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Kinetic Theory of Hydrodynamic Flows. II. The Drag on a Sphere and on a Cylinder

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In this paper we continue the study of solutions of the extended Boltzmann equation started previously. In particular, we study an iterated solution of the equation that can be used to describe the flow of a rarefied gas around a macroscopic object. We discuss the rarefied flow and then show how the iterated solution can be extended into the hydrodynamic regime. The results for the drag force and for the distribution function of the gas molecules are shown to be identical to the results obtained in a previous paper by a generalization of the normal solution method. We also discuss the special properties of both rarefied and continuum flows around a cylinder and show that in both regions one must take into account Oseen-like terms which naturally appear in the extended Boltzmann equation. In the hydrodynamic regime we obtain Lamb's formula for the force on the cylinder. By relating the terms in the iterated expression to dynamical events taking place in the fluid, we are able to discuss the dynamical origin of the results obtained here.

KEY WORDS: Boltzmann equation; rarefied gas flow; continuum flow; flow around spheres; flow around cylinders; drag force; dynamical events; Stokes' law, Lamb formula.

1. INTRODUCTION

In the first paper of this series[1),3] (to be referred to as I) we obtained the drag force on a nonrotating sphere moving in a dilute gas, from an extended Boltzmann equation. This equation, after a linearization about total equilibrium, was of the form

$$(\partial/\partial t + \mathbf{v}\cdot\nabla - L - \bar{\mathbf{T}})\Psi(\mathbf{r}, \mathbf{v}, t) = -\mathbf{T}\beta m[\mathbf{v}\cdot\mathbf{V}(t)] \tag{1.1}$$

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where the symbols have the following meaning: The one-particle distribution function has been linearized about a Maxwellian in the (accelerated) coordinate system where the sphere is at rest at all times. Its form is then given as

$$f(\mathbf{r}, \mathbf{v}, t) = n\varphi(v)(W(r)\{1 - \beta m[\mathbf{v}\cdot\mathbf{V}(t)]\} + \Psi(\mathbf{r}, \mathbf{v}, t)) \tag{1.2}$$

where n is the equilibrium density and $\varphi(v)$ is the Maxwellian,

$$\varphi(v) = (\beta m/2\pi)^{3/2} \exp(-\beta m v^2/2)$$

with $\beta = (k_B T)^{-1}$, T is the temperature, and k_B is Boltzmann's constant. $\mathbf{V}(t)$ is the velocity of the sphere at time t and $W(r)$ is unity for points outside the sphere and zero otherwise. The actions of $(\mathbf{v}\cdot\mathbf{V})$ and the linearized Boltzmann collision operator L on Ψ describe the rate of change of the distribution function due to free streaming and to collisions among gas particles, respectively, while the operator $\bar{\mathbf{T}}$ describes the influence of collisions between gas molecules and the sphere. The precise form of this operator, as well as that of the closely related operator $\mathbf{T}$, is given in I. In the hydrodynamic regime the Knudsen number $\mathscr{K}$ is much less than unity. This quantity is defined as $\mathscr{K} = l/R$, with l the mean free path between collisions of a gas molecule and R the radius of the sphere. Equation (1.1) was solved in I by an extension of the Chapman–Enskog normal solution method. The function Ψ was divided into a hydrodynamic and a nonhydrodynamic part, and after some algebraic manipulations both the general form of the hydrodynamic part and the boundary layer part of the distribution function could be determined.

In this paper we show how Eq. (1.1) can be solved iteratively, both in the Knudsen or rarefied gas regime where $\mathscr{K} \gg 1$, and in the hydrodynamic regime. Here the iteration will consist in expanding the solution of (1.1) in powers of $G_B\bar{\mathbf{T}}$, where G_B is the so-called Boltzmann propagator $G_B = (z + \mathbf{v}\cdot\mathbf{V} - L)^{-1}$; G_B describes the motion of a typical molecule in the gas, taking into account free motion and collisions with other molecules, while $\bar{\mathbf{T}}$ describes collisions of the gas molecules with the sphere. This iterated solution of (1.1) will allow us to determine, in a rather straightforward way, the particular dynamical events that are responsible for the behavior of the drag force in the different regions of interest. Besides providing us with a physical picture of what takes place when a macroscopic sphere moves through a dilute gas, the iterated solution is very useful when one considers the Brownian motion of a sphere in the gas. We will return to this point later on.

We will also discuss the refinements needed in the calculations presented here and in I in order to compute the drag force per unit length on an infinitely long cylinder moving in a direction perpendicular to its axis. Here one would like to understand the dramatic difference between the form of Stokes' law $\mathbf{F} = \zeta\mathbf{V}$ for the force on a sphere, which is linear in the velocity $\mathbf{V}$, and the form of Lamb's formula $\mathbf{F} = a\mathbf{V}/(b + c \ln \mathscr{M})$ for small Mach number $\mathscr{M}$.[3]

The starting point of the iteration is the Laplace transform of (1.1),

$$(z + \mathbf{v}\cdot\nabla - L - \bar{\mathbf{T}})\hat{\Psi} = -\mathbf{T}\beta m[\mathbf{v}\cdot\mathbf{V}(z)] \tag{1.3}$$

where we ignored the initial value term since, as discussed in Section 6 of I, we are not interested in its influence. Equation (1.3) can be solved formally as

$$\hat{\Psi} = -(z + v\cdot\nabla - L - \bar{\mathbf{T}})^{-1}\mathbf{T}\beta m[\mathbf{v}\cdot\mathbf{V}(z)] \tag{1.4}$$

which can be iterated in powers of the Boltzmann propagator G_B as

$$\hat{\Psi} = -\{G_B + G_B\bar{\mathbf{T}}G_B + G_B\bar{\mathbf{T}}G_B\bar{\mathbf{T}}G_B + \cdots\}\mathbf{T}\beta m[\mathbf{v}\cdot\mathbf{V}(z)] \tag{1.5}$$

The drag force exerted by the gas molecules can be obtained as

$$\mathbf{F}(z) = -\int d\mathbf{r}\,\langle m\mathbf{v}\bar{\mathbf{T}}\{-\beta m[\mathbf{v}\cdot\mathbf{V}(z)] + \hat{\Psi}(\mathbf{r}, \mathbf{v}, z)\}\rangle \tag{1.6}$$

where $\langle f(\mathbf{v})\rangle = n\int d\mathbf{v}\,\varphi(v)f(\mathbf{v})$. Equation (1.6) is easily understood if one realizes that

$$n\varphi(v)\bar{\mathbf{T}}\{-\beta m[\mathbf{v}\cdot\mathbf{V}(t)] + \Psi(\mathbf{r}, \mathbf{v}, t)\}$$

is the rate of change of the distribution function at time t due to collisions between gas molecules and the sphere. Multiplying this by $m\mathbf{v}$ and integrating over $\mathbf{r}$ and $\mathbf{v}$, one obtains the total momentum transfer from the sphere to the gas at time t. Substitution of (1.5) into (1.6) yields

$$\mathbf{F}(z) = \langle m\mathbf{v}\{\bar{\mathbf{T}} + \bar{\mathbf{T}}G_B\mathbf{T} + \bar{\mathbf{T}}G_B\bar{\mathbf{T}}G_B\mathbf{T} + \bar{\mathbf{T}}G_B\bar{\mathbf{T}}G_B\bar{\mathbf{T}}G_B\mathbf{T} + \cdots\}\beta m[\mathbf{v}\cdot\mathbf{V}(z)]\rangle \tag{1.7}$$

Equations (1.7) and (1.5) form the starting point for our further analysis.

