Cusp forms for semisimple symmetric spaces

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Représentations et Analyse Harmonique de Groupes Réels et p-adiques ainsi que de leurs Espaces Symétriques

à l'occasion des 60 ans de Patrick Delorme

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Semisimple symmetric spaces

Setting

- G connected semisimple Lie group, finite center
- σ involution of G, i.e., $\sigma \in \mathrm{Aut}(G)$ and $\sigma^2 = I$
- $(G^\sigma)_e \leq H \leq G^\sigma$,
- $X = G/H$ semisimple symmetric space

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Examples

- $G = {}^{\backprime}G \times {}^{\backprime}G$, $H = \mathrm{diag}({}^{\backprime}G)$. Then $X \simeq {}^{\backprime}G$ ('group case')
- $H = K$ (maximal compact) $\quad X = G/K, \quad$ (Riemannian case)
- Hyperbolic spaces $X_{p,q} = \mathrm{SO}(p,q)/\mathrm{SO}(p-1,q) \simeq$

$$\{x \in \mathbb{R}^{p+q} \mid x_1^2 + \cdots x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) = 1\}.$$

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- $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$.

Semisimple symmetric spaces

Structure

- $\exists\ K \leq G$ maximal compact such that $\sigma(K) = K$
- $K = G^\theta, \quad \theta \circ \sigma = \sigma \circ \theta$

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Examples

- Group case: $G = {}^{\backprime}G \times {}^{\backprime}G, \;\; K = {}^{\backprime}K \times {}^{\backprime}K$
- Riemannian case: $\;K = H$
- Hyperbolic: $X = \mathrm{SO}(p,q)/\mathrm{SO}(p-1,q), \;\; K = \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$
- $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R}), \;\; K = \mathrm{SO}(n)$

Parabolic subgroups

σ-parabolic subgroups

- Parabolic $P \leq G$: $P = N_G(P)$ and $\mathrm{Lie(P)}_{\mathbb{C}} \supset$ Borel
- σ-parabolic subgroup: $\sigma(P) = \bar{P} = \theta(P)$
- $\mathcal{P}_\sigma$: the set of σ-parabolic subgroups

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Examples

- Riemannian case: $\sigma = \theta$ ordinary parabolic subgroups
  $X = \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n, \mathbb{R})$:

$$P \sim \left\{ \begin{pmatrix} B_1 & * & \dots & * \\ 0 & B_2 & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{pmatrix} \middle| B_j \in \mathrm{GL}(k_j, \mathbb{R}) \right\}$$

- Group case: $\mathcal{P}_\sigma = \{ {}^\backprime P \times {}^\backprime \overline{P} \mid {}^\backprime P \leq {}^\backprime G \}$

Plancherel decomposition

Abstractly

Left regular rep'n L of G in $L^2(G/H)$ is unitary

$$(L, L^2(G/H)) \simeq \int_{\widehat{G}_H}^{\oplus} m_\pi \, \pi \; d\mu(\pi)$$

Discrete series

$\widehat{G}_{H,\mathrm{ds}}$: the rep's appearing in discrete part $L^2(G/H)_{\mathrm{d}}$

Classification:

- Group: Harish-Chandra, 1960's
- General G/H

  Construction: Flensted-Jensen, early 1980's
  Classification: Oshima & Matsuki, early 1980's

Plancherel decomposition

Plancherel decomposition of $L, L^2(X)$:

Building blocks are the discrete series for X and the induced reps

$$\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$$

with

- $P = M_P A_P N_P \in \mathcal{P}_\sigma, \ \ P < G$
- ξ a discrete series rep of $M_P / M_P \cap H$,
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History

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- General G/H, Patrick Delorme, Annals of Mathematics 1998 (uses work of O & M, J. Carmona, J. Bernstein)
- Different proof by vdB & H. Schlichtkrull (uses work of O & M, J. Carmona, residue calculus)

Schwartz functions

Schwartz space

$$C_c^\infty(G/H) \subset \mathcal{C}(G/H) \subset L^2(G/H)^\infty$$

Generalizes Harish-Chandra's L^2-Schwartz space $\mathcal{C}(G)$.

