"Many particle systems out of equilibrium" Problems, Series 11, 2006-07.

Problem 24. Fluctuating hydrodynamic equations

a) The classical fluctuation-dissipation equation can be generalized to a set of coupled Langevin equations with colored noise as follows: Let $\boldsymbol{a}(t)$ describe a set of n random variables satisfying the Langevin equations

$$\dot{\boldsymbol{a}}(t) = -\int_0^\infty d\tau\, \mathsf{Z}(\tau)\cdot \boldsymbol{a}(t-\tau) + \boldsymbol{\xi}(t), \tag{1}$$

with Z an $n \times n$ matrix and $\boldsymbol{\xi}(t)$ a gaussian noise vector satisfying

$$\begin{aligned} \langle \boldsymbol{\xi}(t)\rangle &= 0, \\ \langle \boldsymbol{\xi}(t_1)\boldsymbol{\xi}(t_2)\rangle &= \mathsf{K}(t_1-t_2). \\ satisfying & \\ \mathsf{K}(t_1-t_2) &= \mathsf{K}^T(t_2-t_1), \end{aligned} \tag{2}$$

with K^T the transposed of K. Here $\langle\rangle$ denotes an average over the distribution of the random noise. If the characteristic decay times for the variables $\boldsymbol{a}$ are large compared to the decay time of $\mathsf{K}(t)$ one may argue that to a good approximation $\langle \boldsymbol{\xi}(t)\boldsymbol{a}(0)\rangle$ vanishes for $t > 0$, whereas for $t < 0$ this is not the case. Under the same condition the integration over τ in (1) may be restricted to the range $[0, t]$ in most of the relevant cases, because $\mathsf{Z}(\tau)$ has decayed to zero already. If Z is a positive definite matrix the system described by these equations will go to a stationary state. Using the above approximations, show that the fluctuations of $\boldsymbol{a}$ in this stationary state satisfy the fluctuation-dissipation theorem

$$\mathsf{K}(t_1-t_2) = \left[\langle \boldsymbol{a}\boldsymbol{a}\rangle \cdot \mathsf{Z}^T(|t_1-t_2|) + \mathsf{Z}(|t_1-t_2|)\cdot\langle \boldsymbol{a}\boldsymbol{a}\rangle\right], \tag{3}$$

Hints: consider the Fourier transform of (2) with respect to both t_1 and t_2. If the Fourier variables are ω_1 and ω_2 respectively, the result is proportional to $\delta(\omega_1+\omega_2)$. Next multiply (1) with itself, for two different time variables, take a double Fourier transform as well and establish a relationship between $\langle \boldsymbol{a}(\omega)\boldsymbol{a}(-\omega)\rangle$ and $\langle \boldsymbol{\xi}(\omega)\boldsymbol{\xi}(-\omega)\rangle$. Find another expression for $\langle \boldsymbol{a}(\omega)\boldsymbol{a}(-\omega)\rangle$ by integrating $\langle \dot{\boldsymbol{a}}(t)\boldsymbol{a}(0)\rangle \exp(i\omega t)$ respectively $\langle \boldsymbol{a}(0)\dot{\boldsymbol{a}}(t)\rangle \exp(-i\omega t)$ from 0 to ∞ and using (1) for $\dot{\boldsymbol{a}}(t)$.
Note that in (1) the integration range for τ can be extended to $[-\infty, \infty]$ by setting $\mathsf{Z}(\tau) = 0$ for $\tau < 0$.
One may conclude that given the fluctuations $\langle \boldsymbol{a}\boldsymbol{a}\rangle$ the matrices K and Z cannot be chosen independently.

b) One may extend this result to the case where $\boldsymbol{a}$ satisfies the equation

$$\dot{\boldsymbol{a}}(t) = \mathsf{L} \cdot \boldsymbol{a} - \int_0^\infty d\tau\, \mathsf{Z}(\tau) \cdot \boldsymbol{a}(t-\tau) + \boldsymbol{\xi}(t), \tag{4}$$

with L an anti-hermitean matrix. Show that (3) remains valid, but with $\mathsf{Z}(t)$ replaced by $\mathsf{Z}(t) - 2\mathsf{L}\delta(t)$ (with the convention $\int_0^\infty \delta(t)dt = 1/2$).

c) (bonus question) Landau and Placzek have proposed extending the usual hydrodynamic equations by adding fluctuating terms to them, analogous to the fluctuating force in the Langevin equation describing Brownian motion. For simplicity, let us consider linearized hydrodynamic equations, but with transport coefficients that do depend on frequency and wave number (compare script section 2.4). In time representation these may be written in the form

$$\begin{aligned}
\frac{\partial \rho(\boldsymbol{k},t)}{\partial t} &= \rho_0 i\boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k},t), & (5)\\
\rho_0 \frac{\partial \boldsymbol{u}(\boldsymbol{k},t)}{\partial t} &= -i\boldsymbol{k}p(\boldsymbol{k},t) - \int_0^\infty d\tau\, k^2 \eta(\boldsymbol{k},\tau)\boldsymbol{u}(\boldsymbol{k},t-\tau) \\
&\quad - \int_0^\infty d\tau\, [\zeta(\boldsymbol{k},\tau) + \frac{1}{3}\eta(\boldsymbol{k},\tau)]\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{u}(\boldsymbol{k},t-\tau)) \\
&\quad -i\boldsymbol{k} \cdot [\boldsymbol{\Sigma}^r_{sh}(t) + \mathbf{1}\Sigma^r_b(t)]\,, & (6)\\
n_0 T_0 \frac{\partial \sigma}{\partial t} &= -\int_0^\infty d\tau\, k^2 \lambda(\boldsymbol{k},\tau) T(t-\tau) - i\boldsymbol{k}\cdot \boldsymbol{j}^r_q(t). & (7)
\end{aligned}$$

Here $\boldsymbol{\Sigma}^r_{sh}$ is a fluctuating traceless tensor, Σ^r_b is a fluctuating scalar and $\boldsymbol{j}^r_q$ is a fluctuating vector, all with zero mean. Using the results of item b) find the relationships between correlations of these fluctuating currents at different times and the transport kernels $\eta(\boldsymbol{k},t)$, $\zeta(\boldsymbol{k},t)$ and $\lambda(\boldsymbol{k},t)$. Relate the results to the Green-Kubo expressions for the transport coefficients. Why are these relationships only correct in the limit $k \to 0$? What is missing in the fluctuating hydrodynamic equations for $k \neq 0$? Hint: compare the formal structure of the hydrodynamic equations and the Green-Kubo expressions, obtained through the projection operator formalism, to that of the fluctuating hydrodynamic equations.