The ABC-conjecture

Frits Beukers

ABC-day, Leiden 9 september 2005

ABC-hits

- The product of the distinct primes in a number is called the *radical* of that number. Notation: $\mathrm{rad}()$. For example,

$$\mathrm{rad}(2^2 \times 3^4) = 2 \times 3 = 6, \qquad \mathrm{rad}(2 \times 3 \times 5^2) = 2 \times 3 \times 5 = 30.$$

ABC-hits

- The product of the distinct primes in a number is called the *radical* of that number. Notation: $\mathrm{rad}()$. For example,

  $$\mathrm{rad}(2^2 \times 3^4) = 2 \times 3 = 6, \qquad \mathrm{rad}(2 \times 3 \times 5^2) = 2 \times 3 \times 5 = 30.$$

- Three positive integers A, B, C are called ABC-triple if they are coprime, $A < B$ and

  $$A + B = C$$

ABC-hits

- The product of the distinct primes in a number is called the *radical* of that number. Notation: $\mathrm{rad}()$. For example,

$$\mathrm{rad}(2^2 \times 3^4) = 2 \times 3 = 6, \qquad \mathrm{rad}(2 \times 3 \times 5^2) = 2 \times 3 \times 5 = 30.$$

- Three positive integers A, B, C are called ABC-triple if they are coprime, $A < B$ and

$$A + B = C$$

- Compute $\mathrm{rad}(ABC)$ and check whether $\mathrm{rad}(ABC) < C$. If this inequality is true we say that we have an ABC-hit!

ABC-hits

- The product of the distinct primes in a number is called the *radical* of that number. Notation: $\mathrm{rad}()$. For example,

$$\mathrm{rad}(2^2 \times 3^4) = 2 \times 3 = 6, \qquad \mathrm{rad}(2 \times 3 \times 5^2) = 2 \times 3 \times 5 = 30.$$

- Three positive integers A, B, C are called ABC-triple if they are coprime, $A < B$ and

$$A + B = C$$

- Compute $\mathrm{rad}(ABC)$ and check whether $\mathrm{rad}(ABC) < C$. If this inequality is true we say that we have an ABC-hit!
- Among all 15×10^6 ABC-triples with $C < 10000$ we have 120 ABC-hits.

ABC-hits

- The product of the distinct primes in a number is called the *radical* of that number. Notation: $\mathrm{rad}()$. For example,

$$\mathrm{rad}(2^2 \times 3^4) = 2 \times 3 = 6, \qquad \mathrm{rad}(2 \times 3 \times 5^2) = 2 \times 3 \times 5 = 30.$$

- Three positive integers A, B, C are called ABC-triple if they are coprime, $A < B$ and

$$A + B = C$$

- Compute $\mathrm{rad}(ABC)$ and check whether $\mathrm{rad}(ABC) < C$. If this inequality is true we say that we have an ABC-hit!
- Among all 15×10^6 ABC-triples with $C < 10000$ we have 120 ABC-hits.
- Among all 380×10^6 ABC-triples with $C < 50000$ we have 276 hits.

More hits

- Theorem: *There are infinitely many ABC-hits.*

More hits

- Theorem: *There are infinitely many ABC-hits.*
- Proof: Let us take $A = 1$ and $C = 3, 3^2, 3^4, 3^8, \ldots, 3^{2^k}, \ldots$.
  We determine how many factors 2 occur in $B = 3^{2^k} - 1$.

More hits

- Theorem: *There are infinitely many ABC-hits.*
- Proof: Let us take $A = 1$ and $C = 3, 3^2, 3^4, 3^8, \ldots, 3^{2^k}, \ldots$.
  We determine how many factors 2 occur in $B = 3^{2^k} - 1$.
- Notice

$$\begin{aligned} 3^{64} - 1 &= (3^{32}+1)(3^{32}-1) \\ &= (3^{32}+1)(3^{16}+1)(3^{16}-1) \\ &\ldots \\ &= (3^{32}+1)(3^{16}+1)(3^{8}+1)\cdots(3+1)(3-1) \end{aligned}$$

So $3^{64} - 1$ is divisible by $2 \cdot 2^8$.