Plancherel decomposition gives

$$\mathcal{C}(G/H) = \bigoplus_{[P]\in\mathcal{P}_\sigma/\sim} \mathcal{C}(G/H)_{[P]},$$

Note: $\mathcal{C}(G/H)_{[G]} = \mathcal{C}(G/H)_{\mathrm{d}}$

Cusp forms

- Harish-Chandra characterizes $\mathcal{C}(G)_d$ as cusp forms
  his proof of the Plancherel formula for G depends on this

Cusp forms for the group

Theorem (Harish-Chandra)

If $P = M_P A_P N_P$ is a parabolic subgroup of G then for all $f \in \mathcal{C}(G)$

$$\int_{N_P} |f(n)| \, dn < \infty.$$

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Cusp form on G

- Is: a function $f \in \mathcal{C}(G)$ such that

$$\forall x, y \in G \ \ \forall P < G : \quad \int_{N_P} f(xny)\, dn = 0.$$

- $\mathcal{C}(G)_{\mathrm{cusp}} \subset \mathcal{C}(G)$: space of cusp forms

Cusp forms

Theorem (HC, 60's)

$$\mathcal{C}(G)_{\mathrm{cusp}} = \mathcal{C}(G)_{\mathrm{d}}.$$

Remark

In theory of automorphic forms, in general

$$\mathcal{C}(G/\Gamma)_{\mathrm{cusp}} \subsetneq \mathcal{C}(G/\Gamma)_{\mathrm{d}}$$

Question

Is there an appropriate notion of cusp form for G/H ??

Cusp forms, first attempt

First attempt

Cusp form: $f \in \mathcal{C}(G/H)$ such that

$$\forall P \in \mathcal{P}_\sigma, P < G \ \forall x \in G : \qquad \int_{N_P} f(xnH)\, dn = 0.$$

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Problems

- Integral need not converge, e.g.,
  - hyperbolic spaces, recent work by N.B. Andersen, M. Flensted-Jensen & H. Schlichtkrull
  - $\mathrm{SL}(n, \mathbb{R})/\mathrm{GL}(n-1, \mathbb{R})$, vdB, J. Kuit, H. Schlichtkrull
- In the group case, new definition differs from old one:

$$\text{New}: \quad \int_{N_P \times \bar{N}_P} f(xn\bar{n}y)\, dn\, d\bar{n} = 0, \quad \text{Old}: \quad \int_{N_P} f(xny)\, dn = 0.$$

Idea of Flensted-Jensen

Minimal σ-parabolics

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q} \quad \pm 1$ eigenspaces for θ, σ
- $\mathfrak{a}_\mathfrak{q} \subset \mathfrak{p} \cap \mathfrak{q}$ maximal abelian;
- $\Sigma(\mathfrak{a}_\mathfrak{q})$ roots of $\mathfrak{a}_\mathfrak{q}$ in $\mathfrak{g}$, fix $\Sigma^+(\mathfrak{a}_\mathfrak{q}) \quad \rightsquigarrow \mathfrak{n}_0,\ N_0$
- $P_0 = Z_G(\mathfrak{a}_\mathfrak{q})N_0$ is a minimal σ-parabolic.

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Minimal parabolics

- Extend $\mathfrak{a}_\mathfrak{q}$ to $\mathfrak{a}$: max abelian in $\mathfrak{p}$; $\quad A = \exp \mathfrak{a}$.
- Fix $P \in \mathcal{P}_{\min}(A)$ with $P \subset P_0$,

Then P has $N_P \cap H$ of minimal dimension and $N_0 \simeq N_P / N_P \cap H$.

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Then P has $N_P \cap H$ of minimal dimension and $N_0 \simeq N_P/N_P \cap H$.

Flensted-Jensen's idea

- Use $Q \in \mathcal{P}_{\min}(A)$ with $N_Q \cap H$ of maximal dimension.