2. THE KNUDSEN REGIME

It is illuminating to make a geometric interpretation of the right-hand side of (1.7). The first term is the *free molecular flow term*, which gives the force on the sphere due to collisions with molecules that have not been perturbed by the presence of the sphere. The second term describes all correlated dynamical processes in which there are two collisions between the gas molecules and the sphere and an arbitrary number of intermediate collisions between the gas molecules. All further terms describe similar processes in which the number of collisions between the gas molecule and the sphere increases by one in each successive term.

In the Knudsen regime the propagation of particles over distances on the order of R is dominated by free streaming. Hence it seems natural to expand

the Boltzmann propagator about the free propagator $G_0 = (z + \mathbf{v}\cdot\nabla)^{-1}$ according to

$$G_B = G_0 + G_0 L G_0 + G_0 L G_0 L G_0 + \cdots \tag{2.1}$$

and to substitute this expansion into (1.7).

The iteration (2.1) corresponds to a decomposition of G_B into sequences of free streaming and collisions between gas molecules. As an illustration, some of the dynamical events contributing to (1.7) after substitution of (2.1) are given schematically in Fig. 1.

Formally the insertion of (2.1) into (1.7) gives rise to an expansion of the drag force in powers of $\mathscr{K}^{-1}$. The nth term in the expansion contains the contribution of all terms containing $n-1$ operators L. It turns out, however,

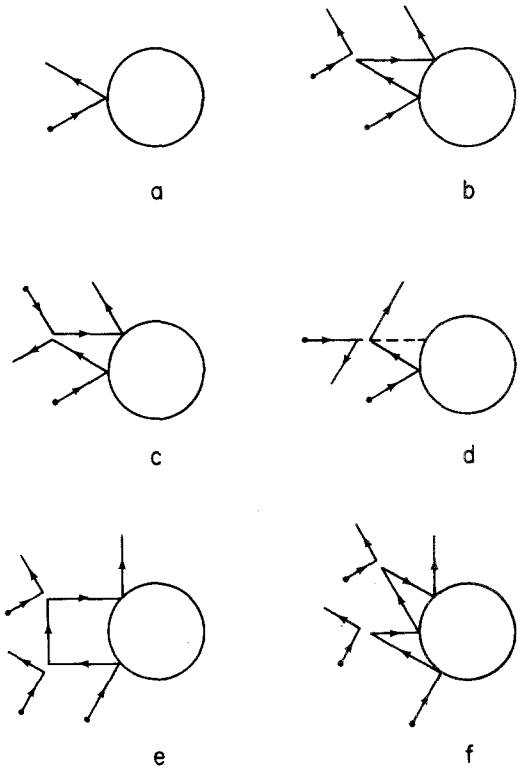

Fig. 1. The collisions of fluid particles with the sphere that are taken into account in the expansion of the force on the sphere in powers of the inverse Knudsen number. (a) The collision that is responsible for the free molecular flow force. (b, c, d) Dynamical events that contribute to the $\mathscr{K}^{-1}$ correction to this value. (d) A process where the second fluid particle does not hit the sphere, but would have done so had the second collision not taken place. (e) Events that contribute to order $\mathscr{K}^{-2}\ln(\mathscr{K}^{-1})$. (f) Events that contribute to order $\mathscr{K}^{-2}$.

that most of the coefficients in this formal expression, which are given in the form of complicated multiple integrals, are divergent.[4] The cause of these divergences is the same as that of the divergences in the density expansion of the transport coefficients[4]: The dynamical events occurring in the nth-order term describe collision processes between n gas molecules and the sphere, in which the gas molecules may travel arbitrarily long distances between subsequent collisions without being perturbed by other particles. In reality this cannot happen, and a collective effect, the mean free path damping, restricts the free trajectories between collisions to be on the order of a few mean free paths at most. For molecules for which the total cross section for binary scattering is finite, it is simple to build this into the formalism by separating the linearized Boltzmann operator into the sum of the (velocity-dependent) collision frequency and an interaction term, $L = -\nu(v) + L_I$. The Boltzmann propagator then becomes $G_B = [z + \mathbf{v}\cdot\mathbf{V} + \nu(V) - L_I]^{-1}$, which can be iterated as

$$G_B = G_0{}^d + G_0{}^d L_I G_0{}^d + G_0{}^d L_I G_0{}^d L_I G_0{}^d + \cdots \tag{2.2}$$

with $G_0{}^d = [z + \mathbf{v}\cdot\mathbf{V} + \nu(\mathbf{v})]^{-1}$. The damped free propagator $G_0{}^d$ takes into account precisely the probability that a gas molecule, before finishing free streaming (in an equilibrium system) over a given track, will be hit by another molecule.

In the literature it is customary to iterate (1.7) with the aid of (2.2), although the notations used mostly differ rather widely from ours.[5–8]

A consequence of the divergences and the subsequent resummation is that one does not obtain a simple expansion of the drag force in powers of $\mathscr{K}^{-1}$ (and it follows from the divergence of the coefficients in the formal expansion that no such expansion is possible); instead, one obtains an expansion of the drag force in the form[5,8,9]

$$F/F_0 = 1 + a_1\mathscr{K}^{-1} + a_2'\mathscr{K}^{-2}\ln\mathscr{K}^{-1} + a_1\mathscr{K}^{-2} + \cdots \tag{2.3}$$

where F_0 is the free molecular flow force; the coefficient a_1 is determined by sequences of three or more collisions taking place among two gas particles and the object, as illustrated in Figs. 1b–1d; a_2' is determined by sequences of four collisions among three particles and the sphere such as those illustrated in Fig. 1e; and a_2 is determined by collision sequences involving any number of particles greater than two and the sphere. For special models, a_1 and a_2' are known.[4] There is a close relation between the form of the expansion of F as given by (2.3) and the density expansion of the transport coefficients. For example, the viscosity of a moderately dense gas is given by

$$\eta/\eta_0 = 1 + \eta_1(n\sigma^3) + \eta_2'(n\sigma^3)^2\ln(n\sigma^3) + \eta_2(n\sigma^3)^2 + \cdots \tag{2.4}$$