More hits

- Theorem: *There are infinitely many ABC-hits.*
- Proof: Let us take $A = 1$ and $C = 3, 3^2, 3^4, 3^8, \ldots, 3^{2^k}, \ldots$.
  We determine how many factors 2 occur in $B = 3^{2^k} - 1$.
- In general $3^{2^k} - 1$ is divisible by 2^{k+2}. So

$$\mathrm{rad}(B) = \mathrm{rad}(3^{2^k} - 1) \leq (3^{2^k} - 1)/2^k < C/2^{k+1}$$

We conclude

$$\mathrm{rad}(ABC) = 3 \cdot \mathrm{rad}(B) < 3C/2^{k+1}.$$

More hits

- Theorem: *There are infinitely many ABC-hits.*
- Proof: Let us take $A = 1$ and $C = 3, 3^2, 3^4, 3^8, \ldots, 3^{2^k}, \ldots$.
  We determine how many factors 2 occur in $B = 3^{2^k} - 1$.
- In general $3^{2^k} - 1$ is divisible by 2^{k+2}. So

$$\mathrm{rad}(B) = \mathrm{rad}(3^{2^k} - 1) \leq (3^{2^k} - 1)/2^k < C/2^{k+1}$$

  We conclude

$$\mathrm{rad}(ABC) = 3 \cdot \mathrm{rad}(B) < 3C/2^{k+1}.$$

- In other words, $C > \mathrm{rad}(ABC) \cdot 2^{k+1}/3$. So when $k \geq 1$ we have an ABC-hit.

More hits

- Theorem: *There are infinitely many ABC-hits.*
- Proof: Let us take $A = 1$ and $C = 3, 3^2, 3^4, 3^8, \ldots, 3^{2^k}, \ldots$. We determine how many factors 2 occur in $B = 3^{2^k} - 1$.
- In general $3^{2^k} - 1$ is divisible by 2^{k+2}. So

$$\mathrm{rad}(B) = \mathrm{rad}(3^{2^k} - 1) \leq (3^{2^k} - 1)/2^k < C/2^{k+1}$$

We conclude

$$\mathrm{rad}(ABC) = 3 \cdot \mathrm{rad}(B) < 3C/2^{k+1}.$$

- In other words, $C > \mathrm{rad}(ABC) \cdot 2^{k+1}/3$. So when $k \geq 1$ we have an ABC-hit.
- But we have shown more. For any number $M > 1$ there exist infinitely many ABC-triples such that $C > M \cdot \mathrm{rad}(ABC)$.

Super hits

- Instead of something linear in $\mathrm{rad}(ABC)$ let us take something quadratic.
  Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^2$?

Super hits

- Instead of something linear in $\mathrm{rad}(ABC)$ let us take something quadratic.
  Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^2$?
- Answer: Unknown

Super hits

- Instead of something linear in $\mathrm{rad}(ABC)$ let us take something quadratic.
  Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^2$?
- Answer: Unknown
- Working *hypothesis*: For every ABC-triple: $C < \mathrm{rad}(ABC)^2$.

Super hits

- Instead of something linear in $\mathrm{rad}(ABC)$ let us take something quadratic.
  Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^2$?
- Answer: Unknown
- Working *hypothesis*: For every ABC-triple: $C < \mathrm{rad}(ABC)^2$.
- Consequence: Let x, y, z, n be positive integers such that $\gcd(x, y, z) = 1$ and $x^n + y^n = z^n$. Then the *hypothesis* implies $n < 6$.

Super hits

- Instead of something linear in $\mathrm{rad}(ABC)$ let us take something quadratic.
  Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^2$?
- Answer: Unknown
- Working *hypothesis*: For every ABC-triple: $C < \mathrm{rad}(ABC)^2$.
- Consequence: Let x, y, z, n be positive integers such that $\gcd(x, y, z) = 1$ and $x^n + y^n = z^n$. Then the *hypothesis* implies $n < 6$.
- Proof: Apply the *hypothesis* to the triple $A = x^n, B = y^n, C = z^n$. Notice that $\mathrm{rad}(x^n y^n z^n) \leq xyz < z^3$. So, $z^n < (z^3)^2 = z^6$. Hence $n < 6$.
  Fermat's Last Theorem for $n \geq 6$ follows!

Formulation

- Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^{1.5}$?

Formulation

- Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^{1.5}$?
- or $C > \mathrm{rad}(ABC)^{1.05}$?

Formulation

- Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^{1.5}$?
- or $C > \mathrm{rad}(ABC)^{1.05}$?
- or $C > \mathrm{rad}(ABC)^{1.005}$?

Formulation

- Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^{1.5}$?
- or $C > \mathrm{rad}(ABC)^{1.05}$?
- or $C > \mathrm{rad}(ABC)^{1.005}$?
- We expect at most finitely many instances.