Cusp forms, II

Group case

Let $G = {}^{\backprime}G \times {}^{\backprime}G$, and ${}^{\backprime}P \subset {}^{\backprime}G$ a minimal parabolic then

- $P_0 = P = {}^{\backprime}P \times {}^{\backprime}\overline{P}, \qquad Q = {}^{\backprime}P \times {}^{\backprime}P.$
- $N_P / N_P \cap H \simeq N_{{}^{\backprime}P} \times \overline{N}_{{}^{\backprime}P}, \qquad N_Q / N_Q \cap H \simeq N_{{}^{\backprime}P}.$

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- $N_P / N_P \cap H \simeq N_{{}^{\backprime}P} \times \overline{N}_{{}^{\backprime}P}, \qquad N_Q / N_Q \cap H \simeq N_{{}^{\backprime}P}.$

Theorem: (Andersen, Flensted-Jensen, Schlichtkrull)

Let $G/H = X_{p,q}$ be hyperbolic. Let Q be a minimal parabolic subgroup with $N_Q \cap H$ of maximal possible dimension. Then

$$\forall f \in \mathcal{C}(X): \qquad \int_{N_Q/(N_Q \cap H)} |f(n)|\, dn < \infty$$

This allows [A, F-J, S] to define a cusp form by the requirement

$$\forall x \in G: \qquad \int_{N_Q/(N_Q \cap H)} f(xn)\, dn = 0.$$

σ-split rank one

From now on $\dim \mathfrak{a}_\mathfrak{q} = 1$ (true for $X_{p,q}$, $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$)

Define $\rho_{Qh} \in \mathfrak{a}^* : \; X \mapsto \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{n}_Q \cap Z(\mathfrak{a}_\mathfrak{q})})$

Theorem (vdB - K)

Let $Q \in \mathcal{P}_{\min}(A)$ be H-compatible, i.e.,

(a) $\langle \rho_{Qh}, \alpha \rangle \geq 0 \quad \forall \alpha \in \Sigma(\mathfrak{a}, \mathfrak{n}_Q)$

Then

(b) $\forall f \in \mathcal{C}(X), \quad \int_{N_Q/N_Q \cap H} |f(n)| \, dn < \infty.$

Remark (vdB - K - S)

Let $Q \in \mathcal{P}_{\min}(A)$ and $N_Q \cap H$ of max dimension.

(1) For $X = {}^{\backprime}G$ or X hyperbolic, (a) is automatic.

(2) If $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$, then (a) is restrictive. Moreover,

$$(a) \iff (b)$$

Outline of proof

Step 1: reduction to K-fixed positive f:

Let $f \in \mathcal{C}(X)$. Then $\quad \exists \varphi \in \mathcal{C}(X)^K : \quad |f| \leq \varphi \text{ on } X.$

Outline of proof

Step 1: reduction to K-fixed positive f:

Let $f \in \mathcal{C}(X)$. Then $\quad \exists \varphi \in \mathcal{C}(X)^K : \quad |f| \leq \varphi \;$ on X.

Step 2: Theorem

The operator $\mathcal{H}_Q : C_c(X)^K \to C^\infty(A_\mathrm{q})$ defined by

$$\mathcal{H}_Q f(a) = a^{\rho_Q} \int_{N_Q / N_Q \cap H} f(an)\, dn$$

extends to a continuous linear operator $\widetilde{\mathcal{H}}_Q : \mathcal{C}(X)^K \to C^\infty(A_\mathrm{q})$.

Proof: uses Plancherel formula for $\mathcal{C}(X)^K$. Discussion postponed.

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Final step

Let $f \in \mathcal{C}(X)^K, \quad f \geq 0. \quad \exists (f_n) \subset C_c^\infty(X)^K$ s.t. $f_n \nearrow f$ in $\mathcal{C}(X)$.

$\Longrightarrow \quad \mathcal{H}_Q(f_n)(e) \nearrow \quad \& \quad \mathcal{H}_Q(f)(e) \to \widetilde{\mathcal{H}}_Q(f)(e)$

$\Longrightarrow \quad \int_{N_Q/(N_Q \cap H)} f(n)\, dn < \infty.$

Eisenstein integrals

For simplicity assume G/P_0 has one open H-orbit.

Let $Q \in \mathcal{P}_{\min}(A)$, $\lambda \in \mathfrak{a}^*_{\mathrm{q}\mathbb{C}}$.
Define

$$\psi_{Q,\lambda} : G \to \mathbb{C}, \quad kan_Q \mapsto a^{\lambda+\rho_{Qh}-\rho_Q}$$

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Definition: Eisenstein integral

$$E(Q,\lambda)(x) := \int_{H/(H \cap N_Q)} \psi_{Q,\lambda}(xh)\, dl_h(e)^{-1*}|\omega|$$

where

- $\omega \in \wedge^{\mathrm{top}} T_e^*(H/H \cap Q) \setminus \{0\}$
- $\mathrm{Re}\lambda$ sufficiently Q-dominant.