[4] For a_1 see Ref. 10; for a_1 and a_2' see Ref. 9.

where n is the number density of the gas, η_0 the viscosity at low density, and σ an effective diameter of the gas molecules. This relation exists because the corresponding coefficients (F_0, η_0), (a_1, η_1), (a_2', η_2'), etc., are determined by similar kinds of dynamical events.[4,9–11]

3. THE HYDRODYNAMIC REGIME

In the hydrodynamic regime ($\mathscr{K} \ll 1$) the iteration expansion for G_B given by (2.2) is not very effective when inserted in Eq. (1.7) because the resulting expansion for the force on the sphere is roughly an expansion in increasing powers of $\mathscr{K}^{-1} = R/l$. Nevertheless, (1.7) is a convenient starting point for calculating the drag force on the sphere if one realizes that, except very close to the surface of the sphere, the distribution function is completely dominated by its hydrodynamic part.[1] We can take advantage of this by separating the Boltzmann propagator into a hydrodynamic and a nonhydrodynamic part. Just as in I, this is done by means of a projection operator P,[5] which projects onto the hydrodynamic eigenfunctions of the operator $\mathbf{v}\cdot\nabla - L$. The separated form of G_B reads

$$G_B = PG_BP + P_\perp G_B P_\perp \tag{3.1}$$

with $P_\perp = 1 - P$.

Next one substitutes (3.1) for all the Boltzmann propagators occurring in (1.5), separates $\hat{\Psi}$ according to $\hat{\Psi} = P\hat{\Psi} + P_\perp\hat{\Psi}$, and makes a resummation of the resulting series for $P\hat{\Psi}$ by collecting all terms containing one, two, three, etc. hydrodynamic propagators, respectively. One then obtains the following result for $P\hat{\Psi}$:

$$P\hat{\Psi} = -\{PG_BP\tilde{\mathbf{T}} + PG_BP\tilde{\bar{\mathbf{T}}}PG_BP\tilde{\mathbf{T}} + \cdots\}\beta m[\mathbf{v}\cdot\mathbf{V}(z)] \tag{3.2}$$

where

$$\tilde{\bar{\mathbf{T}}} = \bar{\mathbf{T}} + \bar{\mathbf{T}}P_\perp G_B P_\perp\bar{\mathbf{T}} + \bar{\mathbf{T}}P_\perp G_B P_\perp\bar{\mathbf{T}}P_\perp G_B P_\perp\bar{\mathbf{T}} + \cdots \tag{3.3a}$$

$$= \bar{\mathbf{T}} + \bar{\mathbf{T}}P_\perp[P_\perp(z + \mathbf{v}\cdot\nabla - L - \bar{\mathbf{T}})P_\perp]^{-1}P_\perp\bar{\mathbf{T}} \tag{3.3b}$$

$$\tilde{\mathbf{T}} = \mathbf{T} + \bar{\mathbf{T}}P_\perp[P_\perp(z + \mathbf{v}\cdot\nabla - L - \bar{\mathbf{T}})P_\perp]^{-1}P_\perp\mathbf{T} \tag{3.3c}$$

The physics of the dynamical processes corresponding to the terms on the right-hand sides of (3.2) and (3.3) can be considerably clarified if one realizes that the hydrodynamic propagator PG_BP describes processes where the particles travel over distances of several mean free paths, and that the propagator $P_\perp G_B P_\perp$ describes processes where the particles travel over only a few mean free paths. Then one can see that the term $\bar{\mathbf{T}}P_\perp G_B P_\perp\bar{\mathbf{T}}$ in Eq. (3.3a)

[5] We purposely do not specify which projection operator in I is being used here, since either the projection operator of I, Section 5 or that of I, Section 7 can be used.

describes a process in which a particle collides with the sphere and travels only a few mean free paths, i.e., remains in the boundary layer, before it collides with the sphere again. Hence the operators $\bar{\bar{\mathbf{T}}}$ and $\tilde{\mathbf{T}}$ can be considered as "renormalized" collision operators containing all dynamical processes that consist of a chain of particle–sphere collisions alternated by nonhydrodynamic propagators. On the other hand, a term such as $\bar{\mathbf{T}}PG_{\mathrm{B}}P\bar{\mathbf{T}}$ in Eq. (3.2) describes processes in which a particle, having collided once with the sphere, makes a long excursion into the fluid before coming back to hit the sphere again. Thus the dynamical events that contribute to Eq. (3.2) can be considered as sequences of collisions of fluid particles with the sphere, described by $\bar{\bar{\mathbf{T}}}$ or $\tilde{\mathbf{T}}$, connected by long excursions of the particles into the fluid. It is by such processes that the hydrodynamic flow field is set up around the sphere.

To continue with the calculation of the force on the sphere, we next notice that the orthogonal part of the distribution function can be expressed, similarly to (3.2), as

$$P_\perp\hat{\Psi} = -[P_\perp(z + \mathbf{v}\cdot\nabla - L - \bar{\mathbf{T}})P_\perp]^{-1}\{\mathbf{T} + \bar{\mathbf{T}}PG_{\mathrm{B}}P\tilde{\mathbf{T}} + \bar{\mathbf{T}}PG_{\mathrm{B}}P\bar{\bar{\mathbf{T}}}PG_{\mathrm{B}}P\tilde{\mathbf{T}} + \bar{\mathbf{T}}PG_{\mathrm{B}}P\bar{\bar{\mathbf{T}}}PG_{\mathrm{B}}P\bar{\bar{\mathbf{T}}}PG_{\mathrm{B}}P\tilde{\mathbf{T}} + \cdots\}\beta m[\mathbf{v}\cdot\mathbf{V}(z)] \tag{3.4}$$

Substitution of (3.2) and (3.4) into (1.6) yields for the drag force on the sphere

$$\mathbf{F}(z) = \langle m\mathbf{v}\{\tilde{\mathbf{T}} + \bar{\bar{\mathbf{T}}}PG_{\mathrm{B}}P\tilde{\mathbf{T}} + \bar{\bar{\mathbf{T}}}PG_{\mathrm{B}}P\bar{\bar{\mathbf{T}}}PG_{\mathrm{B}}P\tilde{\mathbf{T}} + \cdots\}\beta m[\mathbf{v}\cdot\mathbf{V}(z)]\rangle \tag{3.5}$$

where we used the properties

$$\bar{\mathbf{T}} = \mathbf{T} + (\mathbf{v}\cdot\hat{\mathbf{r}})\,\delta(r - R) \tag{3.6}$$

and

$$\langle(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{v}[\mathbf{v}\cdot\mathbf{V}(z)]\rangle = 0 \tag{3.7}$$

The interpretation of (3.5) is completely analogous to the interpretation given to (1.7) at the beginning of Section 2. The only difference is that the full Boltzmann propagator has been replaced by its projection onto the hydrodynamic subspace, and the simple molecule–sphere collision operators have been replaced by the renormalized ones, which describe the full complicated boundary-layer process through which a hydrodynamic mode "colliding" with the sphere can excite a new hydrodynamic mode.