Formulation

- Question: Are there ABC-triples such that $C > \mathrm{rad}(ABC)^{1.5}$?
- or $C > \mathrm{rad}(ABC)^{1.05}$?
- or $C > \mathrm{rad}(ABC)^{1.005}$?
- We expect at most finitely many instances.
- ABC-Conjecture (Masser-Oesterlé, 1985): *Let $\kappa > 1$. Then, with finitely many exceptions we have $C < \mathrm{rad}(ABC)^{\kappa}$.*

Fermat-Catalan

The Fermat-Catalan equation $x^p + y^q = z^r$ in x, y, z *coprime* positive integers. Of course we assume $p, q, r > 1$. We distinguish three cases.

Fermat-Catalan

The Fermat-Catalan equation $x^p + y^q = z^r$ in x, y, z *coprime* positive integers. Of course we assume $p, q, r > 1$. We distinguish three cases.

- 1) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. It is an exercise to show that (p, q, r) is a permutation of one of $(2, 2, k), (2, 3, 3), (2, 3, 4), (2, 3, 5)$. In any such case the number of solutions is infinite.

Fermat-Catalan

The Fermat-Catalan equation $x^p + y^q = z^r$ in x, y, z *coprime* positive integers. Of course we assume $p, q, r > 1$. We distinguish three cases.

- 1) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. It is an exercise to show that (p, q, r) is a permutation of one of $(2, 2, k), (2, 3, 3), (2, 3, 4), (2, 3, 5)$. In any such case the number of solutions is infinite.
- 2) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Again it is an exercise to show that (p, q, r) is a permutation of one of $(2, 4, 4), (2, 3, 6), (3, 3, 3)$. There are finitely many solutions.

Fermat-Catalan

The Fermat-Catalan equation $x^p + y^q = z^r$ in x, y, z *coprime* positive integers. Of course we assume $p, q, r > 1$. We distinguish three cases.

- 1) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. It is an exercise to show that (p, q, r) is a permutation of one of $(2, 2, k), (2, 3, 3), (2, 3, 4), (2, 3, 5)$. In any such case the number of solutions is infinite.
- 2) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Again it is an exercise to show that (p, q, r) is a permutation of one of $(2, 4, 4), (2, 3, 6), (3, 3, 3)$. There are finitely many solutions.
- 3) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. There are infinitely many possible triples (p, q, r). For any such triple the number of solutions is at most finite (Darmon-Granville, 1995).

Numeric results

$$1^k + 2^3 = 3^2 \ (k > 6), \qquad 13^2 + 7^3 = 2^9, \qquad 2^7 + 17^3 = 71^2$$

$$2^5 + 7^2 = 3^4, \qquad 3^5 + 11^4 = 122^2, \qquad 17^7 + 76271^3 = 21063928^2$$

$$1414^3 + 2213459^2 = 65^7, \qquad 33^8 + 1549034^2 = 15613^3$$

$$43^8 + 96222^3 = 30042907^2, \qquad 9262^3 + 15312283^2 = 113^7.$$

Fermat-Catalan conjecture

Consequence of ABC-conjecture:
The set of triples x^p, y^q, z^r with x, y, z coprime positive integers such that $x^p + y^q = z^r$ and $1/p + 1/q + 1/r < 1$, is finite.

Fermat-Catalan conjecture

Consequence of ABC-conjecture:
The set of triples x^p, y^q, z^r with x, y, z coprime positive integers such that $x^p + y^q = z^r$ and $1/p + 1/q + 1/r < 1$, is finite.

- Observation, $1/p + 1/q + 1/r < 1$ implies $1/p + 1/q + 1/r \leq 1 - 1/42$.

Fermat-Catalan conjecture

Consequence of ABC-conjecture:
The set of triples x^p, y^q, z^r with x, y, z coprime positive integers such that $x^p + y^q = z^r$ and $1/p + 1/q + 1/r < 1$, is finite.

- Observation, $1/p + 1/q + 1/r < 1$ implies $1/p + 1/q + 1/r \leq 1 - 1/42$.
- Apply ABC with $\kappa = 1.01$ to $A = x^p, B = y^q, C = z^r$. Notice that $\mathrm{rad}(x^r y^q z^r) \leq xyz < z^{r/p} z^{r/q} z$.

Fermat-Catalan conjecture

Consequence of ABC-conjecture:
The set of triples x^p, y^q, z^r with x, y, z coprime positive integers such that $x^p + y^q = z^r$ and $1/p + 1/q + 1/r < 1$, is finite.

