"Many particle systems out of equilibrium" Problems, Series 5, 2006-07.

Problem 12. The Lorentz gas

The Lorentz gas was introduced by Lorentz in the beginning of the 20th century to describe the conduction of electrons in metals. In its simplest version the model consists of a collection of fixed scatterers modeled as hard spheres of radius a with infinite mass and zero velocity. They are located at random postitions but they cannot overlap each other. In addition there is one light point particle (or equivalently, a large collection of mutually non-interacting point particles) that moves ballistically among the scatterers and makes specular collisions whenever it hits one of them (see figure).

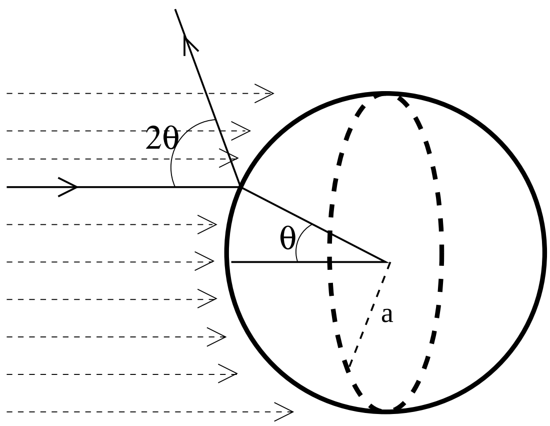

Figure 1: Sketch of a scattering trajectory. The dashed lines represent a homogeneous bundle of light particles.

a) Show that the collisions with the scatterers are *isotropic*, i.e. if a homogeneous bundle of light particles with the same velocity hits a scatterer, the number of particles scattered per unit solid angle is the same in all directions.

For low density of scatterers the process may be approximated by a random flight process with isotropic scattering, as treated in previous problems, but now in three dimensions.

b) If the number density of the scatterers is n, their radius a and the speed of the light particle v, show that the collision frequency is given by $\nu = nv\pi a^2$.

c) Calculate the diffusion coefficient for this process, using the Green-Kubo expression.

d) Show that the function $C(k,z)$ may be found from

$$C(k,z) = v^2 \left[\left\langle \frac{(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}})^2}{z + \nu + i\boldsymbol{k} \cdot \boldsymbol{v}} \right\rangle + \left\langle \frac{\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}}{z + \nu + i\boldsymbol{k} \cdot \boldsymbol{v}} \right\rangle^2 \frac{\nu}{1 - \nu \left\langle \frac{1}{z+\nu+i\boldsymbol{k}\cdot\boldsymbol{v}} \right\rangle} \right]. \tag{1}$$

Here $\langle\rangle$ corresponds to an angular average over $\hat{\boldsymbol{v}}$.

e) Work out the average, give explicit expressions for $C(k,z)$ and $U(k,z)$ and determine the long time behavior of the intermediate scattering function $F(k,t)$.

Problem 13. Free flight time distributions

a) How to resolve the following paradox: The mean free flight time from the initial time to the first collision is $1/\nu$. Similarly the mean free flight time since the last collision equals $1/\nu$. So the average time beteen these two subsequent collisions equals $2/\nu$. On the other hand, if you pick a time at which a collision occurs and ask for the average time till the next collision, this should be $1/\nu$, since what will happen in the future is independent of what happened at $t = 0$. How to reconcile these results?

b) In the random flight model the distribution of free flight times is the exponential $\nu \exp -(\nu t)$. In many other cases free flight time distributions are not exponential. In such cases we may distinguish three different types of free flight time distributions:

   1) The probability distribution $p_{cc}(t)$ for the time beteen two subsequent collisions.
   2) The probability distribution $p_{fc}(t)$ for the time beteen an arbitrary initial point and the subsequent collision. Why is this the same as $p_{cf}(t)$ for the inverse event?
   3) The distribution $p_{ff}(t)$ for the probability that between an arbitrary initial time and a time t later no collision occurs.

   What are the relations between these three distributions? Check that these reproduce the expected results for an exponential distribution!

Problem 14. Lorentz gas with anisotropic scattering

In dimensions d different from 3 the scattering of a light particle by a fixed hard sphere is not isotropic. On may then introduce a *persistence factor* p as the ratio between the average velocity after the collision (averaged in this case over the cross section of the collision) and the velocity before the collision. Show that, in the random flight approximation, the z-dependent diffusion coefficient is given by

$$D(z) = \frac{v^2}{d(z+\nu)(1-p)}. \tag{2}$$