Equations (3.2) and (3.5) can be evaluated by methods similar to those used in I. For a given value of the Laplace variable z, the space of distribution functions generated by the hydrodynamic propagator $PG_{\mathrm{B}}P$ is spanned by a set of six basic functions, and similarly the space of source terms generated by $P\bar{\bar{\mathbf{T}}}P$ or $P\tilde{\mathbf{T}}P$ is spanned by six basic source terms. If the basic functions are labeled $\hat{\Psi}_i$ and the basic source terms are labeled S_i (for details see I), then the

action of $P\tilde{\bar{\mathbf{T}}}P$ can be characterized by a matrix $\mathbf{A}$ such that A_{ij} is the amount of S_i generated by $P\bar{\mathbf{T}}P$ acting on $\hat{\boldsymbol{\Psi}}_j$. Similarly, the action of $PG_{\mathrm{B}}P$ can be characterized by a matrix $\mathbf{B}$ such that B_{ij} is the amount of $\hat{\Psi}_i$ generated by $PG_{\mathrm{B}}P$ acting on S_j. Furthermore the "initial source" $-P\tilde{\mathbf{T}}P\beta m(\mathbf{v}\cdot\mathbf{V})$ can be represented by the vector S^0 such that the element S_i^0 is the amount of source S_i present in $-P\tilde{\mathbf{T}}P\beta m(\mathbf{v}\cdot\mathbf{V})$. Similarly, $P\hat{\Psi}$ can be represented as a vector such that $(P\hat{\Psi})_i$ is the amount of $\hat{\Psi}_i$ present in $P\Psi$. With these conventions (3.2) can be rewritten as

$$P\hat{\Psi} = \{\mathbf{B} + \mathbf{BAB} + \mathbf{BABAB} + \cdots\}S^0 \tag{3.8}$$

The geometric series that occurs on the right-hand side of this expression can be formally summed, with the result

$$P\hat{\Psi} = \mathbf{B}(\mathbb{1} - \mathbf{AB})^{-1}S^0 \tag{3.9}$$

where $\mathbb{1}$ is the unit matrix. This result is identical to either Eqs. (6.33) and (6.34) of I or Eqs. (7.20) and (7.21) of I, depending on the choice of the projection operator P. This proves that the iterative method, applied in the hydrodynamic regime, leads to the same solution for the distribution function as the generalized normal solution method discussion in I, provided one applies the formal summation leading from (3.8) to (3.9).

Similarly, Eq. (3.5) for the force on the sphere can be rewritten formally as

$$\mathbf{F}(z) = -\langle m\mathbf{v}\{\mathbb{1} + \mathbf{AB} + \mathbf{ABAB} + \cdots\}S^0\rangle \tag{3.10a}$$

$$= -\langle m\mathbf{v}\{(\mathbb{1} - \mathbf{AB})^{-1}\}S^0\rangle \tag{3.10b}$$

Again, (3.10b) has been obtained by a formal summation of the geometric series contained in (3.10a), and it coincides with the result we obtained in I.

Let us consider the summation in more detail. From the explicit form of $\mathbf{AB}$ given in I, it follows that this matrix has eigenvalues of order $\mathscr{K}^{-1}$, hence much larger than unity. This means that we have summed the geometric series far outside its area of convergence. Clearly, what we have done is to sum the series inside its radius of convergence and then to analytically continue the sum into the region where $\mathscr{K}$ is small. It is apparent that this procedure is justified since it yields exactly the same result as we obtained before using a noniterative method.

The resummation of the divergent series (1.5) in order to obtain a result that is meaningful in the hydrodynamic regime is an example of a common method to treat collective effects in the kinetic theory of gases. Here, the influence of the sphere on the behavior of the system at small Knudsen numbers is so strong that one must take into account collectively all dynamic processes where there are one, two, three, . . . collisions between particles and the sphere such that the particle motions between these collisions are

described by hydrodynamic propagators. That is, between collisions with the sphere the particles make long excursions into the fluid, where they collide many times with other gas particles before returning to the sphere. The mean free path damping that is incorporated in G_0^d given in Section 2 is another example of a collective effect which needs to be taken into account in the flow of a rarefied gas around a large heavy sphere. In this case the divergent series that needs to be resummed is given by the right-hand side of Eq. (1.7) when the expression for G_B given by Eq. (2.1) is used. The required resummation takes into account the collisions that a molecule suffers with other molecules between its collisions with the sphere.

4. THE DRAG ON A CYLINDER

As a further application of our methods we consider the motion of an infinitely long circular cylinder in a dilute gas in a direction perpendicular to its axis, or, equivalently, the steady motion of a disk in a dilute two-dimensional gas.[6] This system again is described by a nonlinear extended Boltzmann equation of the same form as (I.2.1):

$$(\partial/\partial t)f(\mathbf{r}, \mathbf{v}, t) = (-\mathbf{v}\cdot\nabla)f(\mathbf{r}, \mathbf{v}, t) + [\mathbf{a}(t)\cdot\partial/\partial\mathbf{V}]f(\mathbf{r}, \mathbf{v}, t) + J(f,f) + \bar{\mathbf{T}}f(\mathbf{r}, \mathbf{v}, t) \tag{4.1}$$

and in the coordinate frame where the cylinder is at rest at all times. Again, $\mathbf{a}(t) = d\mathbf{V}(t)/dt$ is the linear acceleration of the cylinder at time t. The collision operator $\bar{\mathbf{T}}$ now describes the effect of collisions between gas particles and the cylinder. We assume again that it is given as a linear combination of a specular and a diffuse reflection operator

$$\bar{\mathbf{T}}_\alpha = \alpha\bar{\mathbf{T}}_{\mathrm{di}} + (1-\alpha)\bar{\mathbf{T}}_{\mathrm{sp}} \tag{4.2}$$

with

$$\bar{\mathbf{T}}_{\mathrm{sp}}f(\mathbf{r}, \mathbf{v}, t) = R\int_{-\infty}^{\infty} dx \int d\hat{\boldsymbol{\sigma}}\, |\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}|\, \delta(\mathbf{r} - R\hat{\boldsymbol{\sigma}} - x\hat{\mathbf{x}}) \times [\Theta(\mathbf{v}\cdot\hat{\boldsymbol{\sigma}})f^+(\mathbf{r}, \mathbf{v} - 2(\mathbf{v}\cdot\hat{\boldsymbol{\sigma}})\hat{\sigma}, t) - \Theta(-\mathbf{v}\cdot\hat{\boldsymbol{\sigma}})f^+(\mathbf{r}, \mathbf{v}, t)] \tag{4.3a}$$