- Observation, $1/p + 1/q + 1/r < 1$ implies $1/p + 1/q + 1/r \leq 1 - 1/42$.
- Apply ABC with $\kappa = 1.01$ to $A = x^p, B = y^q, C = z^r$. Notice that $\mathrm{rad}(x^r y^q z^r) \leq xyz < z^{r/p} z^{r/q} z$.
- Hence, with finitely many exceptions we get

$$z^r < z^{\kappa(r/p + r/q + 1)}$$

Fermat-Catalan conjecture

Consequence of ABC-conjecture:
The set of triples x^p, y^q, z^r with x, y, z coprime positive integers such that $x^p + y^q = z^r$ and $1/p + 1/q + 1/r < 1$, is finite.

- Observation, $1/p + 1/q + 1/r < 1$ implies $1/p + 1/q + 1/r \leq 1 - 1/42$.
- Apply ABC with $\kappa = 1.01$ to $A = x^p, B = y^q, C = z^r$. Notice that $\mathrm{rad}(x^r y^q z^r) \leq xyz < z^{r/p} z^{r/q} z$.
- Hence, with finitely many exceptions we get

$$z^r < z^{\kappa(r/p + r/q + 1)}$$

- This implies $r < \kappa(r/p + r/q + 1)$ and hence $1 < \kappa(1/p + 1/q + 1/r)$. But this is impossible because $\kappa = 1.01$ and $1/p + 1/q + 1/r \leq 1 - 1/42$.

Catalan

As a special case we see that $x^p - y^q = 1$ has finitely many solutions.
But this was shown in 1974 by Tijdeman and completely solved in 2002 by Michailescu.

Mordell's conjecture

Consider a diophantine equation $P(x, y) = 0$ in the unknown rational numbers x, y.
For example
$x^5 + 3x^2y - y^3 + 1 = 0, \; x^4 + y^4 + 3xy + x^3 - y^3 = 0$, etc.

Mordell's conjecture

Consider a diophantine equation $P(x, y) = 0$ in the unknown rational numbers x, y.
For example
$x^5 + 3x^2y - y^3 + 1 = 0,\ x^4 + y^4 + 3xy + x^3 - y^3 = 0$, etc.

Noam Elkies (1991) observed:

The *ABC*-conjecture implies: If $\mathrm{genus}(P) > 1$ then the number of rational solutions to $P(x, y) = 0$ is at most finite.

Mordell's conjecture

Consider a diophantine equation $P(x, y) = 0$ in the unknown rational numbers x, y.
For example
$x^5 + 3x^2y - y^3 + 1 = 0,\ x^4 + y^4 + 3xy + x^3 - y^3 = 0$, etc.

Noam Elkies (1991) observed:

The ABC-conjecture implies: If $\mathrm{genus}(P) > 1$ then the number of rational solutions to $P(x, y) = 0$ is at most finite.

Previously known as Mordell's conjecture (1923) and Faltings' theorem (1983).

Schinzel-Tijdeman theorem

- An integer n is called a *perfect power* if it is either a square, a cube, a fourth power, etc of another integer.

Schinzel-Tijdeman theorem

- An integer n is called a *perfect power* if it is either a square, a cube, a fourth power, etc of another integer.
- Let $P(x)$ be a polynomial with integer coefficients and at least three simple zeros.

Schinzel-Tijdeman theorem

- An integer n is called a *perfect power* if it is either a square, a cube, a fourth power, etc of another integer.
- Let $P(x)$ be a polynomial with integer coefficients and at least three simple zeros.
- Theorem (Schinzel-Tijdeman, 1976) Among the numbers $P(1), P(2), P(3), \ldots$ there are at most finitely many perfect powers.

Schinzel-Tijdeman theorem

- An integer n is called a *perfect power* if it is either a square, a cube, a fourth power, etc of another integer.
- Let $P(x)$ be a polynomial with integer coefficients and at least three simple zeros.
- Theorem (Schinzel-Tijdeman, 1976) Among the numbers $P(1), P(2), P(3), \ldots$ there are at most finitely many perfect powers.
- Example: $P(x) = x^3 + 17$. We have $2^3 + 17 = 5^2,\ 4^3 + 17 = 9^2,\ 8^3 + 17 = 23^2,\ 43^3 + 17 = 282^2$ $52^3 + 17 = 375^2,\ 5234^3 + 17 = 378661^2$.

Schinzel-Tijdeman conjecture

- An integer n is called *powerfull* if all of its prime divisors occur with exponent 2 or higher in the prime factorisation.