Extend $E(Q,\lambda) \in C^\infty(G/H)^K$ meromorphically in $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$.

Fourier transform and Harish-Chandra transform

Fourier transform

Define $\mathcal{F}_Q^{\mathrm{un}} : C_c^\infty(G/H)^K \to \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*)$ by

$$\mathcal{F}_Q^{\mathrm{un}} f(\lambda) := \int_X f(x)\, E(Q : -\lambda)(x)\, dx.$$

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Relation to HC transform

$$\mathcal{F}_Q^{\mathrm{un}}(f)(\lambda) = \mathcal{F}_{\mathrm{eucl}}(\mathcal{H}_Q f)(\lambda)$$

for $\mathrm{Re}\lambda$ sufficiently dominant.

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$$\Longrightarrow \qquad \mathcal{H}_Q f(a) = \int_{\eta + i\mathfrak{a}_{\mathrm{q}}^*} a^\lambda \mathcal{F}_Q^{\mathrm{un}}(f)(\lambda)\, d\lambda$$

for $\eta \in \mathfrak{a}_{\mathrm{q}}^*$ sufficiently dominant.

Residual operators

Residual formula

Let $f \in C_c^\infty(X)^K$. Then

$$\mathcal{H}_Q f = T_Q f + R_Q f$$

where

$$T_Q f(a) := \lim_{\varepsilon \downarrow 0} \int_{i\mathfrak{a}_\mathfrak{q}^* + \varepsilon\eta} a^\lambda \mathcal{F}_Q^{\mathrm{un}} f(\lambda)\, d\lambda$$

and

$$R_Q(f)(a) := 2\pi i \sum_j \operatorname*{Res}_{\lambda=\mu_j} \left(a^\lambda \mathcal{F}_Q^{\mathrm{un}} f(\lambda)\right)$$

Extension of T

Residual formula

Let $f \in C_c^\infty(X)^K$. Then

$$\mathcal{H}_Q f = T_Q f + R_Q f$$

Extension of T

There exists a tempered distribution v_Q on A_q such that

$$T_Q f = v_Q * \mathcal{F}_{\mathrm{eucl}}^{-1} \mathcal{F}_{\bar{P}_0} f$$

for all $f \in C_c^\infty(X)^K$.

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Corollary

The operator T_Q extends to a continuous linear operator

$$\mathcal{C}(X)^K \to C^\infty(A_{\mathrm{q}})_{\mathrm{temp}}$$

Extension of R

Kernel for R

Let $f \in C_c^\infty(X)^K$. Then, for $a \in A_\mathrm{q}$,

$$R_Q f(a) = 2\pi i \sum_j \int_X f(x)\, R_j(a,x)\, dx$$

where $R_j(a,x) = \mathrm{Res}_{\lambda=\mu_j}\, a^\lambda E_Q(-\lambda, x)$

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Theorem

Let Q be H-compatible. Then $R_j \in \mathcal{E}(A_{\mathrm{q}}) \otimes \mathcal{C}(X)_{\mathrm{d}}^K$.

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Theorem

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Corollary

Let Q be H-compatible. Then R_Q extends to a continuous linear map

$$\mathcal{C}(X)^K \to \mathcal{E}(A_{\mathrm{q}}).$$

Final conclusions

Assumption: $\dim \mathfrak{a}_{\mathrm{q}} = 1$.

Theorem

$\mathcal{C}(X)_{\mathrm{cusp}}$ decomposes discretely.

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$$\mathcal{C}(X)_{\mathrm{res}}^{K} = \mathrm{span}\{\underset{\lambda=\mu_j}{\mathrm{Res}}(a^{\lambda} E(Q, -\lambda)) \mid a \in A_{\mathrm{q}}\}$$

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Theorem

$$\mathcal{C}(X)_{\mathrm{res}}^{K} = 0 \quad \Longrightarrow \quad \mathcal{C}(X)_{\mathrm{res}} = 0.$$

Cher Patrick

Cher Patrick

Mes félicitations!