$$\bar{\mathbf{T}}_{\mathrm{di}}f(\mathbf{r}, \mathbf{v}, t) = R\int_{-\infty}^{\infty} dx \int d\hat{\boldsymbol{\sigma}}\, \delta(\mathbf{r} - R\hat{\boldsymbol{\sigma}} - x\hat{\mathbf{x}})|\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}| \times \left[(2\pi\beta_w m)^{1/2}\Theta(\mathbf{v}\cdot\hat{\boldsymbol{\sigma}})\varphi_w(v)\int d\mathbf{v}'\, \Theta(-\mathbf{v}'\cdot\hat{\boldsymbol{\sigma}})|\mathbf{v}'\cdot\hat{\boldsymbol{\sigma}}| \times f^+(\mathbf{r}, \mathbf{v}', t) - \Theta(-\mathbf{v}'\cdot\hat{\boldsymbol{\sigma}})f^+(\mathbf{r}, \mathbf{v}, t)\right] \tag{4.3b}$$

[6] We do not take into account here the various problems that appear in the kinetic description of dense two-dimensional fluids.

These equations are analogous to (I.2.10). Now, however, the $\hat{\sigma}$ integration runs over a unit circle perpendicular to the cylinder axis. The latter is assumed to run through the origin in the x direction. Here R is the radius of the cylinder, and $(k_B\beta_w)^{-1}$ is the wall temperature.

For the case of slow flow of the gas around the cylinder, one might be tempted to compute the force on the cylinder by following the method outlined in the previous section for calculating the force on a sphere. There we obtained the result that the frequency-dependent force on the sphere is related to the velocity of the sphere (in the laboratory frame) by

$$\mathbf{F}(z) = \zeta(z)\mathbf{V}(z) \tag{4.4}$$

Here $\zeta(z)$ is the friction coefficient, which can be expressed as

$$\zeta(z) = \langle m[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\{\bar{\mathbf{T}} + \bar{\mathbf{T}}G_B\mathbf{T} + \cdots\}\beta m[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\rangle \tag{4.5}$$

where $\hat{\mathbf{V}}(z)$ is a unit vector in the direction of $\mathbf{V}(z)$. One might then expect that Eqs. (4.4) and (4.5) could be generalized to hold for the force on a cylinder simply by replacing the $\bar{\mathbf{T}}$ and $\mathbf{T}$ operators for the sphere by the corresponding $\bar{\mathbf{T}}$ and $\mathbf{T}$ operators defined for a cylinder and by computing the force per unit length rather than the total force on the cylinder. However, when one attempts to compute $\zeta(z)$ from Eq. (4.5), one immediately runs into difficulties. To see the origin of the difficulties, consider the various terms in the expansion of $\zeta(z)$ as

$$\zeta(z) = \zeta_0(z) + \zeta_1(z) + \zeta_2(z) + \cdots \tag{4.6a}$$

with

$$\zeta_0(z) = \beta m^2\langle[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\bar{\mathbf{T}}[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\rangle \tag{4.6b}$$

$$\zeta_1(z) = \beta m^2\langle[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\bar{\mathbf{T}}(z + \mathbf{v}\cdot\mathbf{V} - L)^{-1}\mathbf{T}[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\rangle \tag{4.6c}$$

Here $\mathbf{T} = \bar{\mathbf{T}} - (\mathbf{v}\cdot\mathbf{V})W_c(r)$, where $W_c(r)$ is unity for all points r outside the cylinder and zero otherwise. Further,

$$\zeta_2(z) = \beta m^2\langle[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\bar{\mathbf{T}}(z + \mathbf{v}\cdot\mathbf{V} - L)^{-1}\bar{\mathbf{T}} \times (z + \mathbf{v}\cdot\mathbf{V} - L)^{-1}\mathbf{T}[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\rangle \tag{4.6d}$$

and so on. The first term $\zeta_0(z)$ is simply the friction coefficient in the free molecular flow approximation. For the flow around a cylinder $\zeta_0(z)$ is proportional to the length of the cylinder, but is otherwise well defined. As in the expansion (1.7) for the flow around a sphere, the next term $\zeta_1(z)$ is the contribution to the friction coefficient from correlated dynamical processes where there are two collisions between the gas molecules and the cylinder and an arbitrary number of intermediate collisions between the gas molecules.

This term is not well behaved for small z in that it is proportional to $\ln z^{-1}$ for small z.[4] To see this, write $\zeta_1(z)$ as

$$\zeta_1(z) = \beta m^2 n \int d\mathbf{k}\,(2\pi)^{-3} \int d\mathbf{v}\,[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\bar{\mathbf{T}}_{\mathbf{k}}(\mathbf{v}) \times (z + i\mathbf{k}\cdot\mathbf{v} - L)^{-1}\mathbf{T}_{-\mathbf{k}}(\mathbf{v})[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\varphi(v) \tag{4.7}$$

Here $\bar{\mathbf{T}}_{\mathbf{k}}$ and $\bar{\mathbf{T}}_{-\mathbf{k}}$ are given by

$$\bar{\mathbf{T}}_{\mathbf{k}}(\mathbf{v}) = \int d\mathbf{r}\,\bar{\mathbf{T}}(\mathbf{r}, \mathbf{v}) \exp(-i\mathbf{k}\cdot\mathbf{r}) = 2\pi\,\delta(k_x)\bar{\boldsymbol{\Gamma}}(k_y, k_z, \mathbf{v}) \tag{4.8a}$$

and

$$\mathbf{T}_{-\mathbf{k}}(\mathbf{v}) = \int d\mathbf{r}\,\mathbf{T}(\mathbf{r}, \mathbf{v}) \exp(i\mathbf{k}\cdot\mathbf{r}) = 2\pi\,\delta(k_x)\,\boldsymbol{\Gamma}(-k_y, -k_z, \mathbf{v}) \tag{4.8b}$$

where we used Eq. (4.3) to show that these operators are proportional to $\delta(k_x)$ and to define the operators $\bar{\boldsymbol{\Gamma}}$ and $\boldsymbol{\Gamma}$. When inserted into Eq. (4.7), one of the $(2\pi)\,\delta(k_x)$ terms reduces the $\mathbf{k}$ integration to a two-dimensional integral and the other $(2\pi)\,\delta(k_x)$ is replaced by L, the total length of the cylinder. Then $\zeta_1(z)/L$, the friction coefficient per unit length, becomes