Schinzel-Tijdeman conjecture

- An integer n is called *powerfull* if all of its prime divisors occur with exponent 2 or higher in the prime factorisation.
- Gary Walsh (1998) observed that the ABC-conjecture implies the Schinzel-Tijdeman conjecture: *among the numbers* $P(1), P(2), P(3), \ldots$ *there are at most finitely many powerful numbers.*

Schinzel-Tijdeman conjecture

- An integer n is called *powerfull* if all of its prime divisors occur with exponent 2 or higher in the prime factorisation.
- Gary Walsh (1998) observed that the ABC-conjecture implies the Schinzel-Tijdeman conjecture: *among the numbers* $P(1), P(2), P(3), \ldots$ *there are at most finitely many powerful numbers.*
- Example: $P(x) = x^3 + 17$. We have $2^3 + 17 = 5^2,\ 4^3 + 17 = 9^2,\ 8^3 + 17 = 23^2,\ 43^3 + 17 = 282^2$ $52^3 + 17 = 375^2,\ 5234^3 + 17 = 378661^2$.

State of knowledge

What do we know about ABC?

State of knowledge

What do we know about ABC?

Stewart, Kunrui Yu (1996): *For any* $\epsilon > 0$*:*

$$C < \exp\left(\gamma \mathrm{rad}(ABC)^{1/3+\epsilon}\right)$$

where γ *depends on the choice of* ϵ*.*

An analogy

Why do we believe in ABC ?

An analogy

Why do we believe in ABC ?

There is an analogy with polynomials with rational numbers as coefficients: $\mathbb{Q}[x]$.

An analogy

Why do we believe in ABC ?

There is an analogy with polynomials with rational numbers as coefficients: $\mathbb{Q}[x]$.

A polynomial $F(x)$ with rational coefficients and leading coefficient 1 is called *prime* if it cannot be factored into polynomials with rational coefficients and lower degree.

An analogy

Why do we believe in ABC ?

There is an analogy with polynomials with rational numbers as coefficients: $\mathbb{Q}[x]$.

A polynomial $F(x)$ with rational coefficients and leading coefficient 1 is called *prime* if it cannot be factored into polynomials with rational coefficients and lower degree.

Theorem: *Any polynomial with rational numbers as coefficient can be written in a unique way as a constant times a product of prime polynomials.*

Factors of polynomials

For example: x^2+1, whereas x^2-1 is reducible. Example of a factorisation:

$$\begin{aligned} x^{21}-1 &= (x^6+x^5+x^4+x^3+x^2+x+1)\times \\ &\quad (x-1)(x^2+x+1)\times \\ &\quad (x^{12}-x^{11}+x^9-x^8+x^6-x^4+x^3-x+1). \end{aligned}$$

Degree of a polynomial F: $\deg(F)$.

The *radical* of a polynomial $F(x)$ is the product of the prime polynomials dividing $F(x)$. Notation $\mathrm{rad}(F)$.

PQR-Theorem

PQR-Theorem (Hurwitz, Stothers, Mason): Let P, Q, R be coprime polynomials, not all constant, such that $P + Q = R$. Suppose that $\deg(R) \geq \deg(P), \deg(Q)$. Then

$$\deg(R) < \deg(\mathrm{rad}(PQR)).$$

PQR-Theorem

PQR-Theorem (Hurwitz, Stothers, Mason): Let P, Q, R be coprime polynomials, not all constant, such that $P + Q = R$. Suppose that $\deg(R) \geq \deg(P), \deg(Q)$. Then

$$\deg(R) < \deg(\mathrm{rad}(PQR)).$$

Translation to *ABC*: Replace P, Q, R by A, B, C and deg by log. Note the analogy: $\deg(PQ) = \deg(P) + \deg(Q)$ for polynomials and $\log(ab) = \log(a) + \log(b)$ for numbers. We get:

$$\log(C) < \log(\mathrm{rad}(ABC)).$$

Proof of PQR, I

Observe that for any polynomial F,

$$\operatorname{rad}(F) = F/\gcd(F, F')$$

Proof of PQR, I

Observe that for any polynomial F,

$$\mathrm{rad}(F) = F/\gcd(F, F')$$

Example, $F = x^3(x-1)^5$. Then $F' = (8x-5)x^2(x-1)^4$. Hence $\gcd(F, F') = x^2(x-1)^4$ and $F/\gcd(F, F') = x(x-1)$.

