mathrm{Wh}^\circ(P, \psi, \lambda)$ *is regular on $i\mathfrak{a}^*_P$.*

Wave packets

Definition

For $P \in \mathcal{P}_{st}$, $\psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$, $x \in G$,

$$\mathcal{W}_P(\psi)(x) := \int_{i\mathfrak{a}_P^*} \mathrm{Wh}^\circ(P, \psi(\lambda), \lambda, x)\, d\lambda.$$

Theorem ($\sim$)

$$\mathcal{W}_P : \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau} \to \mathcal{C}(\tau, G/N_0, \chi)$$

is continuous linear.

Remark: HC proves this for $\mathcal{W}_P$ restricted to $C_c^\infty(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$.

Proof requires

- the uniformly tempered estimates
- theory of constant term with parameter
- families of type $\mathrm{II}_{\mathrm{hol}}(\Lambda)$ (as in previous joint work with Carmona and Delorme for reductive symmetric space G/H).

Plancherel formula

Normalized Fourier transform

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$ define $\mathcal{F}_P^\circ f : i\mathfrak{a}_P^* \to {}^\circ\mathcal{C}_{P,\tau}$ by

$$\mathcal{F}_P^\circ f(\lambda) = C_P(\lambda)^* \mathcal{F}_P(\lambda).$$

Then $\mathcal{F}_P^\circ : \mathcal{C}(\tau, G/N_0, \chi) \to \mathcal{S}(i\mathfrak{a}_P^*) \otimes {}^\circ\mathcal{C}_{P,\tau}$ is continuous linear.

Lemma

$\mathcal{W}_P \mathcal{F}_P^\circ \in \mathrm{End}(\mathcal{C}(\tau, G/N_0, \chi))$ only depends on $[P] \in \mathcal{P}_{st}/\sim$.

Proof: This follows from the Maass-Selberg relations.

Plancherel theorem

If $f \in \mathcal{C}(\tau, G/N_0, \chi)$*, then*

$$f = \sum_{[P] \in \mathcal{P}_{st}/\sim} \mathcal{W}_P \mathcal{F}_P^\circ f.$$

Completion proof of Plancherel

Thm

$$f = \sum_{[P] \in \mathcal{P}_{st}/\sim} \mathcal{W}_P \mathcal{F}_P^\circ f \qquad (f \in \mathcal{C}(\tau, G/N_0, \chi)).$$

Proof

(1) From HC's results on descent transform, for all $P, Q \in \mathcal{P}_{st}$,

$$\mathcal{F}_Q^\circ \mathcal{W}_P \mathcal{F}_P^\circ = \begin{cases} \mathcal{F}_Q^\circ & \text{if} \quad Q \sim P \\ 0 & \text{otherwise} \end{cases}$$

(2) Put $g = f - \sum \mathcal{W}_P \mathcal{F}_P^\circ f$. Then $g \in \mathcal{C}(\tau, G/N_0, \chi)$ and by (1):

$$\mathcal{F}_Q^\circ g = \mathcal{F}_Q^\circ f - \mathcal{F}_Q^\circ f = 0.$$

(3) From (2), $\mathcal{F}_Q(g) = 0$ for all $Q \in \mathcal{P}_{st}$, hence $g = 0$ (injectivity).