$$\frac{\zeta_1(z)}{L} = \beta m^2 n \int d\mathbf{v}\,(2\pi)^{-2} \int d\mathbf{q}\,[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\bar{\boldsymbol{\Gamma}}(\mathbf{q}, \mathbf{v}) \times (z + i\mathbf{q}\cdot\mathbf{v} - L)^{-1}\boldsymbol{\Gamma}(-\mathbf{q}, \mathbf{v})[\mathbf{v}\cdot\hat{\mathbf{V}}(z)]\varphi(v) \tag{4.9}$$

where $\mathbf{q}$ is a *two*-dimensional wave vector. Now one can analyze the expression for $\zeta_1(z)/L$ in terms of the eigenfunction and eigenvalues of the operator $i\mathbf{q}\cdot\mathbf{v} - L$. These eigenfunctions and eigenvalues have been discussed in I and elsewhere.[1,12,13] The main result of interest to us here is that there are two eigenfunctions, the so-called hydrodynamic shear modes, with eigenvalues proportional to q^2, which contribute to $\zeta_1(z)/L$ given by Eq. (4.9) and lead to contributions of the form

$$\frac{\zeta_1(z)}{L} = \int_{q<q_0} d\mathbf{q}\,(2\pi)^{-2} \frac{1}{z + vq^2} \approx \ln \frac{vq_0^2}{z} \tag{4.10}$$

for small z, where v is the kinematic viscosity of the gas, and q_0^{-1} is on the order of a mean free path length. A similar analysis shows that $\zeta_n(z)/L \approx (\ln vq_0^2/z)^n$ for small z, where $\zeta_n(z)$ is given by the appropriate term in Eq. (4.6). These $\ln z$ divergences are, in fact, very similar to the $\ln z$ divergences that appear in the time correlation function expressions for the transport coefficients in the linearized Navier–Stokes equations for two-dimensional systems.[12]

The fact that only $\zeta_0(z)$ in Eq. (4.6) is well behaved for small z suggests

that the expression given in (4.6) is not suitable for describing the situation near steady flow of the gas around a cylinder. Although one might consider other possibilities, it seems clear that the origin of these difficulties lies in the assumption that the force on the cylinder is linearly proportional to the velocity of the cylinder $\mathbf{V}(z)$. Consequently, it appears that one must take into account somehow the fact that far from the cylinder the gas is moving with velocity $-\mathbf{V}(z)$ with respect to the cylinder. This can be done by linearizing the distribution function f about the equilibrium distribution in the laboratory frame[17] instead of in the rest frame of the cylinder, as we did in Eq. (1.2) for the case of the flow around a sphere.[7] Such a procedure has been carried out for the case of rarefied gas flow around a cylinder and it leads to a well-behaved expansion for the force $\mathbf{F}$ on the cylinder, which, in the case of slow, steady flow reads,[4]

$$F/F_0 = 1 + K^{-1}[a \ln M + O(M^0)] + K^{-2}[b(\ln M)^2 + \cdots] + \cdots \quad (4.11)$$

Here F_0 is the free-molecular-flow force, and M is the Mach number V/c. The coefficient a is known for a few cases, but very few of the other terms in the expansion have been computed yet. The central point of interest, though, is that the force on the cylinder is not linearly proportional to the velocity of the cylinder in the laboratory frame.

In the case that the Knudsen number is very small, we expect to obtain the results of continuum hydrodynamics from the extended Boltzmann equation (4.1). Now it is well known from hydrodynamics[3] that the force on the cylinder cannot be expressed by $\mathbf{F}(z) = \zeta(z)\mathbf{V}(z)$ where $\zeta(z)$ is well behaved for small z, so the difficulties discussed above must persist in the hydrodynamic limit, too. To study the solution of Eq. (4.1) in the hydrodynamic limit, we try to construct a solution that reduces to the Chapman–Enskog normal solution far from the cylinder. To see where the difficulties come from in the hydrodynamic limit, we note that if (4.1) is linearized about the Maxwell–Boltzmann distribution in the rest frame of the cylinder, there can be no stationary solution of the linearized extended Boltzmann equation that reduces to the Chapman–Enskog equation far from the cylinder. This is because the Chapman–Enskog method then leads to a set of linearized hydrodynamic equations that have no stationary solution irrespective of the type of boundary condition imposed on the fluid fields at the surface of the cylinder.[8] In hydrodynamics texts it is argued that one cannot neglect the nonlinear convective terms in the Navier–Stokes equations in describing the region far from the cylinder.[3] Oseen managed to recover solvable stationary

[7] A similar linearization is responsible for the difficulties in two-dimensional hydrodynamics alluded to earlier, and it can be modified to give well-behaved hydrodynamic equations.[14,15]

[8] We have in mind only the usual slip boundary conditions with arbitrary slip coefficient.

Navier–Stokes equations by adding simple approximations to the convective terms, which are asymptotically correct far away from the cylinder.[16] As we argued above, to construct a well-behaved solution of the extended Boltzmann equation we must linearize about the equilibrium distribution in the laboratory frame, and not in the rest frame of the cylinder. This was first noticed by Scharf.[17] Consequently we put

$$f(\mathbf{r}, \mathbf{v}, t) = n(\beta m/2\pi)^{3/2} \exp(-\beta m C^2/2)\,[W(r) + \Psi(\mathbf{r}, \mathbf{v}, t)] \tag{4.12}$$

with $\mathbf{C} = \mathbf{v} + \mathbf{V}$, as in (I.4.1), but we do not linearize the Maxwellian in $\mathbf{V}$. As we are interested in slow, nearly stationary flow, we will restrict ourselves here to the stationary case to simplify the analysis. The (stationary!) linearized extended Boltzmann equation satisfied by Ψ now reads

$$(\mathbf{v}\cdot\nabla - L - \bar{\mathbf{T}}')\Psi(\mathbf{r}, \mathbf{v}) = -\mathbf{T}'\beta m(\mathbf{v}\cdot\mathbf{V}) \tag{4.13}$$

where L is defined by

$$Lh = n \int d\mathbf{v}_1 \int b\, db\, d\varphi\, |\mathbf{v} - \mathbf{v}_1|(\beta m/2\pi)^{3/2}$$
$$\times [\exp(-\beta m|\mathbf{v}_1 + \mathbf{V}|^2/2)][h(\mathbf{v}_1') + h(\mathbf{v}') - h(\mathbf{v}_1) - h(\mathbf{v})] \tag{4.14}$$