Proof of PQR, I

Observe that for any polynomial F,

$$\mathrm{rad}(F) = F/\gcd(F, F')$$

Example, $F = x^3(x-1)^5$. Then $F' = (8x-5)x^2(x-1)^4$. Hence $\gcd(F, F') = x^2(x-1)^4$ and $F/\gcd(F, F') = x(x-1)$.

Start with

$$P + Q = R$$

and differentiate:

$$P' + Q' = R'$$

Proof of PQR, I

Observe that for any polynomial F,

$$\mathrm{rad}(F) = F / \gcd(F, F')$$

Example, $F = x^3(x-1)^5$. Then $F' = (8x-5)x^2(x-1)^4$. Hence $\gcd(F, F') = x^2(x-1)^4$ and $F/\gcd(F, F') = x(x-1)$.

Start with

$$P + Q = R$$

and differentiate:

$$P' + Q' = R'$$

Muliply first equality by P', second equality by P and subtract,

$$P'Q - pQ' = P'R - PR'$$

Proof of PQR, II

$$P'Q - pQ' = P'R - PR'$$

Proof of PQR, II

$$P'Q - pQ' = P'R - PR'$$

So, $\gcd(R, R')$ divides $P'Q - PQ'$. A fortiori, $\gcd(R, R')$ divides

$$\frac{P'Q - PQ'}{\gcd(P, P')\gcd(Q, Q')}.$$

Proof of PQR, II

$$P'Q - pQ' = P'R - PR'$$

So, $\gcd(R, R')$ divides $P'Q - PQ'$. A fortiori, $\gcd(R, R')$ divides

$$\frac{P'Q - PQ'}{\gcd(P, P')\gcd(Q, Q')}.$$

Consequently, if $P'Q - pQ' \neq 0$,

$$\deg(\gcd(R, R') < \deg(\mathrm{rad}(P)) + \deg(\mathrm{rad}(Q)) = \deg(\mathrm{rad}(PQ)).$$

Proof of PQR, II

$$P'Q - pQ' = P'R - PR'$$

So, $\gcd(R, R')$ divides $P'Q - PQ'$. A fortiori, $\gcd(R, R')$ divides

$$\frac{P'Q - PQ'}{\gcd(P, P')\gcd(Q, Q')}.$$

Consequently, if $P'Q - pQ' \neq 0$,

$$\deg(\gcd(R, R') < \deg(\mathrm{rad}(P)) + \deg(\mathrm{rad}(Q)) = \deg(\mathrm{rad}(PQ)).$$

Add $\deg(R/\gcd(R, R')) = \deg(\mathrm{rad}(R))$ to get

$$\deg(R) < \deg(\mathrm{rad}(PQR)).$$

Proof of PQR, II

$$P'Q - pQ' = P'R - PR'$$

So, $\gcd(R, R')$ divides $P'Q - PQ'$. A fortiori, $\gcd(R, R')$ divides

$$\frac{P'Q - PQ'}{\gcd(P, P')\gcd(Q, Q')}.$$

Consequently, if $P'Q - pQ' \neq 0$,

$$\deg(\gcd(R, R') < \deg(\mathrm{rad}(P)) + \deg(\mathrm{rad}(Q)) = \deg(\mathrm{rad}(PQ)).$$

Add $\deg(R/\gcd(R, R')) = \deg(\mathrm{rad}(R))$ to get

$$\deg(R) < \deg(\mathrm{rad}(PQR)).$$

If $P'Q - PQ' = 0$, then P/Q is constant and all of P, Q, R are constant.

The quest

Main questions

The quest

Main questions

- If the ABC-conjecture is true, there should be a minimal number κ such that $C \geq \mathrm{rad}(ABC)^{\kappa}$ for all ABC-triples. What is the value of κ ?

The quest

Main questions

- If the ABC-conjecture is true, there should be a minimal number κ such that $C \geq \mathrm{rad}(ABC)^{\kappa}$ for all ABC-triples. What is the value of κ ?
- How does the number of ABC-hits with $C < X$ grow as $X \to \infty$? Are there distribution laws? How are the ratios $\log(C)/\log(\mathrm{rad}(ABC))$ distributed?

The quest

Main questions

- If the ABC-conjecture is true, there should be a minimal number κ such that $C \geq \mathrm{rad}(ABC)^{\kappa}$ for all ABC-triples. What is the value of κ ?
- How does the number of ABC-hits with $C < X$ grow as $X \to \infty$? Are there distribution laws? How are the ratios $\log(C)/\log(\mathrm{rad}(ABC))$ distributed?

Happy hunting, or fishing!