$\bar{\mathbf{T}}'$ is related to $\bar{\mathbf{T}}$ by (I.4.5), and it has been assumed that $\beta = \beta_w$. As in Section 4 of I, Ψ can be separated into a hydrodynamic and an orthogonal part. The hydrodynamic part now is of the general form

$$P\Psi = \delta n(\mathbf{r}, t)/n + \beta m[\mathbf{C}\cdot\mathbf{u}(\mathbf{r}, t)] + (\tfrac{1}{2}\beta m C^2 - \tfrac{3}{2})\,\delta T(\mathbf{r}, t)/T$$
$$+ A(C^2)\mathbf{C}\cdot\nabla T(\mathbf{r}, t)/T + B(C^2)(\mathbf{CC} - \tfrac{1}{3}C^2\mathbb{1}):\nabla\mathbf{u} + O(\nabla^2) \tag{4.15}$$

where $\mathbf{u}(\mathbf{r}, t)$ is the fluid velocity in the laboratory frame.[9] This is similar to (I.4.7) but with $\mathbf{V}$ replaced by $\mathbf{C}$. As before, the normal solutions of the homogeneous linearized Boltzmann equation are characterized completely by the hydrodynamic flow fields, as one sees from (4.15), but now the latter satisfy the Oseen hydrodynamic equations,

$$-(\mathbf{V}\cdot\nabla)\,\delta n + n(\nabla\cdot\mathbf{u}) = 0 \tag{4.16a}$$

$$-(\mathbf{V}\cdot\nabla)\mathbf{u} + \beta^{-1}[\nabla\delta n + (n/T)\,\nabla\delta T] - \eta[\nabla^2\mathbf{u} + \tfrac{1}{3}\nabla(\nabla\cdot\mathbf{u})] = 0 \tag{4.16b}$$

$$-(\mathbf{V}\cdot\nabla)\,\delta T/T + \tfrac{2}{3}(\nabla\cdot\mathbf{u}) - \tfrac{5}{3}(D_T/T)\,\nabla^2\delta T = 0 \tag{4.16c}$$

Again $P\Psi$ can be obtained, either iteratively from (3.2), but now $G_B = (\mathbf{v}\cdot\nabla - L)^{-1}$, with L defined by (4.11), or from (I.4.3) (without $\partial/\partial t$) by the methods of I, Section 6 or I, Section 7. As we have seen, both methods are completely equivalent and require the same computational efforts.

[9] That is, $\mathbf{u}$ is the average of $\mathbf{v} + \mathbf{V}$ in the rest frame of the cylinder.

For simplicity we will follow the method of I, Section 6 here, although the method of I, Section 7, where the continuum limit was considered, could be applied equally well.

There are six basic hydrodynamic solutions $\Psi_1,\ldots,\Psi_6$ of the homogeneous linearized Boltzmann equation that can be generated in response to an initial momentum source of the form $\hat{\boldsymbol{\rho}}(\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\rho}})\,\delta(\rho - R)$, where $\hat{\mathbf{z}}$ is the direction of the velocity, $\hat{\rho} = (\mathbf{r} - x\hat{\mathbf{x}})/|\mathbf{r} - x\hat{\mathbf{x}}|$ is a unit vector orthogonal to the cylinder axis, and $\rho = (y^2 + z^2)^{1/2}$ is the distance to the cylinder axis. These solutions now are of the form (4.15) and $\Psi_1,\ldots,\Psi_4$ are determined by the following hydrodynamic flow fields:

$$
\begin{aligned}
\Psi_1\colon\quad & \mathbf{u} = W(\rho)u_0\hat{\mathbf{z}}\cdot\{[-C - \ln(Vr/4v)]\mathbb{1} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}\} \\
& \qquad + [1 - W(\rho)]u_0\hat{\mathbf{z}}[-C - \ln(VR/4v) + 1] \\
& \delta n/n = 2W(\rho)\beta m v u_0(1/\rho)(\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\rho}}) \\
& \delta T/T = 0 \\
\Psi_2\colon\quad & \mathbf{u} = W(\rho)u_0(R^2/\rho^2)\hat{\mathbf{z}}\cdot(\mathbb{1} - 2\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}) - [1 - W(\rho)]u_0\hat{\mathbf{z}} \qquad (4.17) \\
& \delta n/n = \delta T/T = 0 \\
\Psi_3\colon\quad & \mathbf{u} = \mathbf{0} \\
& \delta n/n = -\delta T/T = -W(\rho)(R/\rho)(\hat{\mathbf{z}}\cdot\hat{\rho}) \\
\Psi_4\colon\quad & \mathbf{u} = [1 - W(\rho)]u_0\hat{\mathbf{z}} \\
& \delta n/n = \delta T/T = 0
\end{aligned}
$$

Here C is Euler's constant and all the fluid fields have been determined up to corrections of order $R_e = RV/\nu$. As in Section (I.6), Ψ_1 and Ψ_2 have been given a nonvanishing part inside the cylinder to avoid the occurrence of density sources. The precise forms of Ψ_5 and Ψ_6 are irrelevant. As in I, they have a source that is of order l/R compared to the sources of $\Psi_1,\ldots,\Psi_4$, and as in I, Ψ_5 and Ψ_6 contains a part inside the cylinder that cannot be canceled by linear combinations of $\Psi_1,\ldots,\Psi_4$. Since the complete solution Ψ of Eq. (4.13) has to vanish inside the cylinder, one can argue again that Ψ_5 and Ψ_6 may be neglected systematically without affecting the final results. No replacement of Ψ by $\hat{\Psi}$ is needed. (In the stationary case a Laplace transform merely introduces an overall factor $1/z$.) and the Laplace variable z has to be set equal to zero everywhere. The distribution functions Ψ_i are determined by (4.15)

and (4.17). The matrices **B*** and **A*** assume the following forms:

$$\mathbf{A}^* = \begin{bmatrix} -n\lambda_1[-\ln(VR/4\nu) - C + 1]u_0 & n\lambda_1 u_0 & n\lambda_2 & 0 \\ -n\lambda_3[-\ln(VR/4\nu) - C]u_0 & -n\lambda_3 u_0 & -(n\lambda_3/\sigma)D_T & 0 \\ n\lambda_4[-\ln(VR/4\nu) - C + 1]u_0 & -n\lambda_4 u_0 & n\lambda_5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.18}$$

$$\mathbf{B}^* = \begin{bmatrix} R/4\eta u_0 & R/4\eta u_0 & 0 & 0 \\ 0 & R/4\eta u_0 & 0 & 0 \\ 0 & 0 & R/\lambda T & \frac{5}{2}Rk_B/\lambda \\ 0 & 0 & 0 & -1/nu_0 \end{bmatrix} \tag{4.19}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_5$ are given again by Eq. (I.6.14), and λ is the coefficient of thermal conductivity.

The separation of S^0 which was given by (I.6.23) goes according to (I.6.26)–(I.6.27) as before, with the initial sources given by (I.6.28) in the limit $\rho \to \infty$.[10] The source of $P\Psi$ is finally obtained with the aid of (I.6.34) as

$$S^* = S_{\mathrm{I}}^* + S_{\mathrm{II}}^* \tag{4.20a}$$

$$S_{\mathrm{I}}^* = \begin{pmatrix} 1 + n\lambda_3 R/4\eta \\ n\lambda_3 R/4\eta \\ 0 \\ 0 \end{pmatrix} \frac{4\eta V}{R\{(1 + n\lambda_3 R/2\eta)[1 - C - \ln(RV/4\eta)] - n\lambda_3 R/4\eta\}} \tag{4.20b}$$

$$S_{\mathrm{II}}^* = V \begin{pmatrix} 0 \\ 0 \\ \frac{5}{2}n\beta^{-1} \\ n \end{pmatrix} \tag{4.20c}$$

The drag force per unit length is obtained from this as

$$\frac{\mathbf{F}}{L} = -\pi R(C_1 + C_2)\hat{\mathbf{z}}$$

$$= -4\pi\eta\mathbf{V}\left(\frac{\alpha R}{2l_v} + 1\right)\left[\left(\frac{\alpha R}{2l_v} + 1\right)\left(1 - C - \ln\frac{VR}{4\nu}\right) - \frac{\alpha R}{4l_v}\right]^{-1} \tag{4.21}$$

[10] This is not the distance to the cylinder axis, but it is the quantity defined in I as $\rho = (\nu/z)^{1/2}$.

where C_1 and C_2 are defined by (I.6.36), and (I.6.15), (I.6.38), and the analog of (I.6.43) have been used. In the case of stick boundary conditions (α of order unity) this reduces to the well-known result of Lamb.[16] Notice that as the Reynolds number approaches zero, the difference between stick and slip goes to zero, as the logarithmic term in the denominator becomes dominant.

It is interesting to compare the force on the cylinder in the hydrodynamic limit, given by Eq. (4.21), with that in the rarefied gas, given by Eq. (4.11). Roughly speaking, we may say that in the rarefied regime the force per unit length is proportional to $V \ln(V/c)$, while in the hydrodynamic regime the force is proportional to $V[\ln(V/c)]^{-1}$. The relation between these two cases can be understood if we consider what would happen if we derived Eq. (4.21) by an iterative solution of (4.13) similar to that described in Section 1 for the force on the sphere. As is already suggested in (4.11), the iterated solution leads to an expansion of F/L in powers of $(R/l) \ln M$. The summation of this power series, together with the retention of all terms that are dominant for small $\mathscr{K}$, then leads to (4.21). For *extremely* small Mach number this resummation has to be done even in the rarefied gas case.

5. CONCLUSION

The drag force on a sphere or a cylinder may be computed by iterative solution of the extended Boltzmann equation. In the rarefied gas limit this method is closely connected to the standard solution methods with the aid of characteristics,[18] but an advantage of the present method is that the dynamical origin of each contribution is very easy to determine. In the hydrodynamic regime the same iterative method can be used, but because of the dominance of the hydrodynamic part of the Boltzmann propagator in this regime, an entirely different regrouping of terms is needed to calculate the drag. This calculation is, however, completely equivalent to the calculation by the extended normal solution method discussed in I. So one sees that basically the same dynamical processes are responsible for the drag force in both the Knudsen and the hydrodynamic regimes. However, in the Knudsen regime the dominant contribution comes from the free-molecular-flow term, and processes containing correlated collisions between gas particles and the macroscopic object are rare and produce only small corrections. In the hydrodynamic regime the dominant contribution to the drag force arises from the interplay of the hydrodynamic propagations of the particles far into the fluid and their collisions with the sphere.

Loosely speaking, one could say that the drag force exerted on a sphere is of the form $F \sim aR^2/[1 + a(R/l_v)]$, where a is a constant of order unity for all Knudsen numbers. For $R \ll l_v$ this result reduces to the free-molecular-flow

term and corrections may be obtained by expanding in powers of R/l_v. In the hydrodynamic regime Stokes' law is recovered, but one cannot simply obtain corrections by expanding in powers of l_v/R, since a is a complicated function of the Knudsen number depending on the detailed structure of the kinetic boundary layer. For the same reason it is very hard to make precise predictions about the behavior of the drag force for intermediate Knudsen numbers. Cercignani and co-workers[19] did a calculation for general Knudsen numbers on the basis of a variational method, modeling the linearized Boltzmann equation by the BGK equation. The agreement of these results with experiments is remarkably good. Otherwise, no results for intermediate Knudsen numbers are known.

Mehaffey and Cukier[20] calculated the friction coefficient—or equivalently, the diffusion coefficient—for a sphere moving through a dense, hard-sphere gas by an iterative method similar to the one discussed in this paper. Although they essentially used the methods of I, Section 7, they did not take the $O(l/R)$ corrections to the hydrodynamic eigenfunctions into account, and they did not realize that the geometric series of operators appearing in (3.2) and (3.5) has to be represented as a series of matrices. As a result, they found a value of $5\pi\eta RV$ for the drag on a sphere with a slip boundary condition, instead of the correct value $4\pi\eta RV$. Furthermore, they completely neglected the effects of the kinetic boundary layer. Bedeaux and Mazur[21] were able to obtain the correct result for the force on a sphere by applying an iterative method at the level of fluctuating hydrodynamics.

In the case of the cylinder, the most remarkable thing is that the response (the velocity of the cylinder) to a constant outside force is nonlinear, which follows from (4.21). This is intimately connected with the nonexistence of stationary linearized hydrodynamics in two dimensions. Even in the Knudsen regime this remains true, and for sufficiently small Reynolds numbers the "self-interaction" of the cylinder through hydrodynamic propagation dominates the drag. Moreover, as mentioned earlier, there is a close connection between the $\ln z$ divergences that appear in the friction coefficient per unit length for the cylinder and the $\ln z$ divergence that appears in the time correlation function expression for two-dimensional transport coefficients.[4,12–15] It also appears that these divergences must be removed in the same way, i.e., one must take into account some essential nonlinear features.[14,15]

Finally we want to touch on the connections with Brownian motion. It is clear that, according to Onsager's regression hypothesis,[22] the decay to equilibrium of fluctuations in the velocity of a Brownian particle are described on the average by the same hydrodynamic equations that determine the drag force. More explicitly, by the fluctuation-dissipation theorem the friction coefficient for Brownian motion can be expressed as a time integral of a force–

force correlation function.[11] In paper IV we will show that in the case of Brownian motion in a dilute gas, this expression for the friction coefficient reduces to the one we obtained in this paper for the motion of a macroscopic sphere. In addition, we will discuss the connection with the theory of diffusion of a microscopic particle, and to some extent the effects of a finite ratio of the mass of the particles to that of the Brownian particle.

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[11] See, e.g., the discussion of Brownian motion in Ref. 23.

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